Kasahara, Y. and Yano, Y.
Osaka J. Math.
**42** (2005), 1–10

# ON A GENERALIZED ARC-SINE LAW FOR ONE-DIMENSIONAL DIFFUSION PROCESSES

YUJI KASAHARA and YUKO YANO

(Received June 23, 2003)

**Abstract**

Laws of the occupation times on a half line are studied for one-dimensional diffusion processes. The asymptotic behavior of the distribution function is determined in terms of the speed measure.

## 1. Introduction

Let $\{B_t, P^x\}$ be a one-dimensional standard Brownian motion and let $\Gamma_+(t) = \int_0^t 1_{\{B_s>0\}}\,ds$. Thus $\Gamma_+(t)$ $(t \geq 0)$ denotes the sojourn time on the half line $(0, \infty)$ and the following fact is well-known as P. Lévy's arc-sine law:

$$P^0\left(\frac{1}{t}\Gamma_+(t) \leq x\right) = P^0(\Gamma_+(1) \leq x)$$
$$= \frac{2}{\pi}\arcsin\sqrt{x}, \quad 0 \leq x \leq 1.$$

Many authors have tried to extend this result for more general stochastic processes and in the present paper we are interested in linear diffusions.

A typical, interesting example is the case of the skew Bessel diffusion processes and in this case Barlow-Pitman-Yor ([1]) obtained the law of $\Gamma_+(t)/t \overset{d}{=} \Gamma_+(1) =: Y_{\alpha,p}$ explicitly (see Section 2). In connection with their result, S. Watanabe [11] determined all possible limiting distributions as $t \to \infty$ of $\Gamma_+(t)/t$ for general linear diffusion processes. Since they have calculated the double Laplace transform of the distribution function of $\Gamma_+(t)$, we may say that the law of $\Gamma_+(t)$ is already known in a sense. However, it would still be of interest to derive further properties of the law, and our aim of the present paper is to study the asymptotic behavior of

$$P^0(\Gamma_+(t) \leq x) \quad \text{as} \quad x \to 0+ \tag{1.1}$$

for every fixed $t > 0$.

To state our results, we first recall that the generator of a conservative linear diffusion has the following canonical representation: $L = (d/dm(x))(d^+/ds(x))$

2000 *Mathematics Subject Classification* : 60J25, 60F05, 60J05, 60J60.

where $s(x)$ and $dm(x)$ are called the scale and the speed measure, respectively. Changing the scale, if necessary, we may and do assume that the scale $s(x)$ is identically equal to $x$ and thus we shall consider stochastic processes with generator of the form $L = (d/dm(x))(d^+/dx)$. Our main result (Theorem 1) is as follows: the order of infinitesimal of $P^0(\Gamma_+(t) \le x)$ as $x \to 0+$ depends on the asymptotic behavior of $m_+(x) := \int_{[0,x]} dm$ as $x \to 0+$ and the multiplicative constant is determined by $m_-(x) := \int_{[-x,0)} dm$. The proof will be carried out analytically as an easy application of Krein's string theory. In fact we also have a probabilistic proof based on the excursion theory, but we shall not go into details here.

Another result of the present paper is related to a result of S. Watanabe [11] which treats the limiting distributions of $\Gamma_+(t)/t$ as $t \to \infty$, and we shall study the asymptotic behaviour of $P^0((1/t)\Gamma_+(t) \le x)$ when not only $t \to \infty$ but $x \to 0$. This may be regarded as a sort of large deviation problem in the sense of W. Feller ([2, p.548]).

The contents of this paper are as follows. In Section 2, we shall first introduce some notations and review well-known facts not only on linear diffusions but also on Krein's string theory, and then we shall state our main theorem with the proof. In Section 3, we shall study the case that $x = x(t)$ varies with $t$ in S. Watanabe's result.

