Osaka Mathematical Journal
Vol. 4, No. 1, May, 1952

A Class of Topological Spaces

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1. Introduction. It is well known that the Čech's bicompactification $\beta(X)$ for any completely regular space X can be regarded as the completion of X in the uniform structure over X with the basis made up of all "finite" normal covering of X. In this point of view the following question naturally arises: What is the space which is obtained by the completion of the structure over X whose basis consists of all "countable" normal coverings of X?

In the present paper we are concerned with the space mentioned in the above problem. First of all we establish the relation between it and the Q-space introduced by E. Hewitt[1], then investigate the connections between our space and other important spaces. Moreover we discuss the relations between our space and the algebraic systems of the set of all continuous real valued functions on it.

2. Definition. Let us call the structure over a completely regular space X with the basis made up of all countable normal coverings of the space X the *e-structure* over X and denote by eX. Moreover we say the space with the complete e-structure to be *e-complete* and let us call a cardinal number $\mathfrak{m}$ *e-complete* if the discrete space with the potency $\mathfrak{m}$ is *e-copmlete*.

Remark. The notation "eX" was introduced by Tukey[2], but he said 'if the enumerable normal coverings are a basis for a uniformity, then we denote the uniformity by "eX"'. Thereby we shall show that the countable normal coverings are always a basis for a uniformity agreeing with the topology. To see that let X be a completely regular space and let $\mathfrak{U}$ be a countable normal coverings of X. Then we show that there exists a countable normal covering $\mathfrak{V}$ such that $\mathfrak{V}\overset{\Delta}{<}\mathfrak{U}$. Let $\mathfrak{U}=\{U_n\}$. Then since $\mathfrak{U}$ is normal, there exists an open covering $\mathfrak{U}_1$ such that $\mathfrak{U}_1\overset{\Delta}{<}\mathfrak{U}$. For any i, let $F_i=X-S(X-U_i, \mathfrak{U}_1)$. Then $\{F_i\}$ is a closed covering of X such that $F_i\subset U_i$ for any i and such that

1) Cf. [5]
2) Cf. [8, p. 57]

$S(F_i, \mathfrak{U}_1)\cap(X-U_i)=\phi$[3]. Hence there exists a function $f_i \in C(X, R)$[4] such that $f_i(x)=1$ for $x \in F_i$ $f_i(x)=0$ $x \notin U_i$ and $0 \leqq f(x) \leqq 1$ for any $x \in X$. Now we define a continuous mapping h of X into the Hilbert cube I_ω as follows. For any $x \in X$, $h(x)=\{i^{-1}\cdot f_i(x)\} \in I_\omega$. Moreover let $Y=h(x)$ and let $V_i=\{h(x) \mid f_i(x) \neq 0\}$. Then $\{V_i\}$ is an open covering of Y, because for any $x \in X$ there exists an F_i such that $x \in F_i$, hence $f_i(x) \neq 0$, therefore $h(x) \in V_i$. Obviously $h^{-1}(V_i) \subset U_i$. But since Y is separable, hence there exists a countable normal covering $\mathfrak{B}_1$ of Y such that $\mathfrak{B}_1 \overset{\triangle}{<} \{V_i\}$. Let $\mathfrak{B}=\{h^{-1}(V) \mid V \in \mathfrak{B}_1\}$. Then $\mathfrak{B} \overset{\triangle}{<} \mathfrak{U}$ and $\mathfrak{B}$ is a countable normal covering of X. Now let $\{\mathfrak{U}_\alpha\}$ be the family of all countable normal coverings of X. Then obviously for two $\mathfrak{U}_{\alpha_1}, \mathfrak{U}_{\alpha_2} \in \{\mathfrak{U}_\alpha\}$ there exists a $\mathfrak{U}_{\alpha_3} \in \{\mathfrak{U}_\alpha\}$ such that $\mathfrak{U}_{\alpha_3} < \mathfrak{U}_{\alpha_1} \wedge \mathfrak{U}_{\alpha_2}$ and by the above fact there exists a $\mathfrak{U}_{\alpha_4}$ such that $\mathfrak{U}_{\alpha_4} \overset{\triangle}{<} \mathfrak{U}_{\alpha_1}$. Thus we see that $\{\mathfrak{U}_\alpha\}$ is a basis for some uniformity. Obviously this uniformity agrees with the topology.

3. The relations between the *e*-complete space and the *Q*-space.

Theorem 1. *For a completely regular space X, the following conditions are equivalent:*

i) *X is e-complete,*

ii) *for any CZ-maximal family*[4] *of X, the total intersection is non-void,*

iii) *X is homeomorphic to a closed subset of a Cartesian product of the space of real numbers with the usual topology.*

Proof. a) We show first that i) implies ii). Let X be an e-complete space and let $\mathfrak{A}$ be a CZ-maximal family of X. Then we have only to prove that $\mathfrak{A}$ is a Cauchy family of eX. To see it let $\mathfrak{U}$ be a countable normal covering of X. Then we show that there exists a $Z \in \mathfrak{A}$ and a $U \in \mathfrak{U}$ such that $F \subset U$. Let $\mathfrak{U}=\{U_i\}$. Then as we have seen in the above remark there exists a closed covering $\mathfrak{F}=\{F_i\}$ such that for any i $F_i \subset U_i$ holds and such that F_i and $U_i^c(=X-U)$ are completely separated. Hence there exists a Z-set Z_i such that $F_i \subset Z_i \subset U_i$. If for any i $Z_i \notin \mathfrak{A}$ holds, there exists by the properties (c) and (d) of $\mathfrak{A}$ a $Z'_i \in \mathfrak{A}$ such that $Z'_i \cap Z_i=\phi$. Then $\Pi_i Z'_i=\Pi_i Z'_i \cap \Sigma_i Z_i=\Sigma_i(Z'_i \cap Z_i)=\phi$,

3) We denote by ϕ the void set.

4) Cf. [5]. Let $C(X, R)$ be the set of all real-valued continuous functions of X and let f be a function in $C(X, R)$. Then the set of points in X for which f vanishes is said to be a Z-set and is denoted by $Z(f)$. Finally let $Z(X)$ be the family all Z-sets of X. Then a subfamily $\mathfrak{A}$ of the family $Z(X)$ is said to be a CZ-maximal family if $\mathfrak{A}$ enjoys the following four conditions: a) $\mathfrak{A}$ is not empty, b) $\mathfrak{A}$ does not contain a void set, c) $\mathfrak{A}$ never contains countable subfamilies with total void intersection and d) $\mathfrak{A}$ is maximal with respect to (a), (b) and (c).

which contradicts the property (c) of $\mathfrak{A}$. Hence there exists a Z_i such that $Z_i \in \mathfrak{A}$. Thus we see that $\mathfrak{A}$ is a Cauchy family of eX and since eX is complete there exists a limit point of $\mathfrak{A}$, which is the total intersection of $\mathfrak{A}$.

