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The Fundamental Solution of the Parabolic Equation in a Differentiable Manifold

By Seizô ITô

§0. **Introduction.** The existence of the fundamental solution of the parabolic differential equation in the Euclidean m-space has been shown by W. Feller [2] for $m=1$, and by F. G. Dressel [1] for general m. Recently Prof. K. Yosida has generalized the result by an entirely different approach to the case of a Riemannian space. (See the immediately preceding paper by K. Yosida in this issue.) In the present paper, we shall show that the fundamental solution may be constructed for the case of a differentiable manifold, by means of Feller-Dressel's idea, and that the case of a Riemannian space may be deduced from the result.

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§1. **Preliminary notions and main theorems.** Let $\boldsymbol{M}$ be an m-dimensional manifold of C^2-class such that the function of the transformation between two local coordinates has partial derivatives of second order each of which satisfies a Lipschitz condition of order $\gamma(0<\gamma\leqq 1)$ at every point[1], and fix s_0 and t_0 such that $-\infty\leqq s_0<t_0\leqq\infty$.

First we give the following

DEFINITION 1. (Cf. [3] p. 42) Let $f_1(t, x), \dots, f_n(t, x)$ be functions on $(s_0, t_0)\times\boldsymbol{M}$ which depend on the local coordinate around x[2]. The system of functions $\{f_1, \dots, f_n\}$ is said to be *bounded by K* if there exist a canonical coordinate system on $\boldsymbol{M}$ and constant $K>0$ such that

$$|f_i(t, y)|\leqq K, \quad i=1, \dots, n;\ s_0<t<t_0,\ y\in S_x,$$

1) We say that a function $f(x)$ satisfies the *Lipschitz condition of order* $\gamma(>0)$ at x if there exist constants N and $\delta>0$ such that $|f(x)-f(y)|\leqq N\sum_i|x^i-y^i|^\gamma$ whenever $|x^i-y^i|\leqq\delta$, $i=1, \dots, m$, where (x^i) and (y^i) respectively denote the local coordinates; such notion may be defined for functions on $(s_0, t_0)\times\boldsymbol{M}$ (Cf. [1], [2]).

2) Examples of such functions are $a^{ij}(t, x)$ and $b^i(t, x)$ stated below.

for any $x \in \boldsymbol{M}$, where a *canonical coordinate system* should be understood as defined in pp. 41—42 in [3] and S_x denotes the unit sphere with the centre x with respect to the canonical coordinate around x.

We consider the parabolic differential operator L:

$$L \equiv L_{tx} = A_{tx} - \frac{\partial}{\partial t} \tag{1.1}$$

where

$$A \equiv A_{tx} = a^{ij}(t, x)\frac{\partial^2}{\partial x^i \partial x^j} + b^i(t, x)\frac{\partial}{\partial x^i} + c(t, x) \tag{1.2}$$

and $\| a^{ij}(t, x) \|$ is a strictly positive-definite symmetric matrix for any $\langle t, x \rangle$; $a^{ij}(t, x)$ and $b^i(t, x)$ are transformed between two local coordinates (x^i) and $(\bar{x}^i)$ in the following manner:[3)]

$$\bar{a}^{ij}(t, \bar{x}) = \frac{\partial \bar{x}^i}{\partial x^k} \cdot \frac{\partial \bar{x}^j}{\partial x^l} a^{kl}(t, x) \tag{1.3}$$

$$\bar{b}^i(t, \bar{x}) = \frac{\partial \bar{x}^i}{\partial x^k} b^k(t, x) + \frac{\partial^2 \bar{x}^i}{\partial x^k \partial x^l} a^{kl}(t, x) \tag{1.4}$$

We assume further that

I) $\dfrac{\partial a^{ij}(t, x)}{\partial t}, \quad \dfrac{\partial^2 a^{ij}(t, x)}{\partial x^k \partial x^l}, \quad \dfrac{\partial b^i(t, x)}{\partial x^k} \quad (i, j, k, l = 1, \dots, m)$

and $c(t, x)$ satisfy a Lipschitz condition of order $\gamma (0 < \gamma \leq 1)$ at every point $\langle t, x \rangle$ of $(s_0, t_0) \times \boldsymbol{M}$,

II) the system of functions

$$\left\{ \begin{array}{c} a^{ij}, \ \dfrac{\partial a^{ij}}{\partial x^k}, \ \dfrac{\partial^2 a^{ij}}{\partial x^k \partial x^l}, \ \dfrac{\partial a^{ij}}{\partial t}, \ \det \| a^{ij} \|^{-1}, \ b^i, \ \dfrac{\partial b^i}{\partial x^k}, \ c; \\ i, j, k, l = 1, \dots, m \end{array} \right\}$$

is bounded by K (Definition 1).[4)] It follows from this condition that there exist constants c_1 and $c_2 > 0$ such that

$$c_1 \| \xi \|^2 \geq a_{ij}(t, x) \xi^i \xi^j \geq c_2 \| \xi \|^2 \tag{1.5}$$

for any t, x and any $\xi \in R^m$, where $\| a_{ij}(t, x) \|$ denotes the inverse matrix of $\| a^{ij}(t, x) \|$ and $\| \xi \|^2 = \sum_i |\xi^i|^2$.

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3) This transformation rule is connected with the fact that the value of $A \cdot f(t, x)$ is independent of the local coordinate.

4) This condition seems to be closely related with K. Yosida's HYPOTHESIS in [4]. But our condition does not require any restriction for $g(x) = \det \| g_{ij}(x) \|$ even if $\boldsymbol{M}$ is a Riemannian space with the metric $dr^2 = g_{ij}(x) dx^i dx^j$. See Theorem 5 below.

We fix a canonical coordinate system $\mathfrak{S}$ for which the condition II) is satisfied.

Next we fix a real number $s_1(s_0 < s_1 < t_0)$ and put $a_{ij}(x) = a_{ij}(s_1, x)$ and $a(x) = \det \| a_{ij}(x) \|$. Then, by virtue of (1.3), we may define in $\boldsymbol{M}$ the metric $d_a r^2 = a_{ij}(x) dx^i dx^j$ and the measure $d_a x = \sqrt{a(x)} dx^1 \cdots dx^m$. Let L^* and A^* be the (formally) adjoint operator of L and that of A with respect to this metric:

$$\begin{cases} L^* \equiv L^*_{tx} = A^*_{tx} + \dfrac{\partial}{\partial t} \\ A^* \equiv A^*_{tx} = \dfrac{1}{\sqrt{a(x)}} \cdot \dfrac{\partial^2}{\partial x^i \partial x^j} a^{ij}(t, x) \sqrt{a(x)} \\ \qquad\qquad - \dfrac{1}{\sqrt{a(x)}} \cdot \dfrac{\partial}{\partial x^i} b^i(t, x) \sqrt{a(x)} + c(t, x). \end{cases}$$

