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CONSTRUCTION OF VERSAL GALOIS COVERINGS USING TORIC VARIETIES

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Abstract

In this article we give an explicit construction of versal Galois coverings for any given finite subgroup of $GL(n,\mathbb{Z})$. By this construction we give a positive answer to Question 1.4 in [5].

Introduction

Let X and Y be normal projective varieties. Let $\pi: X \to Y$ be a finite surjective morphism. We denote the rational function fields of X and Y by $\mathbb{C}(X)$ and $\mathbb{C}(Y)$, respectively. Under these circumstances, one can regard $\mathbb{C}(Y)$ as a subfield of $\mathbb{C}(X)$ by $\pi^*: \mathbb{C}(Y) \to \mathbb{C}(X)$.

DEFINITION 0.1. π is said to be a Galois covering if $\mathbb{C}(X)/\mathbb{C}(Y)$ is a Galois extension. We call π a G-covering when the Galois group of the field extension is isomorphic to a finite group G.

REMARK 0.2. Note that there exists a natural G-action on X such that $Y = X/G$.

In [2], Namba gave a method for constructing new G-coverings from a given G-covering as follows: Let $\pi: X \to Y$ be a G-covering. Let W be a normal projective variety.

NOTATION 0.3. We denote the stabilizer of $x \in X$ by G_x. Also we define $\mathrm{Fix}(X, G)$ by

$$\mathrm{Fix}(X, G) = \{x \in X \mid G_x \neq \{1\}\}.$$

DEFINITION 0.4. A rational map $\nu: W \dashrightarrow Y$ is called a G-indecomposable rational map to Y if $\nu(W) \not\subset \pi(\mathrm{Fix}(X,G))$ and ν does not factor through $\pi_H: X/H \to Y$ for any H, where X/H is the quotient variety of X by a subgroup $H \subset G$ and π_H is the quotient morphism.

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Fix a G-indecomposable rational map $\nu\colon W \dashrightarrow Y$. Let W_0 be the graph of ν. Then we can obtain a G-covering Z over W by taking the $\mathbb{C}(W_0 \times_Y X)$ normalization of W. We also obtain a G-equivariant rational map from $\mu\colon Z \dashrightarrow X$ such that $\mu(Z) \not\subset \mathrm{Fix}(X, G)$. We can construct many new G-coverings in this manner. However, we may not be able to construct every G-covering by this method, as the construction depends on the existence of a G-indecomposable rational map. This leads us to the notion of a versal G-covering introduced in [5] and [6].

DEFINITION 0.5. $\varpi\colon X \to Y$ is called a versal G-covering if, for any G-covering $\pi'\colon Z \to W$, there exists a G-equivariant rational map $\mu\colon Z \dashrightarrow X$ such that $\mu(Z) \not\subset \mathrm{Fix}(X, G)$.

REMARK 0.6. μ induces a G-indecomposable rational map ν from W to Y, and Z coincides with the G-covering constructed by the method above by using ν. Note that the versal G-covering here is not unique.

By the definition any G-covering can be obtained as a "rational pullback" from a versal G-covering. As for the existence of versal G-coverings, Namba proved the following.

Theorem 0.7 (Namba [2]). *For any finite group G, there exists a versal G-covering.*

Namba explicitly constructed a versal G-covering for each finite group G. However his method of construction gave versal coverings with dimensions equal to the order of the given group G, and it does not seem to be practical to use it in order to construct new Galois coverings. In [6], Tsuchihashi constructed versal G-coverings over the projective space $\mathbb{P}^n$ for the symmetric groups and for a generalization of the symmetric groups using toric varieties. In this paper we generalize Tsuchihashi's result partially and construct versal coverings of dimension n for any subgroup G of $GL(n, \mathbb{Z})$. Our result is the following.

Theorem 0.8. *Let N be a free $\mathbb{Z}$-module, Δ a projective fan in $N_{\mathbb{R}}$. Let $X(\Delta)$ be the toric variety associated to the fan Δ. Let G be a subgroup of $\mathrm{Aut}_{\mathbb{Z}}(N)$ which keeps Δ invariant. Then G acts naturally on $X(\Delta)$ and*

$$\varpi : X(\Delta) \to X(\Delta)/G$$

is a versal G-covering.

1. Construction and proof of versality

In this section we will prove Theorem 0.8. We will first construct projective toric varieties with G-action and construct G-coverings by taking the quotient variety and the quotient morphism. Then we prove that the G-coverings that we have constructed are versal.

