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Ergodic Skew Product Transformations on the Torus

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§ 1. Introduction

It is the purpose of this paper to give examples of ergodic transformations of some special types and to discuss their properties. We begin with the definition of *skew product measure preserving transformations*. Let φ be a measure preserving transformation on a measure space X. Let Y be another measure space, and let us assume that to every point x of the space X, there corresponds a measure preserving transformation ψ_x on Y. Let Ω be the direct product measure space of X and Y:

$$\Omega = X \times Y, \quad \omega = (x, y), \quad \omega \in \Omega, \quad x \in X, \quad y \in Y.$$

Denote the measures on X, Y and Ω by m, μ and ν respectively, ν is the completed direct product measure of m and μ.

If the family of measure preserving transformations $\{\psi_x | x \in X\}$ satisfies certain measurability conditions, it is easy to see that the transformation T which is defind by

$$T(x, y) = (\varphi x, \psi_x y)$$

is a measure preserving transformation on Ω. Then T is called a *skew product measure preserving transformation*. In case the family of transformations $\{\psi_x\}$ consists of the same transformation ψ, the skew product transformation T is the direct product transformation of φ and ψ.

In this paper we assume that φ is an *ergodic* measure preserving transformation on a measure space X, and that Y is the usual Lebesgue measure space of the set of real numbers mod 1, which will be called simply *circle*.

Let A be the set of all Y-valued measurable functions on X. To any $\alpha(x)$ belonging to A we may assign a one-to-one mapping T on Ω in the following way:

$$T(x, y) = (\varphi x, \alpha(x) + y).$$

The function $\alpha(x)+y$ is a Y-valued measurable function on Ω, that is, for any Borel set B in Y, the set $\{(x, y)|\alpha(x)+y \in B\}$ is ν-measurable and by Fubini's theorem the following equality holds:

$$\nu\left\{(x, y)|\alpha(x)+y \in B\right\} = \int \mu\left\{y|y \in B-\alpha(x)\right\} m(dx).$$

If N is a Borel set of μ-measure zero we have the identity: $\mu\{y|y \in N-\alpha(x)\}=0$ for all x. This means $\nu\{(x, y)|\alpha(x)+y \in N\}=0$. This implies that for any μ-measurable set L, the set $\{(x, y)|\alpha(x)+y \in L\}$ is ν-measurable, therefore for any complex-valued μ-measurable function $g(y)$, $g(\alpha(x)+y)$ is a ν-measurable function. Hence we get for any $f(x) \in L_1(X)$, $g(y) \in L_1(Y)$, the following equality:

$$\iint f(\varphi x)\, g(\alpha(x)+y)\, dxdy = \int f(\varphi x)\left\{\int g(\alpha(x)+y)\, dy\right\} dx$$
$$= \int f(\varphi x)\left\{\int g(y)\, dy\right\} dx = \iint f(x)\, g(y)\, dx\, dy.$$

On the other hand it is shown in the same way that $g(-\alpha(\varphi^{-1}x)+y)$[1] is also ν-measurable and that the following equality holds:

$$\iint f(\varphi^{-1}x)\, g(-\alpha(\varphi^{-1}x)+y)\, dx\, dy = \iint f(x)\, g(y)\, dx\, dy.$$

Thus the transformation T is proved to be a measure preserving transformation on Ω, T is called *skew product transformation with the α-function* $\alpha(x)$.

Except in § 2 we treat the case in which the space X is also a circle, therefore the product space Ω is a two-dimensional torus, and the transformation φ is a translation by some irrational number γ mod 1.

The author is much indebted to Professor S. Kakutani for his kind discussions on the whole subjects of this paper, especially we owe him the essential simplification of the proof of Theorem 1 and 2. Further he taught the author that Professor J. von Neumann had proved the following theorem: The ergodic transformation $(x, y) \rightarrow (x+\gamma, x+y)$ on the torus is spectrally isomorphic to the direct product transformation of the translation $x \rightarrow x+\gamma$ on the circle and the shift-transformation on the infinite dimensional torus[2], though these transformations are not spatially isomorphic to each other. This fact has been the stimulation in obtaining the results of § 6.

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1) $(x, y) \rightarrow (\varphi^{-1}x, -\alpha(\varphi^{-1}x)+y)$ is the inverse mapping of $(x, y) \rightarrow (\varphi x, \alpha(x)+y)$.

2) Infinite dimensional torus means the infinite direct product measure space of circles.

§ 2. Proper values and ergodicity

We denote by Ξ the submodule of A, whose elements $\xi(x)\in\Xi$ are of the form $\xi(x)=\theta(x)-\theta(\varphi x)$ for some $\theta(x)\in A$.

Remark. If $\alpha(x)=\alpha'(x)$ holds for almost all x, the corresponding skew product transformations T and T' differ only on a null set, that is $\{\omega\,|\,T\omega\neq T'\omega\}$ is a null set in Ω. Therefore we regard the two α-functions as the same if they differ only on a null set. By the same reason if $\xi(x)=\theta(x)-\theta(\varphi x)$ for almost all x, for some $\theta(x)\in A$, we denote $\xi(x)\in\Xi$. Throughout this paper any equality between functions should be taken as the equality for *almost all* values of the variable.