b) Next we show that ii) implies iii). Let X be a completely regular space satisfying the condition ii). Evidently by the mapping $h: h(x)=\{f(x)\}$, X is homeomorphically mapped into the Cartesian product space $P_f R_f$, where $R_f(=R)$ is the space of reals and the index $f \in C(X, R)$. Now identifying the point x of X with the image $h(x)$, we may assume that $X \subset P_f R_f$. Moreover let gX be the substructure of the usual product structure over $P_f R_f$. Then we shall show that gX is complete and that therefore X is closed in the product space. To see it we have only to prove that any Cauchy family of gX is equivalent to a Cauchy family of gX which is a CZ-maximal family of X. Let $\mathfrak{A}'$ be a Cauchy family of gX. Then assuming that $\mathfrak{A}'$ is a subfamily of the family $Z(X)$, we can find a maximal subfamily $\mathfrak{A}$ of $Z(X)$ containing $\mathfrak{A}'$ with respect to the finite intersection property. Now we shall show that $\mathfrak{A}$ is CZ-maximal. Suppose, on the contrary, that there exists a countable subfamily $\{Z_i\}$ of $\mathfrak{A}$ such that $\Pi_i Z_i=\phi$. For any i there exists an $f_i \in C(X, R)$ such that $Z_i=Z(f_i)$. Let $g_n=\bigvee_{i=1}^{n}|f_i|$ where $\bigvee_{i=1}^{n}|f_i|(x)$ is the maximum of the absolute values of $f_i(x)$ for $i=1, 2, \dots, n$. Moreover assuming that for any i $0 \leqq f_i(x) \leqq 1$, let $g=\Sigma_n 2^{-n} g_n$. Then for any $x \in X$, $g(x)>0$. Hence $g^{-1}(x)(=1/g(x))$ is continuous. Now obviously $\mathfrak{A}$ is a Cauchy family of gX which is equivalent to $\mathfrak{A}'$. Therefore for every positive number ε there exists a $Z \in \mathfrak{A}$ such that the diameter of the set $g^{-1}(Z)$ is less than ε. But if $x \in Z(g_n)$ then $g(x)<2^{-(n-1)}$, hence for sufficiently large n, $g^{-1}(x)>\max\limits_{y \in Z} g^{-1}(y)$ for any $x \in Z(g_n)$. This implies that $Z(g_n) \frown Z=\phi$. But $Z(g_n)=Z(f_1) \wedge \dots \wedge Z(f_n)$, hence $Z(g_n) \in \mathfrak{A}$, and since $Z \in \mathfrak{A}$, $\mathfrak{A}$ does not satisfy the finite intersection property, which is a contradiction. Thus we see that $\mathfrak{A}$ is a CZ-maximal family. Therefore by the condition ii) of X there exists the total intersection of $\mathfrak{A}$ which is a limit point of $\mathfrak{A}$, and so of $\mathfrak{A}'$. This implies that gX is complete and that X is a closed subset of $P_f R_f$.

c) We show finally that the condition iii) implies i). Let X be a closed subset of a Cartesian product $P_\alpha R_\alpha$, where $R_\alpha=R$, then the substructure $g'X$ of the usual product structure over $P_\alpha R_\alpha$ is complete, since the structure over $P_\alpha R_\alpha$ is complete. And obviously we can find the basis for the structure $g'X$ consisting of countable normal coverings, since we can find a basis with the same property for the product structure. Hence the identity mapping from eX to $g'X$ is uniformly continuous.

This implies that eX is complete. Thus the proof of our theorem is complete.

From iii) of Theorem 1 we have evidently

Corollary. *A closed subspace of an e-complete space is e-complete.*

E. Hewitt's Q-space[5] is nothing other than the space satisfying the condition ii) of Theorem 1. Therefore we have

Theorem 2. *A completely regular space is e-complete if and only if it is a Q-space.*

Remark. If for two totally bounded structures over a completely regular space their equivalent relations of Cauchy families are equal, they are isomorphic. But the eX and the gX in the proof of Theorem 1, b) are not always isomorphic in spite of their equivalent relations of Cauchy families being equal.

4. **The relation between the e-complete space with complete structure.** In this section we consider the following problem: On which kind of spaces are there a complete structure? This problem was considered by[5] A. Weil, J. Dieudonné, J. W. Tukey, K. Morita and the author.

Lemma 1. *Let $\mathfrak{A}$ be a CZ-maximal family of a completely regular space X and let f be a non-constant function in $C(X, R)$ such that $F_0=Z(f)\in\mathfrak{A}$. If F_1 is a set $\{x\,|\,f(x)\leqq a\}$, where a is a positive number, the family*

$$\mathfrak{A}'=\{Z(g)\,|\,Z(g)\in Z(F_1)\ \&\ Z(g)\supset Z\cap F_0\neq\phi\ \textit{ for some }\ Z\in\mathfrak{A}\}$$

is a CZ-maximal family of F_1.

Proof. Obviously $\mathfrak{A}'$ enjoys the condition (c), hence we have only to prove that $\mathfrak{A}'$ is a maximal family of $Z(F_1)$ with respect to the finite intersection property. Let $Z(g)$ be compatible with $\mathfrak{A}'$, where $g\in C(F_1, R)$ and let $F_0''=Z(g)\cap F_0$. Then $F_0''\neq\phi$ since $F_0\in\mathfrak{A}'$. Here we may suppose that a is equal to 1. Now for any rational number τ in $[0, 1]$, let $G_\tau=\{x\,|\,f(x)<\tau\}$, $G'=\{x\,|\,g(x)<\tau\}$, $F_\tau=\{x\,|\,f(x)\leqq\tau\}$ and let $F_\tau'=\{x\,|\,g(x)\leqq\tau\}$. Furthermore let $G_\tau''=G_\tau\cap G_\tau''$ and let $F_\tau''=F_\tau\cap F_\tau'$. Then F_τ'' is a closed subset of X and G_τ'' is an open set, since $G_\tau\subset F_1$ and G_τ' is open in X. Moreover if $\tau<\sigma$ $F_\tau''\subset G_\sigma''\subset F_\sigma''$. Hence for any $x\in X$ we set $g'(x)=\sup_{\tau\notin F_\tau''}\tau$. Then $g'\in C(X, R)$ and $Z(X)\ni Z(g')=\Pi_\tau F_\tau''=\Pi_\tau(F_\tau\cap F_\tau')=F_0\cap Z(g)=F_0''$. Therefore $Z(g')\in\mathfrak{A}$, consequently by the definition of $\mathfrak{A}'$, $Z(g)\in\mathfrak{A}'$. Furthermore $\mathfrak{A}'$ has obviously the finite intersection

5) Cf. [5, Theorem 50].

6) Cf. [7], [8], [9], [10] and [12].

property, hence $\mathfrak{A}$ is maximal with respect to this property.

By A. Tarski's definition, Ulam's[7] and Hewitt's Theorem[8] and Theorem 1 we have immediately

Lemma 2. *If a cardinal number* $\mathfrak{m}$ *is weakly accessible from* $\aleph_0$ *in A. Tarski's sense,* $\mathfrak{m}$ *is e-complete.*

From above two lemmas we have

Theorem 3. *If the character*[9] *of a completely regular space* X *is weakly accessible from* $\aleph_0$ *and if there exists a complete structure over* X, X *is e-complete.*

Proof. Let X be a completely regular space with a complete structure gX and let the character $\mathfrak{m}$ of X be weakly accessible from $\aleph_0$. Then the potency $|X|$ of X is also weakly accessible from $\aleph_0$ since $|X| \leqq 2^{\mathfrak{m}}$. Hence X is e-complete. Moreover let $\{\mathfrak{U}_\delta | D\}$ be the uniformity of gX and let $\mathfrak{A}$ be a CZ-maximal family of X. Then we have only to prove that $\mathfrak{A}$ is a Cauchy family of gX.

