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On an Arcwise Connected Subgroup of a Lie Group

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It was recently proved that *an arcwise connected subgroup of a Lie group is a Lie subgroup*[1]. In this note a direct proof for it will be given.

Let A be an arcwise connected subgroup of an r-dimensional Lie group with the Lie algebra l, and we denote U_k a system of neighbourhoods of the identity e such that

$$U_1 \supset U_2 \supset \cdots$$
$$\bigcap_{k=1}^{\infty} U_k = e,$$

and by C_k the arcwise connected component of e in $U_k \cap A$.

We consider the directions $\overrightarrow{e, a_k}$ for $a_k \in C_k$, which converge to a limit direction Δ for a suitable sequence $\{a_k\}$. Let us denote by $X(\Delta)$ one of the corresponding infinitesimal transformations to Δ and by $\mathfrak{G}$ the aggregate of $X(\Delta)$'s.

For a one parameter subgroup $H_x = \{x\,;\ x = \exp \tau X,\ -1 \leqq \tau \leqq 1\}$ for $X \in \mathfrak{G}$, there exists a sequence $\{a_k\}$ so that $\overrightarrow{e, a_k}$ converge to the direction corresponding to X. That is for an arbitrarily small neighbourhood V[2] of e, there exist a pair of integers k and m[3] such that

$$(a_k)^j \subset H_x \cdot V, \qquad (a_k)^m \in (\exp X) \cdot V,$$

where $-m \leqq j \leqq m.$, Put $(a_k)^m = b(1)$ and $(a_k)^{-m} = b(-1)$. Now let us denote by γ_k the continuous curve which is drawn from e to a_k in U_k. Then it is possible to join $b(1)$ and $b(-1)$ by $\Gamma_x = \{(a_x)^j \gamma_k,\ -m \leqq j \leqq m\}$ in such a way that $\Gamma_x \subset H_x \cdot V$. Moreover we can introduce a parameter τ such that

$$\Gamma_x = \{b(\tau),\ -1 \leqq \tau \leqq 1\},\ b(\tau) \in (\exp \tau X) \cdot V.$$

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1) This theorm was proved by Iwamura, Hayashida, Minagawa and Homma when the Lie group is a vector group and by Kuranishi when it is semi simple. Kuranishi, using the above results, proved it for the general case, but the author obtained independently the present proof.

2) In this paper V or V' denotes arbitrarily or sufficiently small neighbourhood of the identity.

3) m depends upon a_k and X.

Now we find by simple calculations that for $X, Y \in l$,

$\lim_{n\to\infty} (\exp X/n \exp Y/n)^{\rho_n} = \exp \rho (X+Y)$,

$\lim_{n\to\infty} (\exp (-X/n) \exp (-Y/n) \exp (X/n) \exp (Y/n))^{\sigma_n} = \exp \sigma [X, Y]$,[4]

where $\rho_n (\leqq n)$ and $\sigma_n (\leqq n^2)$ are integers and ρ_n/n, σ_n/n^2 converge to real numbers respectively. When $X, Y \in \mathfrak{G}$ we can take some n, some large k' and a sufficiently small V',

$$a_k \in C_{k'} \frown (\exp X/n) \cdot V',^{5)}$$
$$b_k \in C_{k'} \frown (\exp Y/n) \cdot V',$$

so that
$$(a_k b_k)^{\rho_n} \in \exp \rho (X+Y) V,$$
$$(a_k^{-1} b_k^{-1} a_k b_k)^{\sigma_n} \in \exp \sigma [X, Y] \cdot V,$$

for all $\rho_n \leqq n$ and $\sigma_n \leqq n^2$. Moreover (a_k, b_k) and $(a_k^{-1} b_k^{-1} a_k b_k)$ belong to C_k, since e and a_k are joined by Γ_x sufficiently near to H_x, e and b_x by Γ_y near to H_y. Therefore

$$(X+Y) \in \mathfrak{G}, \ [X, Y] \in \mathfrak{G},$$

whence we conclude that $\mathfrak{G}$ is a subalgebra of l.

Let the basis of $\mathfrak{G}$ be $X_1, \ldots, X_s$, and let the basis of l be $X_1, \ldots, X_s, X_{s+1}, \ldots, X_r$. We denote by G the corresponding Lie subgroup to $\mathfrak{G}$, and denote for brevity H_{x_i} by H_i, and $\Gamma_{x_i} = \{b_i(\tau_i), \ -1 \leqq \tau_i \leqq 1\}$ by Γ_i for $1 \leqq i \leqq s$. Then we have $\Gamma_i \subset H_k \cdot V$.

Now an element $a_k \in C_k$ can be written uniquely as follows:

$$a_k = (\exp \tau_1 X_1 \ldots \exp \tau_s X_s)(\exp \tau_{s+1} X_{s+1} \ldots \tau_r X_r) \equiv g_k h_k,$$

where $g_k = (\exp \tau_1 X_1 \ldots \exp \tau_s X_s) \in G$, $h_k = (\exp \tau_{s+1} X_{s+1} \ldots \exp \tau_r X_r)$.

If infinitely many h_k are not e, we may take an element $f_k \in A$ so close to g_k that $\overrightarrow{e, f_k^{-1} a_k}$[6]) have the same limit direction Δ_0 as that of $\overrightarrow{e, h_k}$'s, This means that $X(\Delta_0) \in \mathfrak{G}$, which is a contradiction. Therefore $h_k = e$ for a sufficiently large k; $C_k \subset G$. It is clear that C_k generates A, so $A \subset G$. Conversely the continuous mapping φ which maps the cubic neighbourhood $Q = \{x;\ x = \exp \tau_1 X_1 \ldots \exp \tau_s X_s, \ -1 \leqq \tau_i \leqq 1\}$ into A in such a way that to x corresponds an element $u = b_1(\tau_1) \ldots b_s(\tau_s)$ of A, moves the boundary of Q only slightly because $\Gamma_i \subset H_i \cdot V$. So by virtue of the well known theorem of topology a neighbourhood of e with respect to G is contained in $\varphi(Q)$[7]), i.e. in A. As A is a group, $A \supset G$. Thus we have $A = G$.

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[4]) cf. Pontrjagin "Topological groups", p. 236.

[5]) k' taken so large as $(U_{k'})^4 \subset U_k$ then $(a_k b_k)$ and $(a_k^{-1} b_k^{-1} a_k b_k)$ are both in U_k.

[6]) By simple consideration $f_k^{-1} a_k \in C_k$ for a large k'.

[7]) $\varphi(Q)$ is compact and s-dimensional, so it contains a neighbourhood.