Theorem 1. *Let T be a skew product transformation with an α-function $\alpha(x)$. T has the proper value λ if and only if $p\alpha(x)-\lambda\in\Xi$ for some integer p.*

Theorem 2. *T is ergodic if and only if $p\alpha(x)$ never belongs to Ξ unless $p\neq 0$.*

We begin with the proof of Theorem 1. Let $f(x, y)$ be a proper function belonging to the proper value λ. Then we have

$$f(T(x, y)) = f(\varphi x, \alpha(x)+y) = e^{2\pi i\lambda} f(x, y). \tag{1}$$

From (1) it follows

$$\begin{aligned}\int f(\varphi x, \alpha(x)+y)\exp(-2\pi ipy)\,dy \\ = e^{2\pi i\lambda}\int f(x, y)\exp(-2\pi ipy)\,dy.\end{aligned} \tag{2}$$

Put

$$f_p(x) = \int f(x, y)\exp(-2\pi ipy)\,dy. \tag{3}$$

From (2) and (3) we have

$$f_p(\varphi x)\exp(2\pi ip\alpha(x)) = e^{2\pi i\lambda} f_p(x). \tag{4}$$

Take the absolute value of both sides of (4),

$$|f_p(\varphi x)| = |f_p(x)|. \tag{5}$$

The ergodicity of φ implies that (5) is a non-negative constant c_p. If $c_p\neq 0$ there exists a function $\theta_p(x)\in A$ such that

$$f_p(x) = c_p\exp(2\pi i\theta_p(x)). \tag{6}$$

Since $f(x, y)$ is not identically zero, there exists at least an integer p, for which $c_p\neq 0$. Let us assume that p is such an integer. Replacing

$f_p(x)$ in (4) by (6) we have

$$(7) \qquad \theta_p(\varphi x)+p\alpha(x) = \lambda+\theta_p(x),$$

that is

$$(7') \qquad p\alpha(x)-\lambda = \theta_p(x)-\theta_p(\varphi x).$$

Conversely if (7′) holds, $\exp\{2\pi i(\theta_p(x)+py)\}$ is a proper function to the proper value λ. This completes the proof of Theorem 1.

Suppose that T is not ergodic. Then there exists an invariant function $f(x, y)$ which is not a constant:

$$(8) \qquad f(\varphi x, \alpha(x)+y) = f(x, y).$$

Then by the definition of $f_p(x)$ in (3) we have

$$(9) \qquad f_p(\varphi x)\exp(2\pi ip\alpha(x)) = f_p(x).$$

For $p=0$ in (9) we have $f_0(\varphi x)=f_0(x)$. This equality shows that $f_0(x)$ is a constant because of the ergodicity of φ, therefore there must exist an integer $p\neq 0$ for which $f_p(x)$ does not vanish identically. For this p repeating the same argument as in the proof of Theorem 1 we can find a function $\theta(x)\in A$ such that $p\alpha(x)=\theta(x)-\theta(\varphi x)$ holds.

Conversely if $p\alpha(x)=\theta(x)-\theta(\varphi x)$ holds for some $p\neq 0$, then by (7), we see that $\exp\{2\pi i(\theta(x)+py)\}$ is an invariant function, which is not a constant since $p\neq 0$. Hence T is not ergodic.

§ 3. Isomorphism between ergodic skew product transformations

From now on we assume further that X is also a circle and φ is the translation (rotation) by an irrational number γ. Accordingly $\Omega=X\times Y$ is a two-dimensional torus.

Theorem 3. *Let T and S be ergodic skew product transformations with α-functions $\alpha(x)$ and $\beta(x)$ respectively. If T and S are spatially isomorphic, that is, if there exists a measure preserving transformation V of Ω onto itself such that $VTV^{-1}=S$, then between $\alpha(x)$ and $\beta(x)$ there exists the following relation,*

$$\alpha(x)-\beta(x+u)\in\Xi \quad \text{or} \quad \alpha(x)+\beta(x+u)\in\Xi,$$

where u is an element of X. And accordingly V is of the following form:

$$V(x, y) = (x+u, \theta(x)+y) \quad \text{or} \quad V(x, y) = (x+u, \theta(x)-y).$$

Conversely if

$$\alpha(x)-\beta(x+u) = \theta(x)-\theta(x+\gamma)$$

holds for some $u \in X$ *and* $\theta(x) \in A$, *then* $VTV^{-1} = S$ *holds, where* $V(x, y) = (x+u,\ \theta(x)+y)$;
and if

$$\alpha(x)+\beta(x+u) = \theta(x+\gamma)-\theta(x)$$

holds for some $u \in X$ *and* $\theta(x) \in A$, *then* $VTV^{-1} = S$ *holds, where* $V(x,y) = (x+u,\ \theta(x)-y)$.

If $\alpha(x)-\beta(x+u)$ or $\alpha(x)+\beta(x+u)$ belongs to Ξ for some $u \in X$, $\alpha(x)$ and $\beta(x)$ are called *equivalent*.