DEFINITION 2. A function $u(t, x; s, y)$, $s_0 < s < t < t_0$; $x, y \in \boldsymbol{M}$, is called a *fundamental solution of the parabolic equation* $L \cdot f = 0$ if, for any s and any function $f(x)$ uniformly continuous and bounded on $\boldsymbol{M}$, the function

$$f(t, x) = \int_{\boldsymbol{M}} u(t, x; s, y) f(y) d_a y \tag{1.6}$$

satisfies the conditions:

$$L \cdot f(t, x) = 0, \quad s < t < t_0, \quad x \in \boldsymbol{M}, \tag{1.7}$$

$$\lim_{t \downarrow s} f(t, x) = f(x) \quad \text{(uniformly on } \boldsymbol{M}) \tag{1.8}$$

and

$$\begin{cases} \text{both } f(t, x) \text{ and } \dfrac{\partial}{\partial t} f(t, x) \text{ are bounded on } (s', t') \times \boldsymbol{M} \\ \text{for any } s' \text{ and } t',\ s < s' < t' < t_0. \end{cases} \tag{1.9}$$

A function $u^*(s, y; t, x)$, $t_0 > t > s > s_0$; $x, y \in \boldsymbol{M}$, is called a *fundamental solution of the adjoint equation* $L^* f^* = 0$ of the equation $L \cdot f = 0$ if, for any t and any function $f(x)$ continuous and summable on $\boldsymbol{M}$ with respect to the measure $d_a x$, the function

$$f^*(s, y) = \int_{\boldsymbol{M}} u^*(s, y; t, x) f(x) d_a x \tag{1.6*}$$

satisfies the conditions:

$$L^* f^*(s, y) = 0, \quad t > s > s_0, \quad y \in \boldsymbol{M}, \tag{1.7*}$$

$$\lim_{s \uparrow t} f^*(s, y) = f(y) \tag{1.8*}$$

pointwisely and strongly in $L^1(\boldsymbol{M})$ and

$$(1.9^*)\quad \begin{cases} \text{both } \int_{\boldsymbol{M}} |f^*(s, y)|d_a y \text{ and } \int_{\boldsymbol{M}} \left|\frac{\partial}{\partial s} f^*(s, y)\right| d_a y \text{ are} \\ \text{bounded on } (s', t') \text{ for any, } s' \text{ and } t',\ t > t' > s' > s_0 . \end{cases}$$

We state here the main theorems, which will be proved in § 4.

Theorem 1. *There exists a function $u(t, x; s, y)$ of C^1-class in t and s $(s_0 < s < t < t_0)$ and of C^2-class in x and y, with the following properties:*

i) *$u(t, x; s, y)$ is a fundamental solution of the equation $L \cdot f = 0$,*

ii) *$u^*(s, y; t, x) = u(t, x; s, y)$ is a fundamental solution of the adjoint equation $L^* f^* = 0$,*

iii) $L_{tx} u(t, x; s, y) = 0, \quad L^*_{sy} u(t, x; s, y) = 0,$

iv) $\int_{\boldsymbol{M}} u(t, x; \tau, \xi) u(\tau, \xi; s, y) d_a \xi = u(t, x; s, y), \quad s < \tau < t.$

Theorem 2. *Let $u(t, x; s, y)$ and $u^*(s, y; t, x)$ be the functions stated in Theorem* 1.

i) *If a function $f(t, x)$ $(s < t < t_0, x \in \boldsymbol{M})$ satisfies* (1.9), (1.7) *and* (1.8) *where $f(x)$ is continuous and bounded on $\boldsymbol{M}$, then it is expressible by* (1.6).

ii) *If a function $f^*(s, y)$ $(t > s > t_0, y \in \boldsymbol{M})$ satisfies* (1.9*), (1.7*) *and* (1.8*) *where $f(x)$ is uniformly continuous and summable on $\boldsymbol{M}$ with respect to the measure $d_a x$, then it is expressible by* (1.6*).

Theorem 3. (UNIQUENESS OF FUNDAMENTAL SOLUTION) *If a function $v(t, x; s, y)$ is continuous in the region: $s_0 < s < t < t_0$; $x, y \in \boldsymbol{M}$, and satisfies the condition* i) *or* ii) *in Theorem* 1, *then it is identical with $u(t, x; s, y)$ stated in Theorem* 1.

Theorem 4. *If $c(t, x) \leq 0$ in the differential operator A_{tx}, then $u(t, x; s, y) \geq 0$; if especially $c(t, x) \equiv 0$, then $\int_{\boldsymbol{M}} u(t, x; s, y) d_a y = 1$.*

Next, if $\boldsymbol{M}$ is not only an infinitely differentiable manifold but also a Riemannian space with the metric $dr^2 = g_{ij}(x) dx^i dx^j$ *à priori*, then it is natural that we take the measure

$$d_g x = \sqrt{g(x)}\, dx^1 \cdots dx^m \qquad \text{where} \quad g(x) = \det \| g_{ij}(x) \|$$

and consider

$$\begin{cases} L' = A' + \dfrac{\partial}{\partial t} \\ A' = \dfrac{1}{\sqrt{g(x)}} \cdot \dfrac{\partial^2}{\partial x^i \partial x^j} a^{ij}(t, x) \sqrt{g(x)} - \dfrac{1}{\sqrt{g(x)}} \cdot \dfrac{\partial}{\partial x^i} b^i(t, x) \sqrt{g(x)} + c(t, x) \end{cases}$$

as the adjoint operator of L and that of A. But the results of this case may be immediately deduced from the above stated results by means of the function

$$\tilde{u}(t, x; s, y) = u(t, x; s, y)\frac{\sqrt{a(y)}}{\sqrt{g(y)}};^{5)}$$

that is,

Theorem 5. *If* M *is a Riemannian space with the metric* $dr^2 = g_{ij}(x)dx^i dx^j$, *then we may replace* L^*, A^*, u *and* $d_a x$ *in Definition* 2 *and in Theorems* 1, 2, 3 *and* 4 *by* L', A', $\tilde{u}$ *and* $d_a x$ *(stated just above) respectively.*

§2. **Quasi-parametrix.** First we consider L_{tx} and A_{tx} in the Euclidean m-space R^m. Put

$$V(t, x; s, y) = (t-s)^{-\frac{m}{2}} \exp\left\{-\frac{a_{ij}(t, x)\cdot(x^i-y^i)(x^j-y^j)}{4(t-s)}\right\}$$

$$s_0 < s < t < t_0;\ x, y \in R^m,$$

and

$$V_0(t, x) = \int_{R^m} \exp\left\{-\frac{a_{ij}(t, x)\xi^i\xi^j}{4}\right\} d\xi, \quad d\xi = d\xi^1 \cdots d\xi^m.$$

Then we have

Lemma 1. *Let* $f(t, x)$ *be a bounded and continuous function on* $(s, t)\times R^m (s_0 < s < t_0)$, *and put*

$$f(t, x, \tau) = \int_{R^m} V(t, x; \tau, y) f(\tau, y) dy, \quad t > \tau > s. \tag{2.1}$$