We will mostly follow Fulton [1] for notations concerning toric varieties. Let N be a free $\mathbb{Z}$-module of rank n. Let M be the dual module of N. We denote the dual pairing by $\langle u, v\rangle$ for $u \in M$ and $v \in N$. We denote a fan by Δ, and denote the toric variety associated to the fan Δ by $X(\Delta)$. We will say a fan to be a projective fan when $X(\Delta)$ is a projective variety. For basic properties of toric varieties, we refer the reader to Fulton [1] and Oda [3, 4].

A toric variety $X(\Delta)$ with G-action for a given finite subgroup G of $GL(n,\mathbb{Z})$ can be constructed as follows.

Suppose that Δ is a complete G-invariant fan (i.e. for any $g \in G$ and any $\sigma \in \Delta$ there exists $\sigma' \in \Delta$ such that $g(\sigma) = \sigma'$). Then $g\colon N \to N$, for any $g \in G$, induces an automorphism of varieties $g_\sharp\colon X(\Delta) \to X(\Delta)$. Thus we can define a G-action on $X(\Delta)$. We will abuse notation and denote $g_\sharp$ by g. By the following lemma there exists a complete projective invariant fan for any finite subgroup G of $GL(n,\mathbb{Z})$.

Lemma 1.1. *For any finite subgroup G of $GL(n,\mathbb{Z})$, there exists a complete projective G-invariant fan.*

Proof. Take a fan Δ' of $N_{\mathbb{R}}$ corresponding to $(\mathbb{P}^1)^n$. It is a fan obtained by decomposing $N_{\mathbb{R}}$ with hyperplanes. By taking the images of these hyperplanes by G and by decomposing $N_{\mathbb{R}}$ with this new set of hyperplanes , we obtain a G-invariant fan Δ of $N_{\mathbb{R}}$. By the proof of Proposition 2.17 in [3], a complete fan obtained as a hyperplane decomposition is projective, hence Δ is projective. □

By taking the quotient variety X/G of X by G, and taking the quotient morphism $\varpi\colon X \to X/G$ we obtain a G-covering. We will now prove some lemmas in order to show that the G-coverings constructed in the fashion above are versal.

Lemma 1.2. *Let $X(\Delta)$ be a complete projective toric variety with G-action. Then there exists a G-invariant T_N-invariant very ample divisor on $X(\Delta)$.*

Proof. Since $X(\Delta)$ is projective, there exists a T_N-invariant very ample divisor D on $X(\Delta)$. Let D' be

$$D' = \frac{1}{|G_{D'}|}\sum_{g\in G} g(D)$$

where $G_D = \{g \in G \mid g(D) = D\}$. Then D' is a G-invariant T_N-invariant divisor. It remains to show that D' is ample.

For any T_N-invariant ample divisors D_1 and D_2 the sum $D_1 + D_2$ is also ample. This is true since if D_1 and D_2 are ample, the piecewise linear functions ψ_{D_1} and ψ_{D_2} corresponding to D_1 and D_2 respectively are strictly convex. Then $\psi_{D_1+D_2}$ is also strictly convex which implies the ampleness of $D_1 + D_2$.

Each $g(D)$ is ample so D' is an ample divisor and for some m, mD' is a very ample G-invariant T_N-invariant divisor. $\square$

Let $\Delta(1)$ be the set of one dimensional cones of Δ. Let D_{τ_i} be the T_N-invariant divisor corresponding to $\tau_i \in \Delta(1)$. Let v_i be a primitive generator of τ_i. Let $D = \sum_{\tau_i \in \Delta(1)} a_i D_{\tau_i}$ be a G-invariant T_N-invariant cartier divisor (wich implies $a_i = a_j$ if there exists $g \in G$ such that $g(\tau_i) = \tau_j$). Then $P_D = \{u \in M_{\mathbb{R}} \mid \langle u, v_i \rangle \geq -a_i, \forall v_i \in \Delta(1)\} \subset M_{\mathbb{R}}$ is also G-invariant. From [1] p.66, the global sections of the sheaf $\mathcal{O}(D)$ is generated by ω^u, $u \in P_D \cap M$.

$$\mathrm{H}^0(X(\Delta), \mathcal{O}(D)) = \bigoplus_{u \in P_D \cap M} \mathbb{C} \cdot \omega^u.$$

Hence we can define a (right) G-action on the global sections of the sheaf $\mathcal{O}(D)$ by $(\omega^u) \cdot g^* \mapsto \omega^{(u)g*}$.