Let $g(x, y)$ and $h(x, y)$ be the X- and Y-coordinate of $V(x, y)$:

$$V(x, y) = (g(x, y),\ h(x, y)). \tag{10}$$

Then $g(x, y)$ and $h(x, y)$ are X- and Y-valued measurable functions on Ω. Suppose $VTV^{-1} = S$. Then

$$\begin{aligned} VT(x, y) &= V(x+\gamma,\ y+\alpha(x)) \\ &= ((g(x+\gamma,\ y+\alpha(x)),\ h(x+\gamma,\ y+\alpha(x))) \end{aligned} \tag{11}$$

is equal to

$$\begin{aligned} SV(x, y) &= S(g(x, y),\ h(x, y)) \\ &= (g(x, y)+\gamma,\ h(x, y)+\beta(g(x, y))). \end{aligned} \tag{12}$$

In comparing the X- and Y-coordinates of (11) and (12) we have

$$g(x+\gamma,\ y+\alpha(x)) = g(x, y)+\gamma, \tag{13}$$

$$h(x+\gamma,\ y+\alpha(x)) = h(x, y)+\beta(g(x, y)). \tag{14}$$

Since $g(x, y)$ and $h(x, y)$ are quantities on the circle, $g^*(x, y) = \exp\{2\pi i g(x, y)\}$ and $h^*(x, y) = \exp\{2\pi i h(x, y)\}$ are usual complex-valued measurable functions on Ω. From (13) it follows

$$g^*(T(x, y)) = e^{2\pi i\gamma}\, g^*(x, y). \tag{15}$$

This shows that $g(x, y)$ is a proper function to the proper value γ. The function $\exp(2\pi i x)$ is also a proper function to the proper value γ, the ergodicity of T implies

$$g(x, y) = c\exp(2\pi i x), \tag{16}$$

where c is a constant different from zero. If we denote by u the amplitude of c we have from (16)

$$g(x, y) = x+u. \tag{17}$$

Putting (17) in (14) we have

$$h(x+\gamma,\ y+\alpha(x)) = h(x, y)+\beta(x+u). \tag{18}$$

Put

$$h_p(x) = \int h^*(x, y) \exp(-2\pi ipy)\, dy. \tag{19}$$

From (18) and (19) it follows

$$h_p(x+\gamma) \exp\{2\pi ip\alpha(x)\} = h_p(x) \exp\{2\pi i\beta(x+u)\}. \tag{20}$$

Taking the absolute value of (20) we have the equality: $|h_p(x+\gamma)| = |h_p(x)|$, which is a constant and does not vanish identically for some integer p, because $h^*(x, y)$ is not identically zero. For this integer p, there exists a function $\theta(x) \in A$ such that

$$h_p(x) = c\{\exp 2\pi i\theta(x)\}, \tag{21}$$

where c is a positive constant.

Putting (21) into (20) we have

$$\theta(x+\gamma) + p\alpha(x) = \theta(x) + \beta(x+u): \tag{22}$$

If $h_p(x)$ is not identically zero, then $h_q(x)$ must vanish for all $q \neq p$. For otherwise if for some $q \neq p$ $h_q(x)$ does not vanish we obtain the following equality (23) just as we obtained (22), and this leads to a contradiction as follows.

$$\theta'(x+\gamma) + q\alpha(x) = \theta'(x) + \beta(x+u) \tag{23}$$

for some $\theta'(x) \in A$.

Substract (23) from (22). Then we have $(p-q)\alpha(x) \in \Xi$, and $p-q \neq 0$. According to Theorem 2 this contradicts the assumption that T is ergodic. Hence we obtain

$$h^*(x, y) = c \exp\{2\pi i(\theta(x) + py)\}. \tag{24}$$

This implies

$$h(x, y) = \theta(x) + py \tag{25}$$

and

$$V(x, y) = (x+u,\ \theta(x) + py). \tag{26}$$

Since V is a measure preserving transformation on Ω, for almost all x, $\theta(x) + py$ must be a measure preserving transformation on Y. This is valid only if $p = 1$ or $p = -1$. In case $p = 1$ we get from (22)

$$\alpha(x) - \beta(x+u) = \theta(x) - \theta(x+\gamma), \tag{27}$$

and

$$V(x, y) = (x+u,\ \theta(x) + y).$$

In case $p=-1$ we have similarly

$$\alpha(x)+\beta(x+u) = \theta(x+\gamma)-\theta(x) \tag{28}$$

and

$$V(x, y) = (x+u,\ \theta(x)-y).$$

The converse of the proposition is evident.

§ 4. Spectral property of skew product transformations

Let H be the Hilbert space of all functions belonging to $L_2(\Omega)$. Let U be the unitary operator on H which corresponds to the skew product transformation T with the α-function $\alpha(x)$:

$$Uf(x, y) = f(T(x, y)) = f(x+\gamma,\ y+\alpha(x)), \tag{29}$$

where $f(x, y) \in H$.

Unitary-invariant properties of the unitary operator U are called *spectral properties* of the measure preserving transformation T. Two measure preserving transformations T and S are called *spectrally isomorphic* if the corresponding unitary operators are *unitary-equivalent* [3].

Since Ω is the direct product measure space of X and Y, the set of functions $\{\psi_{p,q}(x, y)\}$:

$$\psi_{p,q}(x, y) = \exp\{2\pi i(px+py)\}, \text{ where } p,\ q=0,\ \pm 1,\ \pm 2, \ldots, \tag{30}$$

form a complete orthonormal system of H. Let H_q be the closed linear subspace of H which is spanned by $\{\psi_{p,q}\}$ for fixed q and $p=0$, ± 1, $\pm 2, \ldots$.