Then we have

$$\frac{\partial}{\partial t} f(t, x, \tau) = \int_{R^m} \frac{\partial}{\partial t} V(t, x; \tau, y) f(\tau, y) dy, \tag{2.2}$$

$$\left\{\begin{aligned} \frac{\partial}{\partial x^i} f(t, x, \tau) &= \int_{R^m} \frac{\partial}{\partial x^i} V(t, x; \tau, y) f(\tau, y) dy \\ \frac{\partial^2}{\partial x^i \partial x^j} f(t, x, \tau) &= \int_{R^m} \frac{\partial^2}{\partial x^i \partial x^j} V(t, x; \tau, y) f(\tau, y) dy, \end{aligned}\right. \tag{2.3}$$

and

$$\lim_{\substack{t_1 > t_2 > \tau \\ t_1 \to \tau}} f(t_1, x, t_2) = f(\tau, x) V_0(\tau, x). \tag{2.4}$$

5) It is true that $a(x)$ and $g(x)$ depend upon the local coordinate, but the ratio $a(x)/g(x)$ is independent.

PROOF. The equalities (2.2) and (2.3) may be easily proved from the assumption. We shall prove (2.4). By the substitution: $y^i = (t-\tau)^{\frac{1}{2}}\xi^i + x^i$, in the right-hand side of (2.1), we get

$$f(t, x, \tau) = \int_{R^m} \exp\left\{-\frac{a_{ij}(t, x)\xi^i\xi^j}{4}\right\} f(\tau, (t-\tau)^{\frac{1}{2}}\xi + x) d\xi$$

where $(t-\tau)^{\frac{1}{2}}\xi + x$ means the sum as vectors in R^m. Hence we obtain (2.4) by Lebesgue's convergence theorem, q. e. d.

Similar argument shows that

Lemma 2.

$$\lim_{t \downarrow s} \int_{R^m} f(t, x) V(t, x; s, y) dx = f(s, y) V_0(s, y)$$

for any function $f(t, x)$ bounded and continuous on $[s, t'] \times R^m (s < t' < t_0)$.

Lemma 3. *Let $f(\tau, y)$ be a function defined on $(s, t_0) \times R^m$ which satisfies the following three conditions:* i) $\int_s^t \int_{R^m} |f(\tau, y)| dy d\tau < \infty (s < t < t_0)$, ii) *$f(\tau, y)$ is bounded on $[s', t'] \times R^m$ for any s' and t', $s < s' < t' < t_0$, $f(\tau, y)$ satisfies a Lipschitz condition of order $\gamma (0 < \gamma \leq 1)$ at every point in $(s, t_0) \times R^m$; and define $f(t, x, \tau)$ by (2.1). Then, for any $\langle t, x \rangle \in (s, t_0) \times R^m$, there exists a constant M such that*

$$\left| \frac{\partial}{\partial t'} f(t', x, \tau) \right| \leq M(t-\tau)^{-\left(1-\frac{\gamma}{2}\right)} \tag{2.5}$$

whenever $s < \tau < t \leq t'$; further we have

$$\begin{cases} \displaystyle\int_s^t \left| \frac{\partial}{\partial x^i} f(t, x, \tau) \right| d\tau < \infty, \\ \displaystyle\int_s^t \left| \frac{\partial^2}{\partial x^i \partial x^j} f(t, x, \tau) \right| d\tau < \infty. \end{cases} \tag{2.6}$$

PROOF. By the condition iii), there exist $\delta (0 < \delta \leq 1)$ and $N (> 0)$ such that

$$|f(\tau, y) - f(t, x)| \leq N\{|\tau - t|^\gamma + \sum_i |y^i - x^i|^\gamma\} \tag{2.7}$$

whenever $|\tau - t| \leq \delta$ and $\| y - x \| \leq \delta$. Hence the relation $t - \delta \leq \tau < t \leq t'$ implies that

$$\begin{aligned} &\int_{\|y-x\| \leq \delta} \left| \frac{\partial}{\partial t'} V(t', x; \tau, y) \right| \cdot |f(\tau, y) - f(t, y)| dy \\ &\leq \int_{\|y-x\| \leq \delta} N \cdot \left| \frac{\partial}{\partial t'} V(t', x; \tau, y) \right| \left\{ |\tau - t|^\gamma + \sum_i |y^i - x^i|^\gamma \right\} dy. \end{aligned} \tag{2.8}$$

If we calculate $\frac{\partial}{\partial t'}V(t', x; \tau, y)$ and put

$$y^i - x^i = (t'-\tau)^{\frac{1}{2}}\xi^i, \quad i=1, \dots, m, \tag{2.9}$$

then we may see that the right-hand side of (2.8) is not greater than

$$\int_{R^m} N\left\{\frac{m}{2}(t'-\tau)^{-1} + \frac{1}{4}\left|\frac{\partial}{\partial t}a_{ij}(t', x)\xi^i\xi^j\right| + \frac{1}{4}(t'-\tau)^{-1}\left|a_{ij}(t', x)\xi^i\xi^j\right|\right\} \times$$
$$\times\left\{|t-\tau|^\gamma + |t'-\tau|^{\frac{\gamma}{2}}\sum_i|\xi^i|^\gamma\right\}\exp\left\{-\frac{a_{ij}(t', x)\xi^i\xi^j}{4}\right\}d\xi.$$

Hence, by means of the facts $t'-\tau \geqq t-\tau > 0$ and $t-\tau \leqq \delta \leqq 1$ and by the boundedness of $a_{ij}(t, x)$ and $\frac{\partial}{\partial t}a_{ij}(t, x)$, there exists a constant M_0 such that

$$\left|\int_{\|y-x\|\leqq\delta}\frac{\partial}{\partial t'}V(t', x; \tau, y)\{f(\tau, y)-f(t, x)\}dy\right| \leqq M_0(t-\tau)^{-(1-\frac{\gamma}{2})} \tag{2.10}$$

whenever $t-\delta \leqq \tau < t \leqq t'$. Furthermore, we may easily show by way of the substitution (2.9) that

$$\left|\int_{\|y-x\|\geqq\delta}\frac{\partial}{\partial t'}V(t', x; \tau, y)\cdot\{f(\tau, y)-f(t, x)\}dy\right| \leqq M_1 \tag{2.11}$$

and

$$\left|\frac{\partial}{\partial t'}\int_{R^m}V(t', x; \tau, y)dy\right| = \left|\frac{\partial}{\partial t'}\int_{R^m}\exp\left\{-\frac{a_{ij}(t', x)\xi^i\xi^j}{4}\right\}d\xi\right| \leqq M_1 \tag{2.12}$$

for any $t' > \tau \geqq t-\delta$ for a suitable constant M_1. From (2.10), (2.11) and (2.12) and by (2.2) in Lemma 1, we get