Define $u(\sigma) \in M$ by $\langle u(\sigma), v \rangle = \psi_D(u)|_\sigma$. Then from [1] p.62, $\Gamma(U_\sigma, \mathcal{O}(D)) = \chi^{u(\sigma)} \cdot A_\sigma$. Thus we have local trivialization isomorphisms $\eta_\sigma \colon \Gamma(U_\sigma, \mathcal{O}(D)) \cong A_\sigma$ given by $\omega^u \mapsto \chi^{u-u(\sigma)}$. Let σ and σ' be maximal cones of Δ and suppose there exists $g \in G$ such that $g(\sigma) = \sigma'$. Since D is G-invariant we have $(u(\sigma'))g^* = u(\sigma)$. Then

$$\eta_\sigma(\omega^u \cdot g^*) = \chi^{(u)g^* - u(\sigma)} = \chi^{(u-u(\sigma'))g^a st} = \eta_{\sigma'}(\omega^u) \cdot g^*.$$

Hence this action on the global sections of $\mathcal{O}(D)$ coincides with the geometric action of G on $X(\Delta)$.

Lemma 1.3. *For a finite set of vectors* $\{u_1, \ldots, u_s \in M\}$, *there exists* $v \in N$ *such that* $\{\langle u_i, v \rangle\}_{i=1,\ldots,s}$ *are mutually distinct.*

Proof. We prove this by induction on the rank on M. For $\mathrm{rank}(M) = 1$ take any $u \neq 0$.

Let $\mathrm{rank}(M) = k$. Fix a basis for M and let $u_i = (a_{i_1}, \ldots, a_{i_k})$. Define a projection p onto a lattice of rank $k-1$ by $(a_{i_1}, \ldots, a_{i_k}) \mapsto (a_{i_2}, \ldots, a_{i_k})$. Then by the hypothesis of induction there exists $v' = (b_2, \ldots, b_k)$ such that $\langle p(u_i), v' \rangle$ are distinct for distinct $p(u_i)$. Let $b_1 = 2\max\{|\langle p(u_i), v' \rangle|\}_{i=1,\ldots,s} + 1$. Then $v = (b_1, \ldots, b_s)$ satisfies the desired condition. This can be checked directly.

Let $u_i = (a_{i_1}, \dots, a_{i_k})$, $u_j = (a_{j_1}, \dots, a_{j_k})$, $i \neq j$. If $a_{i_1} > a_{j_1}$ then

$$\begin{aligned}
\langle u_i, v\rangle - \langle u_j, v\rangle &= (a_{i_1} - a_{j_1})b_1 + \left(\sum_{t=2}^{k} a_{i_t} b_t\right) - \left(\sum_{t=2}^{k} a_{j_t} b_t\right) \\
&> b_1 + \left(\sum_{t=2}^{k} a_{i_t} b_t\right) - \left(\sum_{t=2}^{k} a_{j_t} b_t\right) \\
&\geq 1 \quad \text{(by the choice of } b_1\text{)}.
\end{aligned}$$

If $a_{i_1} = a_{j_1}$ then $p(u_i) \neq p(u_j)$ and

$$\begin{aligned}
\langle u_i, v\rangle - \langle u_j, v\rangle &= 0 + \left(\sum_{t=2}^{k} a_{i_t} b_t\right) - \left(\sum_{t=2}^{k} a_{j_t} b_t\right) \\
&\neq 0 \quad \text{(by the choice of } v'\text{)}.
\end{aligned}$$

Hence $\{\langle u_i, v\rangle\}_{i=1,\dots,s}$ are distinct. □

Lemma 1.4. *Let $\pi' \colon Z \to W$ be a G-covering. Let $G = \{g_1, \dots, g_{|G|}\}$.*
(1) *There exists $z \in Z$ such that $z_i = g_i(z)$ $(i = 1, \dots, |G|)$ are mutually distinct.*
(2) *For any $\alpha_1, \dots, \alpha_{|G|} \in \mathbb{C}$ there exists a rational function f on Z such that $f(z_i) = \alpha_i$.*
(3) *If $\alpha_i \neq 0$ for all i, then there exists a G-invariant affine open set U such that there exists a point z in U satisfying* (1) *and a function f satisfying* (2) *and in addition f and f^{-1} are regular on U.*

Proof. Let $U' = \mathrm{Spec}(R)$ be an G-invariant affine open set of Z where G acts freely. Then clearly any point z of U' satisfies (1).