It is clear that H is decomposed into the direct sum of H_q ($q=0$, ± 1, $\pm 2, \ldots$) which are mutually orthogonal and that each H_q is invariant under the unitary operator U: $H=\sum_{q=-\infty}^{\infty} \oplus H_q$. H_q is the set of functions of the form $f(x)\exp(2\pi iqy)$, where $f(x) \in L_2(X)$. Especially H_0 is the set of functions depending only on the value of the X-coordinate. The unitary operator U on H_0 is evidently isomorphic [4] to the unitary operator on $L_2(X)$ which corresponds to the translation by γ on X. We shall denote by $H_0^{\perp}$ the orthocomplement of H_0: $H_0^{\perp}=\sum_{q\neq 0} \oplus H_q$.

It is in $H_0^{\perp}$ where spectral properties of T are to be discussed in connection with the behaviours of the α-function $\alpha(x)$. The property

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3) Unitary operators U and V are called unitary-equivalent if there exists a unitary operator W such that $V=WUW^{-1}$.

4) Here "isomorphic" means "unitary-equivalent".

of U will be completely determined if we know the behaviour of $(U^n\psi_{p,q}, \psi_{p',q})$ as a function of n for every p, p', and q, and which is expressed in the following formula:

$$(31)\qquad (U^n\psi_{p,q}, \psi_{p',q}) = \iint \exp\Big[2\pi i\big\{p\,(x+n\gamma)+q\,(\alpha(x)+\ldots$$
$$+\alpha(x+(n-1)\gamma)+y)\big\}\Big]\exp\big\{-2\pi i\,(p'x+qy)\big\}\,dxdy$$
$$= \exp(2\pi ipn\gamma)\int \exp\Big[2\pi i\big\{(p-p')x+q\,(\alpha(x)+\ldots+\alpha(x+(n-1)\gamma)\big\}\Big]\,dx.$$

§ 5. Point spectrum

Lemma 1. *If a constant function $\alpha(x)=\lambda$ belongs to Ξ, then λ is a multiple of γ.*

From $\lambda=\theta(x)-\theta(x+\gamma)$ for some $\theta(x)\in A$ it follows

$$(32)\qquad \exp\big\{2\pi i\,\theta(x+\gamma)\big\} = e^{-2\pi i\lambda}\exp\big\{2\pi i\,\theta(x)\big\}\,.$$

Therefore $-\lambda$ is a proper value of the translation by γ with the proper function $\exp\{2\pi i\theta(x)\}$. This implies that λ is a multiple of γ.

Theorem 4. *Let Λ be the set of proper values of an ergodic skew product transformation T. Then Λ is an additive group with at most two generators.*

Let Λ^* be the set of integers q for which $q\alpha(x)-\lambda\in\Xi$ holds for some $\lambda\in\Lambda$, where $\alpha(x)$ is the α-function of T:

$$(33)\qquad \Lambda^* = \big\{q\,|\,q\alpha(x)-\lambda\in\Xi \text{ for some } \lambda\in\Lambda\big\}.$$

From the fact that Λ is a subgroup of Y, it is easily verified that Λ^* is a subgroup of the additive group of integers. Consequently Λ^* is a cyclic group with a generator p and to this p there exists an element ρ of Λ such that

$$(34)\qquad p\alpha(x)-\rho\in\Xi\,.$$

According to Theorem 1, for any $\lambda\in\Lambda$, there exists a $q\in\Lambda^*$ such that

$$(35)\qquad q\alpha(x)-\lambda\in\Xi\,.$$

Let n be the quotient of q by p: $q=np$. Multiple (34) by n, and subtract it from (35), then we get $n\rho-\lambda\in\Xi$ which is seen to be a multiple of γ by Lemma 1. Therefore there exists an integer m such that $n\rho-\lambda=m\gamma$, thus ρ and γ are seen to be generators of Λ.

Theorem 5. *An ergodic skew product transformation T with an α-function $\alpha(x)$ has pure point spectrum if and only if $\alpha(x)$ is equivalent with a constant function λ, where λ is an irrational number* *linearly independent of* γ.

Let U be the unitary operator on H which corresponds to T. U on H_0 has always pure point spectrum, the proper values being multiples of γ. If U on H has pure point spectrum, U on $H_0^{\perp}$ must have pure point spectrum, and the proper values must be linearly independent of γ, since T is assumed to be ergodic[5]. By Theorem 4 the additive group of proper values of U has only two generators, one of them is γ, let the other be λ, then the set of proper values of U on $H_0^{\perp}$ is the cyclic group with the generator λ. Since λ is a proper value of T, by Theorem 1 there exists an integer p and a function $\theta(x) \in A$ such that

$$(36) \qquad p\alpha(x) - \lambda = \theta(x) - \theta(x+\gamma).$$

Therefore for any integer n,

$$(37) \qquad np\alpha(x) - n\lambda = n(\theta(x) - \theta(x+\gamma))$$

holds. Then it is easily verified that

$$(38) \qquad \exp\{2\pi in(\theta(x) + py)\}$$

is a proper function belonging to the proper value $n\lambda$. Let M be the closed linear subspace of $H_0^{\perp}$ which is spanned by the set of functions of the form of (38) for $n = \pm 1, \pm 2, \ldots$. Since U has pure continuous spectrum on the orthocomplement of M with respect to $H_0^{\perp}$, $H_0^{\perp}$ must coincide with M, this fact implies that $p = \pm 1$. This means by (36) that $\alpha(x)$ is equivalent with the constant λ. Conversely if $\alpha(x)$ is equivalent with a constant λ: $\alpha(x) - \lambda \in \Xi$, then by Theorem 3 T is isomorphic to the direct product transformation of the translation by γ on X and the translation by λ on Y.