$$\left|\frac{\partial}{\partial t'}f(t', x, \tau)\right| \leqq \left|\int_{R^m}\frac{\partial}{\partial t'}V(t', x; \tau, y)\left\{f(\tau, y)-f(t, x)\right\}dy\right| +$$
$$+\left|f(t, x)\right|\cdot\left|\frac{\partial}{\partial t'}\int_{R^m}V(t', x; \tau, y)dy\right| \leqq M_2(t-\tau)^{-(1-\frac{\gamma}{2})}$$

whenever $t-\delta \leqq \tau < t \leqq t'$, for a suitable constant M_2. On the other hand, $\frac{\partial}{\partial t'}f(t', x, \tau)$ is bounded uniformly in $\langle t', x, \tau\rangle$ such that $t'-\tau \geqq \delta$, as is easily seen from the properties of $V(t', x; \tau, y)$ and $f(\tau, y)$. Hence we conclude (2.5); consequently we get

$$\int_s^t\left|\frac{\partial}{\partial t}f(t, x, \tau)\right| d\tau < \infty.$$

We may prove (2.6) in the similar manner.

Lemma 4. *Let $f(\tau, y)$ and $f(t, x, \tau)$ be as stated in Lemma 3 and put*

$$F(t, x)=\int_s^t f(t, x, \tau)d\tau .$$

Then

$$\frac{\partial}{\partial t}F(t, x)=f(t, x)V_0(t, x)+\int_s^t\int_{R^m}\frac{\partial}{\partial t}V(t, x; \tau, y)f(\tau, y)dyd\tau \tag{2.13}$$

and

$$A_{tx}F(t, x)=\int_s^t\int_{R^m}A_{tx}V(t, x; \tau, y)f(\tau, y)dyd\tau . \tag{2.14}$$

PROOF. For any $\Delta>0$, we have

$$\begin{aligned}&\frac{1}{\Delta}\{F(t+\Delta, x)-F(t, x)\}\\ &\quad=\frac{1}{\Delta}\int_t^{t+\Delta}f(t+\Delta, x, \tau)d\tau+\int_s^t\frac{1}{\Delta}\{f(t+\Delta, x, \tau)-f(t, x, \tau)\}d\tau\\ &\quad=f(t+\Delta, x, t+\theta\Delta)+\int_s^t\frac{\partial}{\partial t}f(t+\theta'\Delta, x, \tau)d\tau,\ 0<\theta,\ \theta'<1 ;\end{aligned}$$

the first term tends to $f(t, x)V_0(t, x)$ bp (2.4), as $\Delta\downarrow 0$, while the second term tends to $\int_s^t\frac{\partial}{\partial t}f(t, x, \tau)d\tau$ by Lebesgue's convergence theorem and (2.5). Hence we have, by (2.2),

$$\begin{aligned}&\lim_{\Delta\downarrow 0}\frac{1}{\Delta}\{F(t+\Delta, x)-F(t, x)\}\\ &\qquad=f(t, x)V_0(t, x)+\int_s^t\int_{R^m}\frac{\partial}{\partial t}V(t, x; \tau, y)f(\tau, y)dyd\tau .\end{aligned}$$

Thus we see that the right derivative $D_t^+F(t, x)$ exists and is continuous in t for any x, and hence $\frac{\partial}{\partial t}F(t, x)$ exists and equals the right-hand side of the above equality; this fact shows (2.13).

(2.14) is obtained from (2.3) and the following relations:

$$\left\{\begin{aligned}&\frac{\partial}{\partial x^i}F(t, x)=\int_s^t\frac{\partial}{\partial x^i}f(t, x, \tau)d\tau ,\\ &\frac{\partial^2}{\partial x^i\partial x^j}F(t, x)=\int_s^t\frac{\partial^2}{\partial x^i\partial x^j}f(t, x, \tau)d\tau ,\end{aligned}\right.$$

which may be proved by virtue of (2.6).

Lemma 5. *Let $\varphi(x)$ be a function of C^2-class on R^m such that*

$$\left|\frac{\partial\varphi(x)}{\partial x^i}\right|,\quad\left|\frac{\partial^2\varphi(x)}{\partial x^i\partial x^j}\right|\leq C_0,\ i, j=1, \ldots, m\ (C_0:\ \textit{constant}).$$

Then

$$|L_{tx}[\varphi(x)V(t,\ x\ ;\ s,\ y)]|\leqq C_1(t-s)^{-\frac{m+1}{2}}\cdot\exp\left\{-\frac{C_2\,\|\,x-y\,\|^2}{4(t-s)}\right\}$$

where C_1 and C_2 are positive constants which depend only on c_2 in (1.5) *and C_0 stated just above.*

This lemma may be proved if we achieve the calculi of differentiations in $L_{tx}[\varphi(x)V(t,\ x\ ;\ s,\ y)]$ considering the assumption for $\varphi(x)$ and the following two facts: 1) $\|\,a_{ij}(t,\ x)\,\|\cdot\|\,a^{ij}(t,\ x)\,\|=$ *identity matrix*, 2) p and C are positive constants and λ is a variable $\geqq 0$, then there exist positive constants C_1 and C_2 such that $\lambda^p\exp(-C\lambda)\leqq C_1\exp(-C_2\lambda)$.

Next we fix a function $\omega(\lambda)$ of C^2-class in $\lambda\geqq 0$ such that $\omega(\lambda)=1$ or 0 if $\lambda\leqq\frac{1}{3}$ or $\lambda\geqq\frac{2}{3}$ respectively and $0\leqq\omega(\lambda)\leqq 1$ for any $\lambda\geqq 0$, and that $\frac{d^2\omega(\lambda)}{d\lambda^2}$ satisfies the Lipschitz condition of order γ at every λ.

For any $z\in\boldsymbol{M}$, we map a neighbourhood $U(z)$ of z onto the unit sphere S in R^m by means of the canonical coordinate $\in\mathfrak{S}$ around z (see § 1), and let $U_1(z)$ and $U_2(z)$ be the inverse images of $S_1=\left\{\xi\ ;\ \|\,\xi\,\|^2<\frac{1}{3}\right\}$ and $S_2=\left\{\xi\ ;\ \|\,\xi\,\|^2<\frac{2}{3}\right\}$ respectively under this mapping. By means of this mapping, we may consider any function $\varphi(\xi)$ defined on S as a function on $U(z)$. We shall denote by $\varphi_z(x)$ the function on $U(z)$ defined in such manner from the function $\varphi(\xi)$ on S.