## 2. Main result and the proof

Let $m\colon [0,l) \to [0,\infty)$ be a right-continuous, nondecreasing function where $0 < l \le \infty$. We put $m(0-) = 0$ and $m(x) = \infty$ for $x \ge l$ when $l < \infty$ so that the Borel measure $dm$ is defined on $[0,l)$. Such $dm$ is referred to as an inextensible measure. Let $\mathcal{M}$ be the class of all such functions $m$. For $\lambda > 0$, let $\phi(x,\lambda)$, $\psi(x,\lambda)$ be the solutions of the following integral equations:

$$\phi(x,\lambda) = 1 + \lambda \int_0^x d\xi \int_{0-}^{\xi+} \phi(u,\lambda)\, dm(u), \tag{2.1}$$

$$\psi(x,\lambda) = x + \lambda \int_0^x d\xi \int_{0-}^{\xi+} \psi(u,\lambda)\, dm(u) \tag{2.2}$$

on the interval $x \in [0,l)$. So $\phi(\,\cdot\,,\lambda)$ and $\psi(\,\cdot\,,\lambda)$ are solutions of the differential equation $(d/dm(x))(d^+/dx)u = \lambda u$ with initial conditions $u(0) = 1$, $u^+(0-) = 0$ and $u(0) = 0$, $u^+(0-) = 1$, respectively. Set

$$h(\lambda) = \lim_{x\uparrow l} \frac{\psi(x,\lambda)}{\phi(x,\lambda)}. \tag{2.3}$$

In particular, if $m(x) \equiv 0$, then $h(\lambda) \equiv \infty$. The correspondence between $m(x)$ and $h(\lambda)$ is called Krein's correspondence and $h(\lambda)$ is called the characteristic function of the string $m$. For the details of Krein's correspondence we refer to Kotani-Watanabe [9].

The function $h$ is known to have the following representation:

$$h(\lambda) = c + \int_{0-}^{\infty} \frac{d\sigma(\xi)}{\lambda + \xi}, \quad \lambda > 0 \tag{2.4}$$

for some $0 \le c \le \infty$ and nonnegative Radon measure $d\sigma$ on $[0, \infty)$ such that

$$\int_{0-}^{\infty} \frac{d\sigma(\xi)}{1 + \xi} < \infty.$$

In fact it holds that $c = \inf\{x > 0 \colon m(x) > 0\}$. Let $\mathcal{H}$ be the class of all functions $h$ of the form (2.4) with $0 \le c \le \infty$. The important result is that Krein's correspondence is one-to-one and onto so that we may write $m \leftrightarrow h$. Let $m^{-1} \in \mathcal{M}$ be the right-continuous inverse of $m \in \mathcal{M}$. If $m \leftrightarrow h$ is Krein's correspondence, then

$$m^{-1}(x) \longleftrightarrow \frac{1}{\lambda h(\lambda)}$$

is also Krein's correspondence. $m^{-1}$ is called the dual string of $m$. Let $\Psi$ be the class of functions $\psi(\lambda)$ of $\lambda > 0$ which have the form

$$\psi(\lambda) = c_0 + c_1\lambda + \int_0^{\infty} (1 - e^{-\lambda u}) n(du) \tag{2.5}$$

where $c_0, c_1 \ge 0$ and $n(du)$ is a nonnegative Radon measure on $(0, \infty)$ with

$$\int_0^{\infty} \frac{u}{1 + u} n(du) < \infty.$$

If $h \in \mathcal{H}$, then $1/h \in \Psi$.