For this we shall show that if $\mathfrak{U}$ is an arbitrary element of $\{\mathfrak{U}_\delta | D\}$, there exist a $Z \in \mathfrak{A}$ and a $U \in \mathfrak{U}$ such that $Z \subset \mathfrak{U}$.

Now since $\mathfrak{U}$ is a normal covering of X there exists a normal sequence $\{U_n\}$ such that $\mathfrak{U} \overset{*}{>} \mathfrak{U}_1 \overset{*}{>} \mathfrak{U}_2 \overset{*}{>} \dots \overset{*}{>} \mathfrak{U}_{n-1} \overset{*}{>} \mathfrak{U}_n \overset{*}{>} \mathfrak{U}_{n+1} \overset{*}{>} \dots$. Let $\mathfrak{U} = \{U_\alpha | A\}$ where A is assumed to be a well ordered set of indices. Then, as A. H. Stone[10] showed, there exists a family $\{F_{n,\alpha} | n=1, 2, \dots \;\&\; \alpha \in A\}$ satisfying the following conditions:

i) $\{F_{n,\alpha}\}$ is a closed covering of X,

ii) every element of $\mathfrak{U}_{n+3}$ does not intersect two elements of $F_n = \{F_{n,\alpha} | \alpha \in A\}$ at the same time, and

iii) $S(F_{n,\alpha}, \mathfrak{U}_{n+1}) \subset U_\alpha$.

Now, let $F_n = \Sigma_\alpha F_{n,\alpha}$. Then by i) and ii) $\{F_m | m=1, 2, \dots\}$ is a closed covering of X. Since $\mathfrak{A}$ satisfies the condition c), there exists a set $F_n \in \{F_n\}$ such that F_n is compatible with $\mathfrak{A}$. Let f be a positive continuous function such that $f(x)=0$ for $x \in F_n$ and $f(x)=2$ for $x \notin S(F_n, \mathfrak{U}_{n+4})$. Moreover let $Z_0 = \{x | f(x) \leqq 0\}$ and let $Z_1 = \{x | f(x) \leqq 1\}$. Then since $Z_0 \supset F_n$, $Z_0 \in \mathfrak{A}$ and hence by Lemma 1 the family $\mathfrak{A}' = \{Z(g) | g \in C(Z_1, R) \;\&\; Z(g) \supset Z \frown Z_0$ for some $Z \in \mathfrak{A}\}$ is a CZ-maximal family of Z_1. As in the proof of Theorem 1, (a), $\mathfrak{A}'$ is a Cauchy family of eZ_1.

Now let $Z_\alpha = Z_1 \frown S(F_{n,\alpha}, U_{n+4})$. Then $Z_1 \subset S(F_n, \mathfrak{U}_{n+4})$ and by ii) $S(F_{n,\alpha}, \mathfrak{U}_{n+4}) \frown S(F_{n,\beta}, \mathfrak{U}_{n+4}) = \phi$, hence there exists a continuous function

7) Cf. [1], [2] and [3, p. 133].

8) Cf. [6, Theorem 16].

9) The character of a space X is the smallest cardinal number of basis for open sets.

10) Cf. [4]. The proof of Theorem 1.

f_α such that $f_\alpha(x)=f(x)$ for $x\in S(F_{n,\alpha}, \mathfrak{U}_{n+4})$ and $f_\alpha(x)=2$ for $x\notin S(F_{n,\alpha}, \mathfrak{U}_{n+4})$. Then $f_\alpha\in C(X,R)$, $\{x\,|\,f(x)\leqq 1\}=Z_\alpha$, i.e., $Z_\alpha\in Z(X)$, $\Sigma_\alpha Z_\alpha=Z_1$, $Z_\alpha \frown Z_\beta=\phi$ for $\alpha,\beta(\alpha\neq\beta)$ and by ii) $S(Z_\alpha, \mathfrak{U}_{n+4})\frown S_\beta, \mathfrak{U}_{n+4})\subset S(F_{n,\alpha}, \mathfrak{U}_{n+3}) \frown S(F_{n,\alpha}, \mathfrak{U}_{n+3})=\phi$. Accordingly if Y is the discrete space $\{\alpha\,|\,Z_\alpha\neq\phi\}$ and h is the mapping from Z_1 to Y such that if $x\in Z_\alpha$, $h(x)=\alpha$, h is a uniformly continuous mapping of eZ into eY.

Since $|Y|\leqq|A|\leqq X$, Y is e-complete and $h(\mathfrak{A}')$ is a Cauchy family of eY. Thereby there exists the limit point α of $h(\mathfrak{A}')$. This implies that for any Z' $h(Z')\ni\alpha$, i.e., $Z'\frown Z_\alpha\neq\phi$. Since for any $Z\in\mathfrak{A}$, $Z\frown Z_1\in\mathfrak{A}'$, $Z\frown Z_\alpha\supset(Z\frown Z_1)\frown Z_\alpha\neq\phi$. Thus Z_α is compatible with $\mathfrak{A}$. Therefore by (c) and (d) of $\mathfrak{A}$ $Z_\alpha\in\mathfrak{A}$ and by iii) $Z_\alpha\subset S(F_{n,\alpha}, \mathfrak{U}_{n+4})\subset U_\alpha$. Since $\mathfrak{U}$ is arbitrary, $\mathfrak{A}$ is a Cauchy family of gX.

By Theorem 1 and Theorem 3 we have

Corollary 1. *The following three statements are equivalent:*

a) *every completely regular space with complete structure is homeomorphic to a closed subset of a Cartesian product of the reals,*

b) *every cardinal number is e-complete,*

c) *every discrete space admits no measure completely additive on all subsets, vanishing for every point, assuming only values* 0 *and* 1 *and equal to* 1 *for the whole space.*

Proof. Every discrete space is metric, hence its α-structure is complete, therefore by Theorem 1 a) implies b). Next by Hewitt theorem[8] and by Theorem 2 b) and c) are equivalent. Finally by Theorem 2 b) implies a).

Corollary 2. *Let X be a fully normal T_1-space. Then if $|X|$ is accessible from $\aleph_0$, X is e-complete.*

For any fully normal space admits a complete structure.

4. **The e-completion.** The completion of the structure eX is an e-complete space. In this section we show that such e-completion is unique in the same sense that $\beta(X)$ and $\omega(X)$ are unique.

Theorem 4. *Let X be any completely regular space. Then there exists a space $e(X)$ which admits the following conditions:*

i) *$e(X)$ is e-complete,*

ii) *$e(X)$ contains X as a dense subset,*

iii) *if $Z_i\in Z(X)$ for $i=1,2,\dots$, $\Pi_{i=1}^\infty \bar{Z}_i=\overline{\Pi_{i=1}^\infty Z_i}$, where $\bar{Z}$ is the closure of Z in $e(X)$.*

Such an e-completion is completely determined up to homeomorphisms.

Proof. We prove first that the completion $\overline{eX}$ of eX satisfies the conditions i) ii) and iii). Obviously $\overline{eX}$ satisfies i) and ii), hence we have only to prove that eX admits the condition iii).