Now we define the *quasi-parametrix*[6] $Z(t,x\ ;\ s,y)$ on $\boldsymbol{M}$ as follows. Since the manifold $\boldsymbol{M}$ satisfies the second countability axiom, there exists a sequence $\{z_1,\ z_2,\ \cdots\}\subset\boldsymbol{M}$ such that $\boldsymbol{M}=\bigcup_{\nu=1}^{\infty}U_1(z_\nu)$, where we may take the sequence in such a manner that every point $z\in\boldsymbol{M}$ is contained in finite number of $U_2(z_\nu)$. We put

$$(2.15)\qquad Z(t,\ x\ ;\ s,\ y)=\frac{\sum_\nu\omega_{z_\nu}(\,\|\,x-z_\nu\,\|^2)\,\omega_{z_\nu}(\,\|\,y-z_\nu\,\|^2)\,W_{z_\nu}(t,\ x\ ;\ s,\ y)}{\sum_\nu\omega_{z_\nu}(\,\|\,x-z_\nu\,\|^2)^2\sqrt{a_{z_\nu}(x)}}\qquad(t>s\ ;\ x,\ y\in\boldsymbol{M})$$

where

$$(2.16)\qquad W(t,\ x\ ;\ s,\ y)=V(t,\ x\ ;\ s,\ y)/V_0(s,\ y)\qquad(x,\ y\in R^m).$$

For any fixed x, $\omega_{z_\nu}(\,\|\,x-z_\nu\,\|^2)=0$ except finite number of ν's and hence

6) The function $Z(t,x\ ;\ s,\ y)$ defined here is somewhat different from the parametrix of K. Yosida [4]. But the former plays a rôle analogous to the latter in the construction of the fundamental solution. So, we call the function $Z(t,\ x\ ;\ s,\ y)$ *quasi-parametrix*.

both $\sum$'s appearing in (2.15) are essentially finite; we may easily see from the definition of $\omega(\lambda)$ that $Z(t, x; s, y)$ is well defined for any $x, y \in \boldsymbol{M}$ and any $t, s (s_0 < s < t < t_0)$, and $\frac{\partial Z}{\partial t}, \frac{\partial Z}{\partial s}, \frac{\partial^2 Z}{\partial x^i \partial x^j}$ and $\frac{\partial^2 Z}{\partial y^i \partial y^j}$ exist and satisfy a Lipschitz condition of order $\gamma > 0$ at every point. (We note that the function $V_0(s, y)$ is bounded above and bounded away from zero and satisfies a Lipschitz condition of or order $\gamma > 0$.)

Whenever x runs over $U(x_0)$ for any fixed x_0, we may consider in (2.15) only such ν's as $U(z_\nu) \cap U(x_0)$ is not empty. From this fact and by (2.15), (2.16) and Lemmas 1, 2, 4 and 5, we may prove the following Lemmas 6, 7 and 8.

Lemma 6.

$$\lim_{t \downarrow s} \int_{\boldsymbol{M}} Z(t, x; s, y) f(y) d_a y = f(x) \qquad \text{(uniformly)}$$

for any bounded and uniformly continuous function $f(x)$ on $\boldsymbol{M}$, and

$$\lim_{t \downarrow s} \int_{\boldsymbol{M}} f(t, x) Z(t, x; s, y) d_a y = f(s, y)$$

for and continuous function $f(t, x)$ on $[s, t') \times \boldsymbol{M} (s < t' < t_0)$.

Lemma 7. *Assume that $f(\tau, y)$ satisfies a Lipschitz condition of order $\gamma (> 0)$ at every point in $(s, t_0) \times \boldsymbol{M}$, that $f(\tau, y)$ is bounded on $[s', t'] \times \boldsymbol{M}$ for any s' and t', $s < t' < t_0$, and that $\int_s^t \int_{\boldsymbol{M}} |f(\tau, y)| d_a y d\tau < \infty$ for any $t > s$; and put*

$$F(t, x) = \int_s^t \int_{\boldsymbol{M}} Z(t, x; \tau, y) f(\tau, y) d_a y d\tau .$$

Then $F(t, x)$ is of C^1-class in $t (> s)$ and of C^2-class in $x (\in \boldsymbol{M})$, and

$$\frac{\partial}{\partial t} F(t, x) = f(t, x) + \int_s^t \int_{\boldsymbol{M}} \frac{\partial}{\partial t} Z(t, x; \tau, y) f(\tau, y) d_a y d\tau , \tag{2.17}$$

$$A_{tx} F(t, x) = \int_s^t \int_{\boldsymbol{M}} A_{tx} Z(t, x; \tau, y) f(\tau, y) d_a y d\tau . \tag{2.18}$$

Lemma 8. *There exists a constant $M > 0$ such that*

$$|Z(t, x; s, y)| \leq M (t-s)^{-\frac{m}{2}},$$

$$\left| \frac{\partial}{\partial t} Z(t, x; s, y) \right| \leq M (t-s)^{-\left(\frac{m}{2}+2\right)}, \quad \left| L_{tx} Z(t, x; s, y) \right| \leq M (t-s)^{-\frac{m+1}{2}},$$

$$\int_{\boldsymbol{M}} |Z(t, \xi; s, y)| d_a \xi \text{ and } \int_{\boldsymbol{M}} |Z(t, x; s, \xi)| d_a \xi \leq M ,$$

$$\int_M |L_{t\xi}Z(t,\ \xi;\ s,\ y)|d_a\xi \ \textit{and} \int_M |L_{tx}Z(t,\ x;\ s,\ \xi)|d_a\xi \leq M(t-s)^{-\frac{1}{2}}$$

for any t, s, x and y.

§ 3. Construction of the fundamental solution. We define functions $J_n(t,\ x;\ s,\ y)$, $n=0,\ 1,\ 2,\ \dots$, by the induction as follows:

$$J_0(t,\ x;\ s,\ y) = L_{tx}Z(t,\ x;\ s,\ y)\,, \tag{3.1}$$

$$J_n(t,\ x;\ s,\ y) = \int_s^t \int_M J_0(t,\ x;\ \tau,\ \xi) J_{n-1}(\tau,\ \xi;\ s,\ y) d_a\xi d\tau\,. \tag{3.2}$$

Then we may prove by Lemma 8 and by the induction that

$$\left.\begin{array}{l} \int_M |J_n(t,\ x;\ s,\ y)|d_a x \\ \int_M |J_n(t,\ x;\ s,\ y)|d_a y \end{array}\right\} \leq M^{n+1}(t-s)^{\frac{n-1}{2}} \prod_{\nu=1}^{n} B\left(\frac{1}{2},\ \frac{\nu}{2}\right)$$

and consequently

$$|J_{n+1}(t,\ x;\ s,\ y)| \leq M^{n+2}(t-s)^{\frac{n-m}{2}-1} \prod_{\nu=1}^{n} B\left(\frac{1}{2},\ \frac{\nu}{2}\right)$$

where $B(\mu,\ \nu)$ is the Beta function. Hence simple calculation shows that there exists a constant M_1 such that

$$\left.\begin{array}{l} \sum_{n=0}^{\infty} \int_M |J_n(t,\ x;\ s,\ y)|d_a x \\ \sum_{n=0}^{\infty} \int_M |J_n(t,\ x;\ s,\ y)|d_a y \end{array}\right\} \leq M_1(t-s)^{-\frac{1}{2}} \exp\{M_1(t-s)^{\frac{1}{2}}\}\,. \tag{3.3}$$

and that

$$\sum_{n=0}^{\infty} |J_n(t,\ x;\ s,\ y)| \leq M_1(t-s)^{-\left(\frac{m}{2}+1\right)} \exp\{M_1(t-s)^{\frac{1}{2}}\}\,. \tag{3.4}$$