Now let $m_+, m_- \in \mathcal{M}$ such that

$$m_\pm \colon [0, l_\pm) \to [0, \infty)$$

and $m_-(0) = 0$. We define the Radon measure $dm(x)$ on $(-l_-, l_+)$ by

$$dm(x) = \begin{cases} dm_+(x) & \text{on} \quad [0, l_+) \\ d\tilde{m}_-(x) & \text{on} \quad (-l_-, 0) \end{cases}$$

where $d\tilde{m}_-(x)$ is the image measure of $dm_-$ under the mapping $x \mapsto -x$. A stochastic process associated with $L = (d/dm(x))(d^+/dx)$ can be constructed as follows. Let $\{B_t\}_{t \ge 0}$ be a standard Brownian motion on $\mathbb{R}$ with $B_0 = 0$ and let $\{l(t, x), t \ge 0, x \in \mathbb{R}\}$ be its local time, i.e., the mapping $(t, x) \mapsto l(t, x)$ is jointly continuous a.s. and

$$\int_0^t 1_A(B_s)\, ds = 2 \int_A l(t, x)\, dx$$

for every $A \in \mathfrak{B}(\mathbb{R})$. Let

$$\phi(t) = \int_{(-l_-,l_+)} l(t,x)\,dm(x)$$

and put

$$X_t = B(\phi^{-1}(t)).$$

Then $\{X_t\}_{t\geq 0}$ is a strong Markov process on the support of $dm(x)$ whose lifetime $\zeta$ is identified with the first hitting time for $l_+$ or $-l_-$. This process is called the generalized diffusion process corresponding to the pair $\{m_+, m_-\}$. Notice that this is one of the standard methods of constructing one-dimensional diffusion processes (and birth and death processes) which allow the killing only on the boundary.

Let $\{X_t, P^x\}$ be a (generalized) diffusion process on $(-l_-, l_+)$ corresponding to the pair $\{m_+, m_-\}$ so that $m_\pm \in \mathcal{M}$ with $m_-(0) = 0$ and let $h_\pm$ be characteristic functions of $m_\pm$, respectively. Set $\psi_\pm = 1/h_\pm$. Since the process is also characterized by the pair $\{h_+, h_-\}$, we may say that $\{X_t, P^x\}$ is the generalized diffusion process corresponding to the pair of characteristic functions $\{h_+, h_-\}$ or to the pair of characteristic exponents $\{\psi_+, \psi_-\}$.

A positive function $L(x)$ is said to vary slowly at 0 [or at $\infty$] if, for every $\lambda > 0$, $\lim_{x\to 0[\infty]} L(\lambda x)/L(x) = 1$ and a function $f(x)$ is said to vary regularly at 0 [$\infty$] with exponent $\rho$ $(-\infty < \rho < \infty)$ if $\lim_{x\to 0[\infty]} f(\lambda x)/f(x) = \lambda^\rho$, $\lambda > 0$. Thus $f$ varies regularly with exponent $\rho$ if and only if $f(x) = x^\rho L(x)$ for some slowly varying $L$. If $\rho \neq 0$, then the (asymptotic) inverse $f^{-1}(y)$ is defined and varies regularly with exponent $1/\rho$.

Now the main result of the present paper is the following.

**Theorem 1.** *Let $\{X_t, P^x\}$ be a diffusion process on $(-\infty,\infty)$ corresponding to the pair $\{m_+, m_-\}$ and $\Gamma_+(t) = \int_0^t 1_{\{X_s>0\}}\,ds$. Let $\varphi(x)$ be a regularly varying function at 0 with exponent $\beta = 1/\alpha$ $(0 < \alpha < 1)$. If*

$$m_+(x) \sim \frac{\varphi(x)}{x}, \quad x \to 0+, \tag{2.6}$$

*then*

$$P^0(\Gamma_+(t) \leq x) \sim \frac{1}{\{\alpha(1-\alpha)\}^\alpha \Gamma(1-\alpha)} c(t)\varphi^{-1}(x), \quad x \to 0+ \tag{2.7}$$

*where $c(t)$, $t > 0$ is a continuous, decreasing function satisfying*

$$\int_0^\infty e^{-\lambda t} c(t)\,dt = \frac{1}{\lambda h_-(\lambda)}, \quad \lambda > 0. \tag{2.8}$$

We postpone the proof and consider, as an example, the case of the skew Bessel diffusion process of dimension $2-2\alpha$, $0 < \alpha < 1$ with the skew parameter $p$,