Let $Z_i \in Z(X)$ for $i=1, 2, \dots$. Then obviously $\Pi_{i=1}^{\infty}\overline{Z}_i \supset \overline{\Pi_{i=1}^{\infty} Z_i}$. Hence we show that $\Pi_i \overline{Z}_i \subset \overline{\Pi_i Z_i}$. Let $\mathfrak{A}_0$ be the Cauchy family of eX such that the limit point is contained in $\Pi_{i=1}^{\infty}\overline{Z}_i$. Then we may assume, as we have shown in the proof of Theorem 1 (b), that $\mathfrak{A}_0$ is a CZ-maximal family of X. And since for any i the limit point of $\mathfrak{A}_0$ is contained in $\overline{Z}_i$, $Z_i \frown Z \neq \phi$ for any $Z \in \mathfrak{A}_0$.

For if $Z_i \frown Z = \phi$, then there exist two $f_1, f_2 \in C(X, R)$ such that $Z(f_1)=Z_i$ and $Z(f_2)=Z$. Let $f=\dfrac{f_1}{f_1+f_2}$. Then $f(x)=0$ for $x \in Z_i$ and $f(x)=1$ for $x \in Z$, i.e., Z_i and Z is completely separated. Hence there exists a countable normal covering $\mathfrak{U}$ of X such that $S(Z_i, \mathfrak{U}) \frown S(Z, \mathfrak{U}) = \phi$, therefore $S(Z_i, \mathfrak{U}^*) \frown S(Z, \mathfrak{U}^*) = \phi$ where $\mathfrak{U}^* = \{U^* \mid U \in \mathfrak{U}\}$ and U^* consists of all the Cauchy equivalent classes $\{\mathfrak{A}\}$ where $\mathfrak{A} \in \{\mathfrak{A}\}$ implies that there exists a $Z \in \mathfrak{A}$ such that $U \supset Z$. Then since $\overline{Z}_i \subset S(Z_i, \mathfrak{U}^*)$ and $\overline{Z} \subset S(Z, \mathfrak{U}^*)$, $\overline{Z}_i \frown \overline{Z} = \phi$. This implies that the limit point of $\mathfrak{A}_0$ is not contained in $\overline{Z}_i$, since it is contained in $\overline{Z}$, which is a contradiction.

Thus by the condition (c) and (d) of $\mathfrak{A}$, $Z_i \in \mathfrak{A}$. Hence $\Pi Z_i \in \mathfrak{A}$, i.e. the limit point of $\mathfrak{A}$ is contained in $\overline{\Pi Z_i}$. Thus we have $\Pi\overline{Z}_i = \overline{\Pi Z_i}$.

To show the uniqueness, let Y satisfy the conditions i), ii) and iii). Then we show that eY is the completion eX. For this we have only to prove that every countable normal covering of X is extended to a countable normal covering of Y. Let $\mathfrak{U}$ be a countable normal covering of X. Then as we have shown in the remark of I, there exists a countable normal sequence $\{\mathfrak{V}_n\}$ such that $\mathfrak{V}_n < \mathfrak{U}$ and such that for any $V \in \mathfrak{V}_n$ $X-V$ is a Z-set. Now for $U \in \mathfrak{U}$ we set $\mu(U) = Y - \overline{Y-U}$ and let $\mu(\mathfrak{U}) = \{\mu(U) \mid U \in \mathfrak{U}\}$. Then obviously $\mu(\mathfrak{V}_n) < \mu(\mathfrak{U})$ and by the condition iii) $\prod_{V \in \mathfrak{V}_n} (Y-\mu(V)) = \prod_{V \in \mathfrak{V}_n} \overline{Y-V} = \overline{\prod_{V \in \mathfrak{V}_n}(Y-V)} = \overline{\phi} = \phi$. Hence $\mu(\mathfrak{V}_n)$ is a covering of Y and thereby so is $\mu(\mathfrak{U})$. Furthermore since $\mathfrak{V}_n \overset{*}{>} \mathfrak{V}_{n+1}$, $\mu(\mathfrak{V}_n) \overset{*}{>} \mu(\mathfrak{V}_{n+1})$. Hence $\mu(\mathfrak{U})$ is a countable normal covering of Y. Thus we see that eX is the substructure of eY and by the uniqueness of the completion, Y is homeomorphic to $e(X)$.

Corollary. *In Theorem* 4 *we may replace the condition* iii) *by the following condition:*

iii′) *every continuous function in* $C(X, R)$ *can be continuously extended on* $e(X)$.

Proof. Obviously the completion $\overline{eX}$ of eX satisfies iii′) since every continuous function in $C(X, R)$ is a uniformly continuous function of eX into the usual structure of the reals R.

Hence we have only to prove that if Y satisfies the conditions i), ii) and iii′), Y enjoys iii). For this let $f_i \in C(X, R)$ such that $Z(f_i)=Z_i$ and such that $|f_i| \leqq 1$, moreover $y \in \Pi\bar{Z}_i - \overline{\Pi_i Z_i}$. Then there exists a Z-set Z of X such that $\bar{Z} \ni y$ and $Z \frown \Pi Z_i = \phi$. Now let $f \in C(X, R)$ such that $Z(f)=Z$ and let $g=\Sigma_{i=1}^{\infty} g_i$ where $g_i=(|f_i|+|f|)\cdot 2^{-i}$. Then g is strictly positive, i.e., $g(x)>0$ for any $x \in X$. Let $\bar{g}$ be the extension of g over Y. Then $\bar{g}(y)=0$, because $g|X=\Sigma_i g_i=\Sigma_i \bar{g}_i|X$, hence by the condition ii) $\bar{g}=\Sigma_i \bar{g}_i$ and also $\bar{g}_i \leqq |\bar{f}_i|+|\bar{f}|$. But there exists $h \in C(X, R)$ such that $h(x)\cdot g(x)=1$ for any $x \in X$ since g is strictly positive. Since $\bar{h}\cdot\bar{g}|X$ $=h\cdot g=\bar{h}\cdot\bar{g}|X$, $\bar{h}\cdot\bar{g}=\overline{h\cdot g}=1$. Hence $\bar{g}(y)\neq 0$, which is a contradiction. Hence $\Pi\bar{Z}_i=\overline{\Pi Z_i}$.

Remark. The above Theorem shows that the e-completion $e(X)$ is characterized by the internal-topological properties, as the Shanin's bicompactification $(\omega, Z(X))$[11] is such one. The corollary shows further that our $e(X)$ and Hewitt's $\nu(X)$ is one and the same thing, and that $\beta(e(X))=\beta(X)$.

5. The combination of topologies.

Theorem 5. *A completely regular space X is e-complete if and only if any closed proper subsets of X are e-complete.*

Proof. From the corollary of Theorem 1 the necessity is obvious. Therefore we have only to prove the sufficiency.

Let X be a space with the potency >1 satisfying the condition of Theorem 4 and let $\mathfrak{A}$ be a CZ-maximal family of X and moreover let Z be a Z-set in $\mathfrak{A}$ such that $Z \neq X$. Then there exists a function f such that $Z(f)=Z$, and such that for some $p \in X$ $f(p)=2$. Now let $Z_1=\{x \mid f(x) \leqq 1\}$. Then by Lemma 1, the family $\mathfrak{A}'=\{Z(g) \mid g \in C(Z_1, R)$ & $Z(g) \supset Z(f') \frown Z(f)$ for some $Z(f') \in \mathfrak{A}\}$ is a CZ-maximal family of Z_1 and Z_1 is a proper closed subset of X, hence by our assumption Z_1 is e-complete. Thereby there exists the total intersection of $\mathfrak{A}'$ which is obviously also the total intersection of $\mathfrak{A}$. Thus we see that the space X is e-complete.