Hence we may define

$$f(t,\ x;\ s,\ y) = \sum_{n=0}^{\infty} J_n(t,\ x;\ s,\ y) \tag{3.5}$$

where the series in the right-hand side converges absolutely and uniformly in $\langle t,\ x;\ s,\ y\rangle$ whenever $0<\varepsilon\leq t-s\leq\eta$, and consequently we get

$$L_{tx}Z(t,\ x;\ s,\ y) + \int_s^t \int_M L_{sx}Z(t,\ x;\ \tau,\ \xi) f(\tau,\ \xi;\ s,\ y) d_a\xi d\tau = f(t,\ x;\ s,\ y) \tag{3.6}$$

and

$$(3.7)\qquad \left.\begin{array}{l}\int_{M}|f(t, x; s, y)|d_a x \\ \int_{M}|f(t, x; s, y)|d_a y\end{array}\right\} \leqq M_1(t-s)^{-\frac{1}{2}} \exp\{M_1(t-s)^{\frac{1}{2}}\} .$$

Now we put

$$(3.8)\qquad \begin{aligned} u(t, x; s, y) = Z(t, x; s, y) + \\ + \int_s^t \int_M Z(t, x; \tau, \xi) f(\tau, \xi; s, y) d_a\xi d\tau .\end{aligned}$$

For any fixed $\langle s, y \rangle$, we may prove from the properties of $Z(t, x; s, y)$ stated in §2 that the function $f(\tau, \xi) = f(\tau, \xi; s, y)$ satisfies the assumptions of Lemma 7. Hence, by (2.17) and (2.18), we get

$$L_{tx}\left[\int_s^t \int_M Z(t, x; \tau, \xi) f(\tau, \xi; s, y) d_a\xi d\tau\right]$$
$$= \int_s^t \int_M L_{tx} Z(t, x; \tau, \xi) f(\tau, \xi; s, y) d_a\xi d\tau - f(t, x; s, y) .$$

By means of (3.6) and the above equality, the function $u(t, x; s, y)$ in (3.8) satisfies

$$(3.9)\qquad L_{tx} u(t, x; s, y) = 0 .$$

If we apply Lemma 8 and (3.7) to (3.8), we get by simple calculations

$$(3.10)\qquad \left.\begin{array}{l}\int_{M}|u(t, x; s, y)|d_a x \\ \int_{M}|u(t, x; s, y)|d_a y\end{array}\right\} \leqq M \cdot \exp\{2M_1(t-s)^{\frac{1}{2}}\} ,$$

$$(3.11)\qquad \left.\begin{array}{l}\int_{M}\left|\frac{\partial}{\partial t} u(t, x; s, y)\right|d_a x \\ \int_{M}\left|\frac{\partial}{\partial t} u(t, x; s, y)\right|d_a y\end{array}\right\} \leqq M(t-s)^{-2} \exp\{2M_1(t-s)^{\frac{1}{2}}\}$$

and

$$(3.12)\qquad \begin{aligned} |u(t, x; s, y)| &\leqq M(t-s)^{-\frac{m}{2}} \exp\{M_1(t-s)^{\frac{1}{2}}\} , \\ \left|\frac{\partial}{\partial t} u(t, x; s, y)\right| &\leqq M(t-s)^{-(\frac{m}{2}+2)} \exp\{2M_1(t-s)^{\frac{1}{2}}\} .\end{aligned}$$

Therefore we may show by (3.7), (3.8) and by Lemmas 6 and 8 that

$$(3.13)\qquad \lim_{t \downarrow s} \int_M u(t, x; s, y) f(y) d_a y = f(x) \quad \text{(uniformly)}$$

for any bounded and uniformly continuous function $f(x)$ on $\boldsymbol{M}$, and that

$$(3.14) \qquad \lim_{t \downarrow s} \int_{\boldsymbol{M}} f(t, x) u(t, x; s, y) d_a x = f(s, y)$$

for any continuous fuaction $f(t, x)$ on $[s, t') \times \boldsymbol{M}$ such that $\int_{\boldsymbol{M}} |f(t, x)| d_a x$ is bounded on $[s, t')$ for any t', $s < t' < t_0$.

Next, we consider the adjoint equation $L^* f^* = 0$. If we expand the terms in $A^* f^*(t, x)$ and consider the conditions I) and II) in §1 for A, we see that the coefficients of A^* also satisfy the conditions I) and II). Hence we may construct a fundamental solution $u^*(s, y; t, x)$ of $L^* f^* = 0$, which has the similar properties to those of $u(t, x; s, y)$ stated just above. For the later use, we note especially that

$$(3.9^*) \qquad L^*_{s,y} u^*(s, y; t, x) = 0$$

$$(3.13^*) \qquad \lim_{s \uparrow t} \int_{\boldsymbol{M}} u^*(s, y; t, x) f(x) d_a x = f(y)$$

pointwisely and also in $L^1(\boldsymbol{M})$ for any function $f(x)$ continuous and summable on $\boldsymbol{M}$ with respect to the measure $d_a x$, and

$$(3.14^*) \qquad \lim_{s \uparrow t} \int_{\boldsymbol{M}} f(s, y) u^*(s, y; t, x) d_a y = f(t, x)$$

for any continuous function $f(s, y)$ on $(s, t] \times \boldsymbol{M}$, bounded on $[s', t] \times \boldsymbol{M}$ for any s', $s_0 < s' < t$.

§4. **Proof of Theorems.** Let $u(t, x; s, y)$ and $u^*(s, y; t, x)$ be the functions defined in §3. We first prove the following

Lemma 9. *There is a constant M_0 such that, for any compact set $\Gamma \subset \boldsymbol{M}$, there exists a function $\varphi(x)$ of C^2-class on $\boldsymbol{M}$ with the following three properties*: 1) *$\varphi(x) = 1$ on Γ,* 2) *the support of $\varphi(x)$ is compact,* 3) *for any z_ν (stated in* §3), *the functions $|\varphi(x)|$, $\left|\frac{\partial \varphi(x)}{\partial x^i}\right|$, $\left|\frac{\partial^2 \varphi(x)}{\partial x^i \partial x^j}\right|$; $i, j = 1, \ldots, m$, are $\leqq M_0$ for $x \in U(z_\nu)$ with respect to the local coordinate ($\in \mathfrak{S}$) around z_ν.*