For any finite number of distinct points $z_i \in U'$, $i = 1, \dots, s$ and for any $\alpha_i \in \mathbb{C}$, $i = 1, \dots, s$, there exists a regular function f on U satisfying $f(z_i) = \alpha_i$. This is proved by induction on the number of points. The case where $s = 1$ is trivial. Let $s = k$ and let $\mathfrak{m}_i \subset R$ be the maximal ideal corresponding to the point z_i. Then $\mathfrak{m}_i \setminus \bigcup_{j \neq i} \mathfrak{m}_j \neq \emptyset$. For each i take a regular function $f_i \in \mathfrak{m}_i \setminus \bigcup_{j \neq i} \mathfrak{m}_j$. Then

$$f_1 \cdots f_{k-1}(z_i) \begin{cases} = 0, & i = 1, \dots, k-1 \\ \neq 0, & i = k \end{cases}.$$

By the hypothesis of induction, there exists regular functions h, h' satisfying $h(z_i) = \xi_i$ for $i = 1, \dots, k-1$, and $h'(z_k) = (\alpha_k - h(z_k))/(f_1 \cdots f_{k-1}(z_k))$. Then $f = h + f_1 \cdots f_k \cdot h'$ satisfies $f(z_i) = \alpha_i$ for $i = 1, \dots, k$. Hence we have a regular function satisfying (2).

Let $V = \mathrm{Spec}(R_f)$. Then $U = \bigcap_{g \in G} g(V)$ satisfies (3). □

Let $\pi'\colon Z \to W$ be any G-covering. Let f be a rational function on Z. For f, $u \in M$, $v \in N$, define $f^{u,v}$ by

$$f^{u,v} = \prod_{g \in G} f^{\langle u, g(v)\rangle} \cdot (g^{-1}).$$

Then $f^{u,v}$ satisfies the following properties (1) and (2) for any u_1 and $u_2 \in M$ and any $g' \in G$.

$$\begin{aligned} f^{u_1,v} \cdot f^{u_2,v} &= \prod_{g \in G} f^{\langle u_1+u_2, g(v)\rangle} \cdot (g^{-1}) \\ &= f^{u_1+u_2,v} \end{aligned} \tag{1}$$

$$\begin{aligned} f^{u,v} \cdot (g') &= \prod_{g \in G} f^{\langle u, g(v)\rangle} \cdot (g^{-1}(g')) \\ &= \prod_{g'' \in G} f^{\langle (u)g', g''(v)\rangle} (g''^{-1}) \\ &= f^{(u)g',v}. \end{aligned} \tag{2}$$

Let $V = \mathrm{Spec}(R)$ be a G-invariant affine open set of Z where f and $1/f$ are regular. Define a ring homomorphism $\mu_f^v\colon R \to \mathbb{C}[M]$ by $\mu_f^v(\chi^u) = f^{u,v}$. Then from equations (1) and (2) above, μ_f^v is a G-equivariant ring homomorphism. Thus we obtain a G-equivariant morphism of varieties $\mu_f^{v\,\sharp}\colon V \to T_N = \mathrm{Spec}(\mathbb{C}[M])$.

We will show that we can choose a rational function f of Z and $v \in N$ so that $\mu_f^v(Z) \not\subset \mathrm{Fix}(X(\Delta), G)$.

Let D be a G-T_N invariant very ample divisor of $X(\Delta)$. Then

$$\mathrm{H}^0(X(\Delta), \mathcal{O}(D)) = \bigoplus_{u \in P_D \cap M} \mathbb{C} \cdot \omega^u$$

as before. Put $h = \dim(\mathrm{H}^0(X(\Delta), \mathcal{O}(D)))$, and $\{u_1 = 0, u_2, \ldots, u_h\} = P_D \cap M$. Put $g(i) = j$ when $(u_i)g = u_j$. Note again that P_D is G-invariant.