§ 6. Discussion of the case $\alpha(x) = mx$ (strongly mixing case on $H_0^{\perp}$)

In this § we use the notations in § 4.

Theorem 6. *Let Ω' be the infinite dimensional torus,*[6] *and S be the usual shift transformation on Ω'. Let V be the unitary operator on $L_2(\Omega')$ corresponding to S. Let C_0 be the one-dimensional subspace of $L_2(\Omega')$ consisting of constant functions. Let us denote by M the orthocomplement of C_0 in $L_2(\Omega')$. On the other hand let T_m be the skew product transformation with the α-function $\alpha(x) = mx$ respectively,*

5) The point spectrum of an ergodic transformation must be simple.

6) See the footnote 2)

where m is an arbitrary integer different from zero. Let U_m be the unitary operator corresponding to T_m. Then for every m, U_m on $H_0^{\perp}$ and V on M are isomorphic to each other.

By the definition of U_m and $\psi_{p,q}$ we have the equality:

$$U_m\psi_{p,q} = \psi_{p,q}(x+\gamma,\ y+mx) = e^{2\pi i p\gamma}\,\psi_{p+mq,\,q}\,. \tag{39}$$

Let $\{\gamma_{p,q}\}$ $(p,\ q=0,\ \pm 1,\ \pm 2,\ldots)$ be a system of constants satisfying the following relations:

$$|\gamma_{p,q}| = 1\,, \quad \gamma_{p+mq,\,q} = e^{2\pi i p\gamma}\,\gamma_{p,q} \tag{40}$$

Put

$$\psi'_{p,q} = \gamma_{p,q}\psi_{p,q} \tag{41}$$

then we have

$$U_m\psi'_{p,q} = \psi'_{p+mq,\,q} \tag{42}$$

For any integer k such that $1\leq k\leq mq$ let $M_k^{(p)}$ be the subspace of H_q, which is spanned by $\{\psi_{p,q}\}$, where p runs through every integer congruent to k modulo mq. Obviously $M_k^{(q)}$ is spanned by $\{\psi'_{p,q}\}$, where $p\equiv k\ (mod\ mq)$. Since $H_q = \sum_{k\ (mod\ mq)}\oplus M_k^{(q)}$, we have

$$H_0^{\perp} = \sum_{q\neq 0}\ \sum_{k\ (mod\ mq)}\oplus M_k^{(q)}\,. \tag{43}$$

Every $M_k^{(q)}$ is infinite dimensional and in each of them there exists a complete orthonormal system $\{\psi'_{p,q}\}$, $(p\equiv k\ (mod\ mq))$, which is transitive under U_m by (42). This fact tells that U_m on $H_0^{\perp}$ and V on M are isomorphic to each other.

Corollary. *For any integer m different from zero, T_m is an ergodic transformation which has pure continuous spectrum on $H_0^{\perp}$, and every spectrum on $H_0^{\perp}$ is absolutely continuous.*

Theorem 7. *Let T_m $(m=1,2,\ldots)$ be the skew product transformations defined above. They are mutually spectrally isomorphic but not spatially isomorphic.*

We have proved in Theorem 6 that all U'_ms are isomorphic to each other on $H_0^{\perp}$. Since they are clearly isomorphic to each other on H_0, they are isomorphic on the whole space $H=H_0\oplus H_0^{\perp}$. Namely all T'_ms are spectrally isomorphic to each other. But for $m\neq n$ $(m,\ n>0)$ T_m and T_n are not spatially isomorphic. This is because if they were spatially isomorphic, then by Theorem 3 $mx \pm n(x+u) \in \Xi$ for some $u \in X$. This means that $(m \pm n)x - nu \in \Xi$, therefore the α-function $(m \pm n)x$ is equivalent with a constant nu. From the preceding corollary we see that this is a contradiction.

§ 7. Discussion of the case when $\alpha(x)$ takes two different values (weakly mixing case in $H_0^{\perp}$)

In this § we discuss spectral properties of a skew product transformation T with the α-function $\alpha(x) = \rho c_E(x)$, where ρ is an irrational number and E is an interval on X such that $0 < m(E) < 1$, $\rho c_E(x)$ [7] is defined as follows:

$$\rho c_E(x) = \begin{cases} \rho & \text{if} \quad x \in E \\ 0 & \text{if} \quad x \in E. \end{cases} \tag{44}$$

It will be shown that there appears *singular continuous spectrum* for some ergodic skew product transformations of the above type. This fact is to be compared with the result that the transformations discussed in § 6 have pure absolutely continuous spectrum on $H_0^{\perp}$.

For this purpose we need some preliminary considerations on the density of the set of all points of the form $s\gamma$ on X where s is an integer such that $1 \leq s \leq N$.