Proof. We may consider that the sequence $\{z_1, z_2, \cdots\}$ stated in §3 is so chosen that each point $z \in \boldsymbol{M}$ belongs to at most m_0 neighbourhoods $U_2(z_{\nu_n})$, $n = 1, \ldots, m_0$, where m_0 is a constant depending only on the dimension of $\boldsymbol{M}$. Now we take z_{ν_n}, $n = 1, \ldots, n_0$, so that $\Gamma \subset \bigcup_{n=1}^{n_0} U_1(z_{\nu_n})$ (see §3), and put

$$\varphi(x) = \frac{\sum_{n=1}^{n_0} \omega_{\nu_n}(\| x - z_{\nu_n} \|^2)}{\sum_{\nu=1}^{\infty} \omega_\nu(\| x - z_\nu \|^2)}$$

where $\omega_\nu(\cdot)=\omega_{z_\nu}(\cdot)$ which is defined in § 3. Then we may easily show that the function $\varphi(x)$ has the properties 1), 2) and 3) stated above; the constant M_0 is determined by means of m_0 and supremum of $|\omega(\lambda)|$, $\left|\frac{\partial\omega(\lambda)}{\partial\lambda}\right|$, $\left|\frac{\partial^2\omega(\lambda)}{\partial\lambda^2}\right|$, $0\leqq\lambda\leqq 1$ (see § 3).

Lemma 10. *Assume that $f(x)$ and $h(x)$ are function of C^2-class on $\boldsymbol{M}$ such that $\int_{\boldsymbol{M}}|f(x)|d_ax<\infty$, $\int_{\boldsymbol{M}}|A^*f(x)|d_ax<\infty$ and that $h(x)$, $\frac{\partial}{\partial x^i}h(x)$, $\frac{\partial^2}{\partial x^i\partial x^j}h(x)$; $i, j=1, \dots, m$, are bounded on $\boldsymbol{M}$. Then*

$$\int_{\boldsymbol{M}} f(x)\cdot A_{tx}h(x)d_ax=\int_{\boldsymbol{M}} A^*_{tx}f(x)\cdot h(x)d_ax.\text{[7]}$$

PROOF. Let Γ_n, $n=1, 2, \dots$, be compact subsets of $\boldsymbol{M}$ such that $\Gamma_1\subset\Gamma_2\subset\cdots\to\boldsymbol{M}$. Then, for each n, there exists a function $\varphi_n(x)$ of C^2-class on $\boldsymbol{M}$ with the properties 1), 2) and 3) stated in Lemma 9 where we read Γ_n for Γ in 1) while the constant M_0 is independent of n. Hence, by means of Lebesgue's convergence theorem, we may show that

$$\int_{\boldsymbol{M}} f(x)\cdot A_{tx}h(x)d_ax=\lim_{n\to\infty}\int_{\boldsymbol{M}} f(x)\cdot A_{tx}[\varphi_n(x)h(x)]d_ax$$
$$=\lim_{n\to\infty}\int_{\boldsymbol{M}} A^*_{tx}f(x)\cdot\varphi_n(x)h(x)d_ax=\int_{\boldsymbol{M}} A^*_{tx}f(x)\cdot h(x)d_ax,\quad \text{q. e. d.}$$

Lemma 11. *If a function $f^*(s, y)$, $s_0<s<t$, $y\in\boldsymbol{M}$, has the properties* (1.9*) *and* (1.7*) *(t: fixed), then*

$$\int_{\boldsymbol{M}} f^*(\tau, \xi)u(\tau, \xi; s, y)d_a\xi=f^*(s, y) \tag{4.1}$$

for any $\tau(s<\tau<t)$.

PROOF. From (3.9) and the assumption (1.7*) for $f^*(\tau, \xi)$, the relation $s<\tau_1<\tau_2<t$ implies

$$0=\int_{\tau_1}^{\tau_2}d\tau\int_{\boldsymbol{M}}\{L^*f^*(\tau, \xi)\cdot u(\tau, \xi; s, y)-f^*(\tau, \xi)\cdot L_{\tau\xi}u(\tau, \xi; s, y)\}d_a\xi$$
$$=\int_{\tau_1}^{\tau_2}d\tau\int_{\boldsymbol{M}}\{A^*f^*(\tau, \xi)\cdot u(\tau, \xi; s, y)-f^*(\tau, \xi)\cdot A_{\tau\xi}u(\tau, \xi; s, y)\}d_a\xi$$
$$+\int_{\tau_1}^{\tau_2}d\tau\int_{\boldsymbol{M}}\left\{\frac{\partial\cdot f^*(\tau, \xi)}{\partial\tau}\cdot u(\tau, \xi; s, y)+f^*(\tau, \xi)\frac{\partial u(\tau, \xi; s, y)}{\partial\tau}\right\}d_a\xi;$$

7) Any function on $\boldsymbol{M}$ is considered as a function on $(s_0, t_0)\times\boldsymbol{M}$, so we may operate A_{tx} to the function.

the first term equals zero by Lemma 10 since $f(\xi)=f^*(\tau, \xi)$ and $h(\xi)=u(\tau, \xi; s, y)$ satisfy the assumptions of Lemma 10 for any fixed τ, s and ξ, while we may apply Fubini's theorem to the second term by virtue of (3.12) and the assumption (1.9*) for $f^*(\tau, \xi)$, and hence the second term equals

$$\int_M \{f^*(\tau_2, \xi)u(\tau_2, \xi; s, y)-f^*(\tau_1, \xi)u(\tau_1, \xi; s, y)\}d_a\xi .$$

Thus we see that $\int_M f^*(\tau, \xi)u(\tau, \xi; s, y)d_a\xi$ is independent of τ, $s<\tau<t$. From this fact and (3.14), we obtain (4.1).

Now we shall prove Theorems stated in §1.

PROOF OF THEOREM 1. For any fixed t and x, $u^*(s, y; t, x)$ satisfies the assumption of Lemma 11 as a function of s and y. Hence we have

$$\int_M u^*(\tau, \xi; t, x)u(\tau, \xi; s, y)d_a\xi = u^*(s, y; t, x), \quad s<\tau<t. \tag{4.2}$$

Taking the limit as $\tau \uparrow t$, we obtain by (3.10) and (3.14*) that

$$u(t, x; s, y) = u^*(s, y; t, x). \tag{4.3}$$

From the relations (3.9—14), (3.9*, 13*, 14*), (4.2) and (4.3), we may easily show that the function $u(t, x; s, y)$ has all properties i), ii), iii) and iv) in Theorem 1.

PROOF OF THEOREM 2. Assume that $f^*(s, y)$ satisfies (1.9*), (1.7*) and (1.8*). Then, by Lemma 11, we have

$$\int_M f^*(\tau, x)u(\tau, x; s, y)d_a x = f^*(s, y), \quad s_0<s<\tau<t. \tag{4.4}$$

Taking the limit as τ tends to t and considering (4.3) and (1.8*) (strong convergence in $L^1(\boldsymbol{M})$), we get (1.6*), which proves ii) in Theorem 2.

Similar argument shows i) in Theorem 2.