From Theorem 5 the following question arises: What is the space whose subsets are always e-complete? For this we have

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11) Cf. [13] and [14].

Theorem 6. *The following conditions on a completely regular space X are equivalent:*

i) *if a completely regular space Y is the domain of a continuous one-to-one mapping onto X, then Y is e-complete,*

ii) *every subset of X, is e-complete,*

iii) *for any point p of X, the complementary $X-p$ is e-complete,*

Proof. First we will prove that i) implies ii). For this, suppose that a space X satisfies the condition i) and let F be an arbitrary subset of X. Moreover let Y be the space such that the set of points of Y is the same as X and such that the topology of F in Y is the same relative topology of F in X and every point belonging to $Y-F$ is isolated point in Y. Then obviously the topology of Y is finer than the topology of X. Hence by the condition i) Y is e-complete and since F is a closed subset of Y, F is e-complete by the corollary of Theorem 1.

Second ii) implies evidently iii). Finally we prove that iii) implies i). For this purpose suppose that a space X enjoys the condition iii) and let Y be a space which is the domain of a continuous one-to-one mapping h onto X. Then we show that Y is e-complete. Let $\mathfrak{A}$ be a CZ-maximal family of Y. Then $\mathfrak{A}$ is a Cauchy family of eY and h is a uniformly continuous mapping of eY onto eX, hence $h(\mathfrak{A})=\{h(Z)\,|\,Z\in\mathfrak{A}\}$ is a Cauchy family of eX. Since X is e-complete by Theorem 5, $h(\mathfrak{A})$ has a limit point x_0 in X. Suppose that $h^{-1}(x_0)=y$ is not a limit point of $\mathfrak{A}$. Then there exists a Z-set $Z\in\mathfrak{A}$ such that $Z=Z(f)$ for some $f\in C(Y, R)$ and such that $f(y)=2$. Let $Z_1=\{y\,|\,f(y)\leq 1\}$. Then by Lemma 1 the family $\mathfrak{A}'=\{Z'\,|\,Z'\in Z(Z_1)\ \&\ Z'\supset Z\frown Z_0$ for some $Z\in\mathfrak{A}\}$ is a CZ-maximal family of Z_1. As we have shown above, the family $h(\mathfrak{A}')$ is a Cauchy family of $eh(Z_1)$ and $eh(Z_1)$ is finer than the substructure over $h(Z_1)$ of eX' where $X'=X-\{x_0\}$. Hence $h(\mathfrak{A}')$ is a Cauchy family of eX'. Since $X-\{x_0\}$ is e-complete by the assumption, $h(\mathfrak{A}')$ has a limit point x_1 in $X-\{x\}$. But $\mathfrak{A}'\supset\{Z\frown Z_1\,|\,Z\in\mathfrak{A}\}$ hence $x_0=\prod\limits_{Z\in\mathfrak{A}}\overline{h}(Z)=\prod\limits_{Z'\in\mathfrak{A}'}\overline{h}(Z')=x_1$, which is a contradiction. Thus we see that y is a limit point of $\mathfrak{A}$. Since $\mathfrak{A}$ is an arbitrary CZ-maximal family, Y is e-complete.

Corollary 1. *If X is e-complete and if every point of X is a G_δ-set, then X satisfies the equivalent conditions of Theorem 6.*

Proof. Let X be a space satisfying the assumptions of Corollary 1. Then we show that X enjoys the condition iii) of Theorem 6. Evidently if p is an isolated point of $X, X-\{p\}$ is e-complete. Hence let p be an interior point of X and let $\mathfrak{A}$ be a Cauchy family of eY where $Y=X-\{p\}$. Then as we have seen in the proof of the above theorem, $\mathfrak{A}$ is a Cauchy family of eX. Since X is e-complete, there exists a limit point q of $\mathfrak{A}$.

Now we will show that $p \neq q$. Since p is a G_δ-open set of X, it is Z-set, consequently there exists a function $f \in C(X, R)$ such that $Z(f) = \{p\}$. Then there exists a $g \in C(X-\{p\}, R)$ such that $g(x) \cdot f(x)=1$ for $x \in X-\{p\}$ and $g(p)=\infty$. Since $\mathfrak{A}$ is a Cauchy family of eY, there exists a $Z_\varepsilon \in \mathfrak{A}$ such that $\delta(g(Z_\varepsilon)) < \varepsilon$. Hence for some n $Z_\varepsilon \frown \{x \mid g(x) > n\} = \phi$. This implies that $\bar{Z}_\varepsilon \not\ni p$, and $\bar{Z}_\varepsilon \ni q$, i.e., $p \neq q$. Thus q is a limit point of $\mathfrak{A}$ in $X-\{p\}$. Since $\mathfrak{A}$ is arbitrary, we see that X satisfies the condition iii).

Corollary 2. *Separable spaces and bicompact spaces with the first axiom of countability enjoys the conditions of Theorem 6.*

Finally we shall consider the weaker topology than the e-complete topology.

Theorem 7. *The following conditions on a fully normal space X are equivalent:*

i) *X possesses the Lindelöf property,*

ii) *if a completely regular space Y is an image of a continuous one-to-one mapping of X, then Y is e-complete.*

Proof. When a regular space X possesses the Lindelöf property, X is a fully normal space[12] and if a regular space Y is an image of a continuous mapping of X, then Y possesses also the Lindelöf property, and obviously the regular space possessing this property is e-complete. This shows that i) implies ii).

Now we will show that ii) implies i). For this, let X be a fully normal space without the Lindelöf property. Then there exists an open covering whose subcoverings are always uncountable. Since X is a fully normal space, there exists an open covering $\mathfrak{U}$ whose subcoverings are always uncountable. Since X is a fully normal space, there exists an open covering $\mathfrak{W}$ such that $\mathfrak{U} \overset{**}{>} \mathfrak{W}$. Accordingly there exists a subset A of X with potency $\geq \aleph_1$ such that for any point p of X there exists at most an $a \in A$ such that $S(p, \mathfrak{W}) \frown S(a, \mathfrak{W}) = \phi$. Then we may suppose that A is a subset $\{a_\mu \mid 1 \leq \mu \lneqq \omega_1\}$ of X.

Using this set A, we will construct a space Y such that Y is an image of a continuous one-to-one mapping of X and such that Y is not e-complete. Let points of Y coincide with those of X. For each μ, let $\{U_{\mu,\alpha} \mid \alpha \in \Gamma_\mu \ \& \ U_{\mu,\alpha} \subset S(a_\mu, \mathfrak{W})\}$ be a complete system of neighbourhoods of a_μ in X where Γ_μ is a set of indices. Now let a complete system of neighbourhoods in Y of points which are not in A be the same as in X. For a point a_μ, let $U(a_\mu, \Delta(\lambda, \mu)) = \Sigma_{\Delta(\lambda,\mu)=\alpha} U_{\lambda',\alpha}$ where $\Delta(\lambda, \mu)$

12) Cf. [11].

is a one-to-one correspondence from an interval $(\lambda, \mu]$ into $\Sigma_{\lambda'\in(\lambda,\mu]}\Gamma_{\lambda'}$ such that $\Delta(\lambda, \mu)(\lambda')\in\Gamma_{\lambda'}$. Then, let a complete system of neighbourhoods in Y of a_μ be $\{U(a_\mu, \Delta(\lambda, \mu))\}$ where $\Delta(\lambda, \mu)$ runs over all correspondences described above.