$0 \le p \le 1$; SKEWBES$(2-2\alpha, p)$ in notation. This is the case of the diffusion process on $(-\infty, \infty)$ corresponding to the pair $\{m_+, m_-\}$ is given by

$$m_+(x) = p^{1/\alpha} x^{1/\alpha - 1}, \quad l_+ = \infty,$$
$$m_-(x) = (1-p)^{1/\alpha} x^{1/\alpha - 1}, \quad l_- = \infty.$$

SKEWBES$(2-2\alpha, p)$ corresponds to the pair $\{h_+, h_-\}$ given by

$$h_+(\lambda) = D_\alpha p^{-1} \lambda^{-\alpha}, \quad h_-(\lambda) = D_\alpha (1-p)^{-1} \lambda^{-\alpha}$$

where $D_\alpha = \{\alpha(1-\alpha)\}^{-\alpha}\Gamma(1+\alpha)/\Gamma(1-\alpha)$. In the case of SKEWBES$(2-2\alpha, p)$, we put

$$\frac{1}{t}\Gamma_+ \overset{d}{=} \Gamma_+(1) =: Y_{p,\alpha}$$

particularly. Note that, because of the self-similarity of the skew Bessel diffusion process, the law of $\Gamma_+(t)/t$ is independent of $t$, i.e., $\Gamma_+(t)/t \overset{d}{=} \Gamma_+(1)$, so that $\Gamma_+(t)/t \overset{d}{=} Y_{\alpha,p}$. Therefore Theorem 1 implies

$$P^0(Y_{\alpha,p} \le x) \sim \frac{1-p}{p} \frac{1}{\Gamma(1-\alpha)\Gamma(1+\alpha)} x^\alpha, \quad x \to 0+.$$

This fact can be also obtained directly from J. Lamperti's density formula (see [10]):

$$f_{p,\alpha}(x) = \frac{\sin \alpha\pi}{\pi} \frac{p(1-p)x^{\alpha-1}(1-x)^{\alpha-1}}{p^2(1-x)^{2\alpha} + (1-p)^2 x^{2\alpha} + 2p(1-p)x^\alpha(1-x)^\alpha \cos \alpha\pi}$$

for $0 < x < 1$. If $p = \alpha = 1/2$, then the skew Bessel diffusion process is in fact the usual Brownian motion up to a multiplicative constant and the above result implies that

$$P^0\left(\frac{1}{t}\Gamma_+(t) \le x\right) = P^0(\Gamma_+(1) \le x) \sim \frac{2}{\pi} x^{1/2}, \quad x \to 0+.$$

This is of course compatible with P. Lévy's arc-sine law.

For the proof of Theorem 1, we introduce the following result, which is due to Barlow-Pitman-Yor ([1]) and Watanabe ([11]).

**Lemma 1** (Barlow-Pitman-Yor, Watanabe). *Let $\{X_t, P^x\}$ be a diffusion process on $(-l_-, l_+)$ corresponding to the pair $\{m_+, m_-\}$ and let $h_\pm$ be the characteristic function of $m_\pm$ and $\psi_\pm = 1/h_\pm$, respectively. Then, for $\lambda > 0$, $\mu > 0$,*

$$\int_0^\infty e^{-\mu t} E^0[e^{-\lambda \Gamma_+(t)}]\, dt = \frac{\psi_+(\lambda+\mu)/(\lambda+\mu) + \psi_-(\mu)/\mu}{\psi_+(\lambda+\mu) + \psi_-(\mu)}. \tag{2.9}$$