PROOF OF THEOREM 3. If $v(t, x; s, y)$ is continuous in the region: $s_0<s<t<t_0$; $x, y \in \boldsymbol{M}$, and satisfies i) (or ii)) in Theorem 1, then, for any continuous function $f(x)$ on $\boldsymbol{M}$ with a compact support, we get

$$\int_M v(t, x; s, y)f(y)d_a y = \int_M u(t, x; s, y)f(y)d_a y$$

(or $\int_M v(t, x; s, y)f(x)d_a x = \int_M u(t, x; s, y)f(x)d_a x$ respectively) by virtue of Definition 2 and Theorem 2. It follows from this relation and the continuity of u and v that $v(t, x; s, y) \equiv u(t, x; s, y)$.

PROOF OF THEOREM 4. By virtue of Theorem 3 and by the arguments in § 3 and the proof of Theorem 1, we may put

$$u(t, x; s, y) = \sum_{n=0}^{\infty} u_n(t, x; s, y) \tag{4.6}$$

where

$$\begin{cases} u_0(t, x; s, y) = Z(t, x; s, y) \\ u_n(t, x; s, y) = \int_s^t \int_{\boldsymbol{M}} Z(t, x: \tau, \xi) J_{n-1}(\tau, \xi; s, y) d_a\xi d\tau, \quad n \geqq 1. \end{cases}$$

From the definition and properties of Z and J_n (§ 3), we see that each $u_n(t, x; s, y)$ vanishes outside a compact set of x for any fixed t, s and y, and that the right-hand side of (4.6) converges absolutely and uniformly in $\langle t, x; s, y \rangle$ whenever $0 < \varepsilon \leqq t-s \leqq \eta$. Hence we may prove that, for any function $f(x)$ which is continuous on $\boldsymbol{M}$ and vanishes outside a neighbourhood of a point in $\boldsymbol{M}$, the function

$$f(t, x) = \int_{\boldsymbol{M}} u(t, x; s, y) f(y) d_a y = \sum_{n=0}^{\infty} \int_{\boldsymbol{M}} u_n(t, x; s, y) f(y) d_a y$$

tends to zero as x tends to the point at infinity[8], for any fixed $t > s$. Moreover, we get

$$L \cdot f(t, x) = 0$$

and

$$|f(t, x)| \leqq M_0 (t-s)^{-\frac{1}{2}} \exp\{M_0 (t-s)^{\frac{1}{2}}\}$$

for a suitable constant $M_0 > 0$;

the last relation follows from (3.7), (3.8) and the boundedness of $f(x)$. Hence, by the well known method, we may prove that $c(t, x) \leqq 0$ and $f(x) \geqq 0$ imply $f(t, x) \geqq 0$; and consequently we get $u(t, x; s, y) \geqq 0$.

In the case $c(t, x) = 0$, if we apply i) in Theorem 2 to the function $f(t, x) \equiv 1$, then we get

$$\int_{\boldsymbol{M}} u(t, x; s, y) d_a y = 1.$$

Thus Theorem 4 is proved.

PROOF OF THEOREM 5. We shall call by Theorems 1′, 2′, 3′ and

8) This expression means that, for any $\varepsilon > 0$, there exists a compact set $\Gamma \subset \boldsymbol{M}$ such that $x \notin \Gamma$ implies $|f(t, x)| < \varepsilon$.

4′ respectively Theorems 1, 2, 3 and 4 which are modified as stated in Theorem 5. We note the relations

$$(4.7)\qquad A'_{sy}\frac{\sqrt{a(y)}}{\sqrt{g(y)}}=\frac{\sqrt{a(y)}}{\sqrt{g(y)}}A^*_{sy},\ A^*\frac{\sqrt{g(y)}}{\sqrt{a(y)}}=\frac{\sqrt{g(y)}}{\sqrt{a(y)}}A'_{sy}$$

and

$$(4.8)\qquad d_gx=\frac{\sqrt{g(x)}}{\sqrt{a(x)}}d_ax,\ d_ax=\frac{\sqrt{a(x)}}{\sqrt{g(x)}}d_ax\,,$$

which immediately follow from the definition of A', A^*, d_ax and d_ax. By means of these relations and Theorems 1, 2 and 4, we may easily prove Theorems 1′, 4′ and i) in Theorem 2′. If $f_1(s, y)$ satisfies the the assumption of ii) in Theorem 2′, then the function

$$f^*(s,\ y)=\frac{\sqrt{g(y)}}{\sqrt{a(y)}}f'(s,\ y)$$

satisfies (1.9*), (1.7*) in the sense of initial definition, and

$$\lim_{s\,.\,t} f^*(s,\ y)=\frac{\sqrt{g(y)}}{\sqrt{a(y)}}f(y)\,,$$

where $\frac{\sqrt{g(y)}}{\sqrt{a(y)}}f(y)$ is continuous and summable on $\boldsymbol{M}$ with respect to the measure d_ax. Hence, by Theorem 2, it follows that

$$\begin{aligned} f'(s,\ y)&=\frac{\sqrt{a(y)}}{\sqrt{g(y)}}f^*(s,\ y)\\ &=\int_{\boldsymbol{M}}\frac{\sqrt{a(y)}}{\sqrt{g(y)}}u^*(s,\ y\,;\ t,\ x)\frac{\sqrt{g(x)}}{\sqrt{a(x)}}f(x)d_ax\\ &=\int_{\boldsymbol{M}}\tilde{u}(t,\ x\,;\ s,\ y)f(x)d_ax\,, \end{aligned}$$

which proves ii) in Theorem 2′.

Theorem 3′ may be proved from Theorem 2′ by the same argument as the proof of Theorem 3.

Thus Theorem 5 is established.

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Mathematical Institute,
Nagoya University

Added in proof. It may not be of no use to state a relation between the results of Feller [2] and Dressel [1] and the result of the present paper.

The boundedness of a^{ij}, b^i, $\partial a^{ij}/\partial x^k$ etc. is assumed in [1], but the assumption II) stated in §1 in the present paper does not require the boundedness (in the usual sense) of these functions. For example, consider the equation

$$(1) \qquad u_t(t, x) = a(x)u_{xx}(t, x) + b(x)u_x(t, x) + c(x)u(t, x), \ a(x) > 0,$$
$$-\infty < t < \infty, \ (-\infty \leqq) r_1 < x < r_2 (\leqq \infty).$$

If a, b and c satisfy the assumption I) in §1, if

$$\int_{r_1}^{c} a(x)^{-\frac{1}{2}}\, dx = \int_{c}^{r_2} a(x)^{-\frac{1}{2}}\, dx = \infty, \ r_1 < c < r_2$$

(cf. (29), (30), in [2]),

and if $\tilde{b}$, $\tilde{b}_x$ and c are bounded where $\tilde{b} = (b - a/2)a^{-\frac{1}{2}}$, then the equation (1) satisfies also the assumption II) even if a, $1/a$ and b are unbounded.