We may regard the irrational number γ on X as a real irrational number between 0 and 1. Put $\delta_0 = 1$, $\delta_1 = \gamma$, and continue the division-process as follows:

$$\delta_0 = k_1\delta_1 + \delta_2, \quad \delta_1 = k_2\delta_2 + \delta_3, \dots, \quad \delta_{n-1} = k_n\delta_n + \delta_{n+1}, \dots . \tag{45}$$

where k_n is a non-negative integer and $0 < \delta_{n+1} < \delta_n$ for any positive integer n. Namely the irrational number γ is expressed in the form of the continued fraction:

$$\gamma = \cfrac{1}{k_1 + \cfrac{1}{k_2 + \ddots + \cfrac{1}{k + \ddots}}} \tag{46}$$

Let $\{p_n\}$ be the sequence of integers defined by the following equality

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7) As a real-valued function $C_E(x)$ is the usual characteristic function of the interval E, but as a Y-valued function $C_E(x)$ is a constant. Here "$\rho C_E(x)$ as a Y-valued function" is to be considered as a real-valued function $\rho C_E(x)$ modulo 1.

(47) $$p_n = k_n p_{n-1} + p_{n-2}, \qquad n = 1, 2, \ldots.$$

It is evident that p_n is a function of $k_1, k_2, \ldots, k_n$, we use as usual the following notation:

(48) $$p_n = [k_1, k_2, \ldots, k_n].$$

It is easily verified that the following equality holds.

(49) $$\delta_0 = p_n\delta_n + p_{n-1}\delta_{n+1}.$$

Put $m_n = p_n + p_{n-1}$, and let M_n be the set of all points of the form $s\gamma$ on X, where s is an integer such that $0 \leqq s \leqq m_n - 1$. Let N_n be the set of intervals whose end point belong to M_n and whose inner points never belong to M_n. If we represent intervals by their length we have successively the following schema:

$$N_1: \quad \underbrace{\delta_1, \delta_1, \ldots, \delta_1}_{k_1}\ \delta_2$$

$$N_2: \quad \underbrace{\delta_3\ \underbrace{\delta_2 \ldots \delta_2}_{k_2}\ \ \delta_3\ \underbrace{\delta_2 \ldots \delta_2}_{k_2}\ \ \ldots\ \ \delta_3\ \underbrace{\delta_2 \ldots \delta_2}_{k_2}}_{k_1-1}\ \ \delta_3\ \underbrace{\delta_2 \ldots \delta_2}_{k_2+1}$$

$$N_3: \quad \underbrace{\delta_3 \ldots \delta_3}_{k_3+1}\ \delta_4\ \ \underbrace{\underbrace{\delta_3 \ldots \delta_3}_{k_3}\ \delta_4\ \ \ldots\ \ \underbrace{\delta_3 \ldots \delta_3}_{k_3}}_{k_2-1}\ \delta_4\ \ —\ —$$

$$—\ \ \underbrace{\delta_3 \ldots \delta_3}_{k_3+1}\ \delta_4\ \ \underbrace{\underbrace{\delta_3 \ldots \delta_3}_{k_3}\ \delta_4\ \ \ldots\ \ \underbrace{\delta_3 \ldots \delta_3}_{k_3}}_{k_2}\ \delta_4$$

By induction we obtain the following:

Lemma 2. *N_n consists of intervals of length δ_n and δ_{n+1}, and in N_n every δ_{n+1} is isolated and the number of successive δ_n's is k_n or k_n+1.*

Now the following principal lemma is to be proved.

Lemma 3. *Let s be any integer between 1 and m_n, let Γ be a chain of s successive intervals belonging to N_n, and let Γ' be another chain of s successive intervals belonging to N_n. Let us denote by $N(\Gamma)$ the number of δ_{n+1}'s contained in Γ. Then $|N(\Gamma) - N(\Gamma')| \leqq 1$.*

Proof. Γ may be considered as a chain of the letters δ_n's and δ_{n+1}'s placed on the circle. If we shift Γ step by step in the definite direction Γ is finally brought to Γ' after some steps. Suppose that Γ loses a δ_{n+1} at the tail while gaining a δ_n at the head once during this shift. Then it is impossible that the tail of Γ loses another δ_{n+1} before the head of Γ gains a δ_{n+1}, because this is contradictory to Lemma 2. Nevertheless it is possible that when the head gains a δ_{n+1} the tail loses a δ_{n+1} simultaneously, and the same situation takes place several times. But at the instant when this situation breaks the head must gain a δ_{n+1} while the tail must lose a δ_n. This is because otherwise there exists a certain integer $m<n$ such that Lemma 2 is not true for N_m.

The above argument implies that if $N(\Gamma)$ decreases by 1 at a certain moment during the shift, the next instant when the value of $N(\Gamma)$ changes it must increase by 1, and vice versa. This completes the proof of the lemma.