Obviously Y is a T_1-space and Y is an image of a continuous one-to-one mapping of X.

We must prove that Y is a completely regular space. For this we show, for example, that if $x_\mu\in U$ there exists a continuous function $f\in C(Y, R)$ such that $f(a_\mu)=1$ and $f(y)=0$ $y\in Y-U(a_\mu, \Delta(\lambda, \mu))$ for some $U(a_\mu, \Delta(\lambda, \mu))\subset U$. Suppose $U=U(a_\mu, \Delta(\lambda, \mu))=\Sigma\, U_{\lambda', \alpha_{\lambda'}}$. Then for any $\lambda', \lambda<\lambda'\leqq\mu$, there exists an $f_{\lambda'}\in C(X, R)$ such that $f_{\lambda'}(x_{\lambda'})=1$, $f_{\lambda'}(y)=0$ for $y\in X-U_{\lambda', \alpha_{\lambda'}}$ and $0\leqq f_{\lambda'}\leqq 1$. Let f be the function $\Sigma_{\lambda<\lambda'\leqq\mu} f_{\lambda'}$, then obviously $f(a_\mu)=1$ and $f(y)=0$ for $y\in Y-U$. Now we must prove that $f\in C(Y, R)$. For any point $x\notin A$, there exists at most one a_x such that $S(x, \mathfrak{W})\frown S(a_{\lambda'}, \mathfrak{W})\neq\phi$, hence $f\,|\,S(x, \mathfrak{W})$ is equal to $f\,|\,S(a_{\lambda'}, \mathfrak{W})$ or to 0. For $a_\alpha : \alpha>\mu$ or $\alpha\leqq\lambda$, $f\,|\,U(a_\alpha, \Delta(\alpha', \alpha))=0$ for some neighbourhood of a_α, which implies that f is continuous at $x\notin\{a_\alpha\}\lambda<\alpha\leqq\mu\}$. Finally for $a_\alpha : \lambda<\alpha\leqq\mu$ and for an arbitrary positive number $\varepsilon>1$, let $U_{\lambda', \alpha(\varepsilon, \lambda')}=\{x\,|\,f_{\lambda'}(x)>1-\varepsilon\}$ and let $\Delta(\lambda, \alpha)(\lambda')=\alpha(\varepsilon, \lambda')$ for $\lambda' : \lambda<\lambda'\leqq\alpha$, then $|f(y)-f(a_\alpha)|<\varepsilon$ for $y\in U(a_\alpha, \Delta(\lambda, \alpha))$. This implies that f is continuous at a_μ. Thus we see that Y is a completely regular space.

Finally we will show that the set A is a closed subset of Y and that A is not e-complete. Since for any $x\notin A$ there exists a neighbourhood U such that $U\subset S(x, \mathfrak{W})$ and $U\frown A=\phi$, A is closed in Y, and by our construction A is homeomorphic with the space $[1, \omega_1)$ of ordinal numbers with the usual order topology, but the space $[1, \omega_1)$ is not e-complete since the space admits no complete structures.

Corollary. *A metrizable space X is separable if and only if any continuous one-to-one image of X has a complete structure.*

6. Translation lattice $C(X, R)$. In the remaining section we investigate the relation between the topological properties of the e-complete space and the algebraic systems of $C(X, R)$ and we shall give certain extension of results[13] in case of bicompact spaces.

We first begin with the translation lattice[14]. By a *translation lattice* L we shall mean a lattice where for every $a\in L$ and for real numbers α a sum $a+\alpha$ is defined and which satisfies the following conditions:

13) Cf. [17], [19], [20] and [21].

14) Cf. [20].

1. $a+0=a$,
2. $(a+\alpha)+\beta=a+(\alpha+\beta)$,
3. If $\alpha\geqq 0$ then $a+\alpha\geqq a$,
4. If $a\geqq b$ then $a+\alpha\geqq b+\alpha$.

Obviously $C(X, R)$ can be considered as a translation lattice by setting $(f+\alpha)(x)=f(x)+\alpha$ for a real number α and for a function $f\in C(X, R)$.

By a homomorphism of L into the reals we shall mean a mapping φ such that

1) $\varphi(a\cup b)=\max[\varphi(a), \varphi(b)]$,
2) $\varphi(a\cap b)=[\varphi(a), \varphi(b)]$,
3) $\varphi(a+\alpha)=\varphi(a)+\alpha$.

Lemma. *Let φ be a homomorphism of $C(X, R)$ into the reals such that $\varphi(0)=0$, where X is a completely regular space and 0 in $\varphi(0)$ is the function such that $0(x)=0$ for every $x\in X$. Then $Z(\varphi^{-1}(0))=\{Z(f)\mid f\in\varphi^{-1}(0)\}$ is a CZ-maximal family of X.*

Proof. We show first that every function $f\in\varphi^{-1}(0)$ is not strictly positive. Suppose, on the contrary, there exists a function f such that $f\in\varphi^{-1}(0)$ and f is strictly positive. Then there exists a $g\in C(X, R)$ such that $g(x)\cdot f(x)=1$ for every $x\in X$. Let $\varphi(g)=\alpha$. Then $\alpha\neq 0$. For suppose $\varphi(g)=0$, then $\varphi(f\cup g)=0$. But $f\cup g\geqq 1$, hence $\varphi(f\cup g)\geqq 1$, which is a contradiction. Now $\varphi(g-\alpha)=0$, therefore $\varphi((g-\alpha)\cup f)=0$. But, for $\varepsilon>2\alpha$ let $\delta=\min(\alpha, \varepsilon^{-1})$. Then $(g-\alpha)\cap f\geqq\delta$ hence $\varphi((g-\alpha)\cap f)\geqq\delta\neq 0$, which is also a contradiction.

Therefore for any $f\in\varphi^{-1}(0)$, $|f|\in\varphi^{-1}(0)$, where $|f|(x)=|f(x)|$ for every $x\in X$. Accordingly we see that the subfamily $Z(\varphi^{-1}(0))$ is a maximal family of $Z(X)$ with respect to the finite intersection property, since for two $f, g\in\varphi^{-1}(0)$, $Z(f)\cap Z(g)=Z(|f|)\cap Z(|g|)=Z(|f|\cup|g|)$, $|f|\cup|g|\in\varphi^{-1}(0)$, and $Z(f-\alpha)\cap Z(f)=0$.