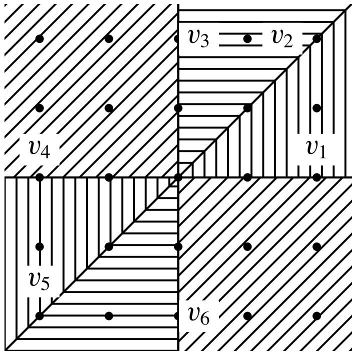
Let $\Phi_{|D|}$ be the morphism associated to the divisor D and embed $X(\Delta)$ into $\mathbb{P}^{h-1}$. For $x \in X(\Delta)$, $\Phi_{|D|}(x)$ is given by

$$\Phi_{|D|}(x) = [\omega^0(x) : \omega^{u_2}(x) : \cdots : \omega^{u_h}(x)].$$

Restricting to T_N,

$$\Phi_{|D|}|_{T_N}(x) = [1 : \chi^{u_2}(x) : \cdots : \chi^{u_h}(x)]$$

since $\omega^0 \neq 0$ on T_N.



Take $z \in Z$ so that $\{g_i z \mid g_i \in G\}$ are distinct. Take $f \in \mathbb{C}(Z)$ so that $|f(z)| \neq 1, 0$ and $f(gz) = 1$ for $g \neq 1_G$. Let $V = \operatorname{Spec}(R)$ be an affine G-invariant open set where f, f^{-1} are regular. Take $v \in N$ so that $\{\langle u_i, v\rangle = c_i\}_{i=1,\ldots,h}$ are distinct. Then

$$\begin{aligned}\Phi_{|D|} \circ \mu_f^v(z) &= [1 : f(z)^{c_2} : \cdots : f(z)^{c_h}]\\ \Phi_{|D|} \circ \mu_f^v(gz) &= [1 : f(gz)^{c_2} : \cdots : f(gz)^{c_h}]\\ &= [1 : f(z)^{c_{g(2)}} : \cdots : f(z)^{c_{g(h)}}]\end{aligned}$$

and we can see that $\{\mu_f^v(gz)\}_{g\in G}$ are distinct so $\mu_f^v(z) \notin \operatorname{Fix}(X(\Delta), G)$.

Thus we have proved Theorem 0.8.

2. Examples

Here we give some examples of versal G-coverings. Generally it is difficult to compute the quotient, but in some cases it is possible.

EXAMPLE 2.1 (Namba). We will restate Namba's construction of versal G-coverings from our point of view. Let $G = \{g_1, \ldots, g_n\}$ be any finite group of order n. Let N be a lattice of rank n and let $\{e_{g_1}, \ldots, e_{g_n}\}$ be a basis of N. Then G can be identified to a subgroup of $\operatorname{Aut}(N)$. The action of G on N is defined by $g(e_{g_i}) = e_{gg_i}$. Let Δ be the complete fan of N consisting of cones generated by $\{\pm e_{g_1}, \ldots, \pm e_{g_n}\}$. Then Δ is a complete projective G-invariant fan and $X(\Delta) \cong (\mathbb{P}^1)^n$. Then $\varpi\colon X(\Delta) \to X(\Delta)/G$ is a versal galois covering from Theorem 0.8. Thus a versal G-covering exists for any finite group.

EXAMPLE 2.2. Let N be a lattice of rank 2 and $\{e_1, e_2\}$ be a basis of N. Let Δ be the complete fan of N generated by $v_1 = v_1$, $v_2 = e_2$, $v_3 = -e_1 + e_2$, $v_4 = -e_1$, $v_5 = -e_2, v_6 = e_1 - e_2$, as in the figure above. Thus $X(\Delta)$ is isomorphic to $\mathbb{P}^2$ blown-up along three points.

Let G be the subgroup of $\mathrm{Aut}(N)$ generated by

$$\alpha = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}, \quad \beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Then $G = \langle \alpha, \beta \mid \alpha^6 = \beta^2 = (\alpha\beta)^2 = 1 \rangle \cong D_{12}$ where D_{12} is the dihedral group of order 12. Δ is an invariant fan of G and by Theorem 0.8, $\varpi : X(\Delta) \to X(\Delta)/G$ is a versal Galois covering. One can compute the quotient as the weighted projective space $\mathbb{P}(1, 1, 2)$. This is done by taking the very ample divisor $D = \sum_{i=1}^{6} D_{v_i}$ and compute the D_{12}-invariant ring of

$$\bigoplus_{i=1}^{\infty} \mathrm{H}^0(X, \mathcal{O}(iD)).$$

It is generated by algebraically independent elements of weight 1, 1, and 2.

Proposition 2.3. Example 2.2 *gives a positive answer to* Question 1.4 *in* [5].

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