Lemma 4. *Let us denote by $\nu_n(E)$ the number of the points of M_n which fall on a closed interval E on X. If the length of the closed interval E is equal to that of a closed interval E', then $|\nu_n(E)-\nu_n(E')|\leqq 3$ holds for every positive integer n.*

Proof. Suppose that

(50) $$\nu_n(E') \geqq \nu_n(E)+4,$$

while the length of E is equal to that of E'. Let p and q be the points which do not belong to E and which are nearest to the left and the right end points of E, respectively. Let E_1 be the interval $[p, q]$, then we have

(51) $$\nu_n(E_1) = \nu_n(E)+2 .$$

Let p' be a point of $E' \frown M_n$ which is nearest to one of the end points of E', and let q' be the point of $E' \frown M_n$ such that

(52) $$\nu_n(E_1') = \nu_n(E_1)$$

where $E_1' = [p', q']$.

Then we obtain from (50), (51) and (52)

(53) $$\nu_n(E_1') \leqq \nu_n(E')-2$$

By (52) and by the fact that the length of E_1' is smaller than the length of E_1, if the number of δ_{n+1}'s contained in E_1 is t the number of δ_n's contained in E_1' is $t+1$, and if the number of δ_n's contained in E_1 is s the number of δ_n's contained in E_1' is $s-1$. Since by Lemma 2 δ_{n+1} is isolated we may conclude from (53) that $m(E'-E_1') \geqq \delta_n+\delta_{n+1}$. Hence we get the following inequalities, where l is the length of E and E':

(54) $$s\delta_n+t\delta_{n+1} > l$$

(55) $$(s-1)\delta_n+(t+1)\delta_{n+1}+\delta_n+\delta_{n+1} \leq l.$$

(54) and (55) are not consistent, therefore the assumption (50) has been proved to be false.

Theorem 8. *Let ρ and γ be arbitrary irrational numbers and let E be an arbitrary interval on X such that $0<m(E)<1$. Then non-absolutely-continuous spectrum appears in $H_0^{\perp}$ for the skew product transformation T with the α-function $\alpha(x)=\rho C_E(x)$.*

Proof. Let q be a positive integer such that

(56) $$|q\rho| < \frac{1}{6} \pmod 1.$$

Let us put

(57) $$f(x, y) = \exp(2\pi iqy),$$

this function belongs to H_q. Let U be the unitary operator corresponding to T. Then we have

(58) $$U^{m_n} f(x, y) = \exp(2\pi iqy)\exp\left\{2\pi iq\rho \sum_{j=0}^{m_n-1} c_E(x+j\gamma)\right\}.$$

It follows

(59) $$(U^{m_n} f, f) = \int \exp\left\{2\pi iq\rho \sum_{j=0}^{m_n-1} c_E(x+j\gamma)\right\} dx.$$

It is obvious that

(60) $$\sum_{j=0}^{m_n-1} c_E(x+j\gamma) = \nu_n(E-x).$$

It follows from Lemma 4 that the function $\nu_n(E-x)$ can take at most four different values. From this fact and (56) we see that all values of $\exp\{2\pi iq\rho\nu_n(E-x)\}$ lie on a one side of the plane with respect to a certian line through the origin. Hence it is impossible that

(61) $$\lim_{n\to\infty} (U^{m_n} f, f) = 0$$

holds. But if contrary to the statement of the theorem all spectra of T are absolutely continuous on $H_0^{\perp}$ the equality (61) must hold. The proof of the theorem is thus completed.

Theorem 9. *There exist an irrational number γ and an interval E such that the skew product transformation T with the α-function $\alpha(x)=\rho c_E(x)$ is ergodic and has pure continuous spectrum on $H_0^{\perp}$.*

Proof. Let $\{k_n\}$ be a sequence of positive even numbers satisfying the following conditions:

$$\lim_{n\to\infty} k_n = \infty, \tag{62}$$

$$\lim_{n\to\infty} \frac{[k_1 k_2, \dots, k_{n-1}]}{k_{n+1}} = 0, \tag{63}$$

$$\lim_{n\to\infty} \frac{k_{n-1} + \sum_{i=1}^{n-2} k_i [k_{i+2}, \dots, k_{n-1}]}{k_{n+1}} = 0. \tag{64}$$

Let γ be the irrational number defined by the formula (46). Let l be the quantity defined by

$$l = \sum_{i=1}^{\infty} \frac{k_i}{2} \delta_i. \tag{65}$$

Since $k_n \geqq 2$ for every n, it follows that $2\delta_{n+1} < \delta_n$ for every n. Therefore the right hand side of (65) is convergent and its value is smaller than 1. We have by (45) and (62)

$$\lim_{n\to\infty} \frac{\delta_{n+1}}{\delta_n} = 0. \tag{66}$$

In the same way as we obtained (49) we have

$$\delta_i = [k_{i+1}, \dots, k_n]\, \delta_n + [k_{i+1}, \dots, k_{n-1}]\, \delta_{n+1} \tag{67}$$

for every $1 \leq i \leq n-2$. For $i = n-1$ we have

$$\delta_{n-1} = k_n \delta_n + \delta_{n+1} \tag{45}$$

Therefore we have the following equality:

$$\begin{aligned} \sum_{i=1}^{n+1} \frac{k_i}{2} \delta_i &= \sum_{i=1}^{n-1} \frac{k_i}{2} \delta_i + \frac{k_n}{2} \delta_n + \frac{k_{n+1}}{2} \delta_{n+1} \\ &= \frac{1}{2} \Big\{ k_n (k_{n-1}+1) + \sum_{i=1}^{n-2} k_i [k_{i+1}, \dots, k_n] \Big\} \delta_n \\ &\quad + \frac{1}{2} \Big\{ k_{n-1} + \sum_{i=1}^{n-2} k_i [k_{i+1}, \dots, k_{n-1}] \Big\} \delta_{n+1} + \frac{k_{n+1}}{2} \delta_{n+1}. \end{aligned} \tag{68}$$