Finally we prove that $Z(\varphi^{-1}(0))$ satisfies the condition (c). Suppose on the contrary that there exists a countable subfamily $\{Z(g'_n)\}$ of $Z(\varphi^{-1}(0))$ without the total intersection. Setting $g_n=2^{-n}\cdot g'_n$, let $g=\Sigma_{n=1}^{\infty}g_n$. Then g is strictly positive, hence there exists an $f\in C(X, R)$ such that $g(x)\cdot f(x)=1$ for all $x\in X$. Then, as we have seen above, $g\notin\varphi^{-1}(0)$ and $f\notin\varphi^{-1}(0)$. Now let $\varphi(f)=\alpha>0$ and let n be the integer such that $2^n>\alpha$. Then $\varphi(f-\alpha)=0$, hence $(f-\alpha)\cup 0\in\varphi^{-1}(0)$. If $x\in Z((f-\alpha)\cap 0)$ $f(x)\leqq\alpha$, hence $g(x)\geqq\alpha^{-1}$. This implies that $\Sigma_{i=1}^{n}g_i(x)\neq 0$, i.e., that $Z(\Sigma_{i=1}^{n}g_i)\not\ni x$. Thus $Z(g_1)\cap\dots\cap Z(g_n)\cap Z((f-\alpha)\cup 0)\subset Z(\Sigma_{i=1}^{n}g_i)\cap Z((f-\alpha)\cup 0)=\phi$. This contradicts the finite intersection property of $Z(\varphi^{-1}(0))$.

Theorem 8. *Let X be e-complete and let φ be a homomorphism of the translation lattice $C(X, R)$ into the reals. Then there exist a point $x \in X$ and a real number α such that for any $f \in C(X, R)$*

$$\varphi(f) = f(x) + \alpha \ .$$

Proof. Supposing $\varphi(0) = \alpha$, let φ' be a homomorphism of the reals into itself such that $\varphi'(s) = s - \alpha$ for any real number s, and let $\varphi'' = \varphi' \cdot \varphi$. Then φ'' is also a homomorphism of $C(X, R)$ into the reals such that $\varphi''(0) = 0$. Hence by the above lemma $Z(\varphi''^{-1}(0))$ is a CZ-maximal family. Since X is e-complete, there exists a point x which is the total intersection of $Z(\varphi''^{-1}(0))$. Thus if for any $f \in C(X, R)$, $\varphi''(f) = \beta$, then $\varphi''(f - \beta) = 0$, i.e., $Z(f - \beta) \ni x$. Accordingly $\varphi''(f) = \beta = f(x)$, hence $\varphi(f) = f(z) + \alpha$.

Theorem 9. *Any two e-complete spaces X and Y are homeomorphic if and only if the translation lattice $C(X, R)$ is isomorphic to the translation lattice $C(Y, R)$.*

Proof. Let $T(X, R)$ be the set of all homomorphisms of the translation lattice $C(X, R)$ into the reals. Now we introduce a topology into the set $T(X, R)$ as follows. For a positive number ε and for a finite set $f_1, f_2, \dots, f_n$ of $C(X, R)$ let $U(\varphi; f_1, f_2, \dots, f_n; \varepsilon) = \{\varphi' \mid |\varphi'(f_i) - \varphi(f_i)| < \varepsilon$ for $i = 1, 2, \dots, n\}$ where $\varphi, \varphi' \in T(X, R)$ and let $\{U(\varphi; f_1 \dots f_n; \varepsilon)\}$ be a complete system of neighbourhood of φ in $T(X, R)$. Furthermore choosing an arbitrary function $f_0 \in C(X, R)$ let $X^*_{f_0}$ be the subspace $\{\varphi \mid \varphi(f_0) = 0\}$ of $T(X, R)$. Then we obtain by the usual methods that $X^*_{f_0}$ is homeomorphic to X, and since $X^*_{f_0}$ can expressed in terms of the translation lattice $C(X, R)$, the isomorphism between $C(X, R)$ and $C(Y, R)$ implies that X and Y are homeomorphic.

Corollary. *Let Φ be a lattice automorphism of $C(X, R)$ as the translation lattice. Then there exists a homeomorphism $\tilde{\Phi}$ of X onto itself and there exists a continuous function $h \in C(X, R)$ such that*

$$\Phi(f)(x) = f(\tilde{\Phi}(x)) + h(x) \, ,$$

where X is e-complete.

Proof. For a fixed f_1, let f_0 be $\Phi(f_1)$. Then for any $x \in X$ there exists exactly one homomorphism φ such that $\varphi(f) = f(x) - f_0(x)$. Then by the proof of Theorem 8 we can find exactly a point x' such that $(\varphi\Phi)(f) = f(x') - f_1(x')$. Then obviously if $\tilde{\Phi}(x) = x'$, $\tilde{\Phi}$ is a homeomorphism of X onto itself. Furthermore $(\varphi\Phi)f = \varphi(\Phi f)$, hence $f(\tilde{\Phi}(x)) - f_1(\tilde{\Phi}) = \Phi(f(x)) - f_0(x)$. Setting $h(x) = f_0(x) - f_1(\tilde{\Phi}(x))$ we have the corollary.

7. The lattice ordered group $C(X, R)$.

Theorem 10. *If for an e-complete space X, φ is a homomorphism of the lattice ordered group $C(X, R)$ onto the reals, then there exists a real number $\alpha \neq 0$ and a point x of X such that $\varphi(f)=\alpha f(x)$ for any $f \in C(X, R)$. Moreover if φ is a homomorphism of the ring $C(X, R)$ onto the reals, there exists a point x such that $\varphi(f)=f(x)$.*

Proof. We show first that $\varphi(1)\neq 0$. We assume on the contrary that $\varphi(1)=0$. Since φ is an onto mapping, there exists a $g \in C(X, R)$ such that $\varphi(g)\gneqq 0$. Let $h=|g|+1$. Then there exists a real number λ such that $\varphi(h^2)=\lambda\varphi(h)$. Obviously $\varphi(\lambda h)=\lambda\varphi(h)$. Hence $\varphi(h^2-\lambda h)=0$. Let $f=h^2-\lambda h$. Then $f \geqq f/h \geqq h-\lambda$, i.e., $f+\lambda \geqq h$. Therefore $\lambda(f+\lambda) \geqq \varphi(h)$, but $\varphi(f+\lambda)=\varphi(f)+\varphi(\lambda)=0+0=0$ since $\varphi(\lambda) \leqq \varphi(n)=n\varphi(1)=0$ for $n(>\lambda)$. Thus we have $\varphi(1)=\alpha \neq 0$.

Let φ' be a homomorphism of the reals onto itself such that $\varphi'(\mu)=\mu/\alpha$ and let $\varphi''=\varphi' \cdot \varphi$. Then $\varphi''(1)=1$ and hence for any real λ and for any $f \in C(X, R)$, $\varphi''(f+\lambda)=\varphi''(f)+\lambda$. Accordingly, φ'' is a homomorphism of the translation lattice $C(X, R)$ onto the reals such that $\varphi''(0)=0$. Hence there exists a point x of X such that $\varphi''(f)=f(x)$. Then obviously $\varphi(f)=\alpha f(x)$. In the case of a ring φ is obviously a homomorphism as the translation lattice and $\varphi(1)=1$ and $\varphi(0)=0$. Hence there exists a point x of X such that $\varphi(f)=f(x)$.