From (64) it is clear that

$$\lim_{n\to\infty} \Big\{ k_{n-1} + \sum_{i=1}^{n-2} k_i [k_{i+1}, \dots, k_{n-1}] \Big\} \frac{\delta_{n+1}}{\delta_n} = 0. \tag{69}$$

From (62) it is clear that

$$\lim_{n\to\infty} \frac{k_{n+1}\delta_{n+1}}{2\delta_n} = \frac{1}{2}, \tag{70}$$

On the other hand we have

$$(71) \qquad \sum_{i=n+2}^{\infty} \frac{k_i}{2}\delta_i < \frac{\delta_{n+1}}{2} + \frac{\delta_{n+2}}{2} + \cdots\cdots < \delta_{n+1},$$

Let us denote by $F(a)$ the fractional part of a real number a. Then it follows from (66), (68), (69), (70) and (71) that

$$(72) \qquad \lim_{n\to\infty} F\left(\frac{l-\delta_n/2}{\delta_n}\right) \equiv 0 \quad (\text{mod } 1)$$

Let E be the interval $[0, l]$ on X, where l is defined by (65). Let γ be the irrational number defined above. Let T be the skew product transformation with the α-function $\rho C_E(x)$ where ρ is an arbitrary irrational number.

In order to show that T is ergodic and has pure continuous spectrum on $H_0^{\perp}$, by Theorem 1, 2 and by the argument in § 5, it is sufficient to prove that the following equality does not hold for any integer $p \neq 0$, $\lambda \in Y$ and $\theta(x) \in A$ [8]:

$$(73) \qquad p\rho c_E(x) - \lambda = \theta(x) - \theta(x+\gamma).$$

Let us suppose that (73) holds for some integer $p \neq 0$, for some $\lambda \in Y$, and for some $\theta(x) \in A$. Then we have the following equality for every positive integer n:

$$(74) \qquad p\rho \sum_{i=0}^{m_n-1} c_E(x+i\gamma) - m_n\lambda = \theta(x) - \theta(x+m_n\gamma).$$

From (60) and (74) we have

$$(75) \qquad p\rho\nu_n(E-x) - m_n\lambda = \theta(x) - \theta(x+m_n\gamma).$$

It is possible to find a subsequence $\{m_n'\}$ of the sequence $\{m_n\}$ such that

$$(76) \qquad \lim_{n\to\infty} m_n'\lambda = \lambda_0, \quad \lim_{n\to\infty} m_n'\gamma = \gamma_0,$$

where λ_0 and γ_0 are certain elements of the circle, and such that

$$(77) \qquad \lim_{n\to\infty} \theta(x+m_n'\gamma) = \theta(x+\gamma_0)$$

holds for almost all x.

Therefore it would lead to a contradiction if we show that no subsequence of the sequence of functions $\{p\rho\nu_n(E-x)\}$ is convergent

8) For $\lambda=0$ the non-existence of $p\neq0$ and $\theta(x)\in A$ implies the ergodicity of T, for $\lambda\neq0$ the non-existence of $p\neq0$ and $\theta(x)\in A$ implies that λ is not a proper value of T.

for almost all x. Let I_n be the sum of the intervals of length δ_{n+1} belonging to N_n. Then it follows from (49)

$$m(I_n) = p_{n-1}\delta_{n+1}. \tag{78}$$

By the condition (63) we have

$$\lim_{n\to\infty} \frac{m(I_n)}{\delta_n} = \lim_{n\to\infty} \frac{p_{n-1}\delta_{n+1}}{\delta_n} = 0. \tag{79}$$

By the definition of the function $\nu_n(E-x)$, this function is constant on every interval whose end points belong to $M_n \cup (M_n - l)$ and whose inner points do not belong to $M_n \cup (M_n - l)$. Let K_n be the set of intervals whose end points belong to $I_n \cup (I_n - l)$ and whose inner points do not belong to $I_n \cup (I_n - l)$. For any interval J belonging to K_n, $p\rho\nu_n(E-x)$ takes alternatively two different values on J (the difference of the values is $p\rho$) with constant intervals[9] with length very near to $\delta_n/2$, this is because (72) holds and $\lim\limits_{n\to\infty} \frac{m\{I_n \cup (I_n - l)\}}{\delta_n} = 0$ holds. If we denote the sum of the intervals belonging to the family K_n again by the same letter K_n, we have $m(K_n) = 1 - m\{I_n \cup (I_n - l)\} \geq 1 - 2p_{n-1}\delta_{n+1}$, which tends to 1 as $n \to \infty$. Therefore it is impossible that any subsequence of $\{p\rho\nu_n(E-x)\}$ is convergent almost everywhere. This completes the proof of the theorem. Combining the results of Theorem 8 and 9 we have shown that there exists an ergodic skew product transformation T which has pure continuous spectrum on $H_0^{\perp}$ and which has singular continuous spectrum.

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9) We mean here by "constant intervals" intervals on which the values of $\nu_n(E-x)$ are constants.