Corollary 1. *For an e-complete space X a proper subgroup M of the lattice ordered group $C(X, R)$ is a maximal ideal if and only if there exists a point x of X such that M consists of all functions satisfying the condition $f(x)=0$.*

Proof. If M is a maximal ideal there exists a homomorphism φ onto the reals such that $\varphi(1)=1$ and $\varphi^{-1}(0)=M$ since the facter group $C(X, R)/M$ is isomorphic to the reals. Hence by Theorem 10 there exists a point x of X such that $\varphi(f)=f(x)$, i.e., $f \in M$ if and only if $f(x)=0$. The sufficiency is obvious.

Corollary 2. *For a completely regular space X a positive function $g \in C(X, R)$ is strictly positive if and only if g is not contained in any maximal ideal of the lattice ordered group $C(X, R)$.*

Proof. If X is e-complete, it is obvious from the above corollary. Since $C(X, R)$ and $C(e(X), R)$ are algebraically isomorphic, we have this corollary in general as well.

Theorem 11. *An e-complete space X is determined by the lattice ordered group $C(X, R)$, and accordingly is determined by the ring $C(X, R)$.*

Proof. If $L(X, R)$ is the space of all homomorphisms of $C(X, R)$ onto the reals and if $X^*_{f_0}$ is the subset $\{\varphi \mid \varphi(f_0)=1\}$ for a fixed but arbitrary strictly positive function f_0 of $C(X, R)$, $X^*_{f_0}$ is homeomorphic to X by Theorem 10 and by the method used in the proof of Theorem 8.

Corollary. *Let Φ be a automorphism of $C(X, R)$ as lattice ordered group or ring. Then there exists a homeomorphism $\tilde{\Phi}$ of X onto itself and there exists a strictly positive function h such that $\Phi(f)(x) = h(x)f(\tilde{\Phi}(x))$ or $\Phi(f)(x) = f(\tilde{\Phi}(x))$ respectively.*

Finally we consider the representation of the vector lattice in a special type. Now the foregoing theorem shows that for a completely regular space X the strictly positive function is characterized by the following property: it is contained in no maximal ideals. Obviously for any vector lattice an Archimedian unit enjoys this property. We consider in the following the vector lattice with such strictly positive elements.

Theorem 12. *Let L be a vector lattice which enjoys the following conditions:*

i) *the intersection of the maximal ideals is* 0,

ii) *there exists an element e contained in no maximal ideals.*

Then there exists the unique e-complete space X_L satisfying the following conditions: L is embedded in $C(X_L, R)$ in such a manner that any point x and any closed set F not containing x are separated by some element of L, that L contains at least one strictly positive function of X_L and that any maximal ideal of L can be extended to a maximal ideal of $C(X_L, R)$.

Proof. Let X_L be the space whose points are maximal ideals of L and whose basis for open sets is $\{U(f)\}$ where $U(f) = \{M \mid \not\ni f\}$. Obviously $U(f) \frown U(g) = U(|f| \frown |g|)$ and $U(f) = U(|f|)$. Moreover if $M_1 \neq M_2$ then there exists a $f \in L$ such that $f \in M_1 - M_2$ hence $U(f) \not\ni M_1$ and $U(f) \ni M_2$. Therefore X is a T_1-space. Further let φ_M be the homomorphism of L onto the reals such that $f \in M$ if and only if $\varphi_M(f) = 0$ and such that $\varphi_M(e) = 1$. Then $f^*(M) = \varphi_M(f)$ is a continuous function of X_L. For let $f^*(M) = \alpha$ and let $g = |f - \alpha e| \frown \varepsilon e - \varepsilon e$. Then obviously $U(g) \ni M$ and if $M' \in U(g)$ $|f^*(M') - \alpha| < \varepsilon$. Thus for any f, f^* is a continuous function such that $f^*(M) = 0$ for $M \notin U(f)$ and $f^*(M) \neq 0$ for $M \in U(f)$. Hence X is completely regular and for any x and for any closed set F not containing x there exists an $f \in L$ such that f^* separates x and F. By the condition i) the correspondence: f to f^* implies the isomorphism between L and $L^* = \{f^* \mid f \in L\}$. Now X is e-complete. To see this we may assume that X_L is contained in the product space PR_{f^*} where $f^* \in L^*$. Then by the usual method[15] every limit point of X_L in PR_{f^*} corresponds

15) Cf. [18].

to the maximal ideal of L and every maximal ideal of L corresponds to a point of X_L. Hence X_L is closed in PR_{f^*}, i.e., X_L is e-complete. Moreover obviously any maximal ideal of L can be extended to a maximal ideal of $C(X_L, R)$. Thus we see that X_L satisfies the conditions of Theorem.

To prove the remainder of the theorem let Y be the e-complete space satisfying the conditions of Theorem. Now for any point $y \in Y$ there exists the unique maximal ideal M_y of $C(Y, R)$ such that $M_y = \{f \mid f(y)=0 \ \& \ f \in C(Y, R)\}$. Then $M_y \frown L$ is a maximal ideal $M(y)$ of L, because, since L contains at least one strictly positive element, $M \frown L \neq L$. Thus by the corresdondence: y to $M(y)$, we have a mapping h of Y into X_L. Conversely for any point $M \in X_L$, considering $M \subset L \subset C(Y, R)$ M can be extended to a maximal ideal M' of $C(Y, R)$ and Y is e-complete, hence there exists a point y such that $M_y = M'$. Then $M = M(y)$. Thus h is onto and obviously h is one-to-one. Finally we show that h is homeomorphic. If $U(y)$ is a neighbourhood of y there exists an $f \in L$ such that $f(y) \neq 0$ and $f(x)=0$ for $x \notin U(y)$. Then $U(f) \ni M(y)$ and if $h(y')=M(y') \in U(f)$, $f(y') \neq 0$, i.e., $y' \in U(y)$. Conversely let $h(y) \in U(f)$ and let $U(y)=\{y' \mid f(y') \neq 0\}$, then if $y' \in U(y)$, $M(y') \not\ni f$, i.e. $h(y') \in U(f)$. Thus the proof is complete.

Remark 1. If the last condition for X_L is omitted in Theorem 12, then any completely regular space satisfying the first three conditions is topologically embedded in X_L as a dense subset.

Moreover if L is the lattice ordered additive group satisfying the conditions i) and ii) of Theorem 12 and if L contains the image of an isomorphism of the reals where the unit 1 goes to the element e of the condition ii), L can be represented as in Theorem 12.

Remark 2. If the element e in the condition ii) of Theorem 12 is an Archimedian unit, then X is bicompact and in this case our theorem coincides with Yosida's Theorem[16].

Remark 3. Recently K. Fan[17] has introduced the concept of the direct extension of the partially ordered additive group and has characterized the partially ordered additive group of (bounded) continuous functions on a bicompact space. By the same method he used and by Corollary of Theorem 10 we can characterize the lattice ordered additive group which is a power of the reals as follow: For a lattice ordered additive group L, if the potency of the set of all elements of L is weakly accessible from $\aleph_0$, it is a power of the reals if and only if it satisfies

16) Cf. [17].
17) Cf. [21].

the following conditions:

i) the intersection of the maximal ideals is 0,

ii) if L' is an extension of L such that there exists the one-to-one correspondence between maximal ideals of L and L' by the relation of inclusion, then $L=L'$.

(Received December 1, 1951)

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Added in proof: While in proof reading I had access to a paper of M. Katětov: Measures in fully normal spaces. Fund. Math. 38 (1951). His theorem 3 follows immediately from ours.