In particular, if $\{X_t\}_{t\geq 0}$ is SKEWBES$(2-2\alpha, p)$, then

$$\int_0^\infty e^{-\mu t} E^0[e^{-\lambda \Gamma_+(t)}]\,dt = \frac{p(\lambda+\mu)^{\alpha-1} + (1-p)\mu^{\alpha-1}}{p(\lambda+\mu)^{\alpha} + (1-p)\mu^{\alpha}}, \tag{2.10}$$

and if $\{X_t\}_{t\geq 0}$ is the Brownian motion, then

$$\int_0^\infty e^{-\mu t} E^0[e^{-\lambda \Gamma_+(t)}]\,dt = \frac{(\lambda+\mu)^{-1/2} + \mu^{-1/2}}{(\lambda+\mu)^{1/2} + \mu^{1/2}} = \frac{1}{\sqrt{\lambda+\mu}\sqrt{\mu}}. \tag{2.11}$$

We refer to Watanabe ([11]) for the proof but we remark that this formula can also be shown by using the Feynman-Kac formula (cf. Itô [5], Karatzas-Shreve ([7])) as follows. Let $\mu > 0$, $\lambda > 0$ and define

$$z(x) = E^x\left[\int_0^\infty \exp\left\{-\mu t - \lambda\int_0^t 1_{(0,\infty)}(X_s)\,ds\right\} dt\right].$$

Then $z(x)$ is a bounded positive solution of

$$\left(\mu + \lambda\cdot 1_{\{x>0\}} - \frac{d}{dm(x)}\frac{d^+}{dx}\right) z = 1.$$

So if we solve this equation, then $z(0)$ is the right-hand side of (2.9).

**Lemma 2.** *Let $u_n(x), x > 0$ $(n = 1, 2, \ldots)$ and $u(x)$, $x > 0$ be nonnegative monotone functions and let $\omega_n$, $\omega$ be their Laplace transforms, i.e.,*

$$\omega_n(\lambda) = \int_0^\infty e^{-\lambda x} u_n(x)\,dx, \quad \lambda > 0,$$
$$\omega(\lambda) = \int_0^\infty e^{-\lambda x} u(x)\,dx, \quad \lambda > 0.$$

*We assume that $\omega_n(\lambda)$ and $\omega(\lambda)$ are finite for all $\lambda > 0$ and that $\omega_n(\lambda) \to \omega(\lambda) + c$ for every $\lambda > 0$ and for some constant $c$. Then $u_n(x) \to u(x)$ for all continuity points $x$ of $u$.*

Proof. By the well-known continuity theorem for Laplace transforms (see Feller ([2])), we have

$$\int_0^x u_n(\xi)\,d\xi \to \int_0^x u(\xi)\,d\xi + c, \quad x > 0$$

and hence

$$\int_x^y u_n(\xi)\,d\xi \to \int_x^y u(\xi)\,d\xi, \quad 0 < x < y.$$

By monotonicity of $u_n$, it is easy to complete the proof. $\square$

We are now ready to prove Theorem 1. We start with proving the existence of the unique continuous function $c(t)$ which satisfies (2.8). Since $1/(\mu h_-(\mu))$ is the characteristic function of the dual string $m^*$ of $m_-$ (i.e., $m^*(x) = m_-^{-1}(x)$), there exists a nonnegative Radon measure $d\sigma^*$ on $[0,\infty)$ such that $\int_0^\infty d\sigma^*(\xi)/(1+\xi) < \infty$ and

$$\frac{1}{\mu h_-(\mu)} \left( = \frac{\psi_-(\mu)}{\mu} \right) = \int_0^\infty \frac{d\sigma^*(\xi)}{\mu + \xi}.$$

Here, we used the assumption that $m_-(0) = 0$ and hence $\inf\{x > 0 \colon m^*(x) > 0\} = 0$, although this will not play any essential role. Put

$$c(t) = \int_{0-}^\infty e^{-t\xi} d\sigma^*(\xi), \quad t > 0.$$

Then, it is easy to see that $c(t)$ is continuous, nonincreasing and satisfies (2.8). Now by Lemma 1, we have

$$\int_0^\infty e^{-\mu t} E^0[e^{-\lambda \Gamma_+(t)}]\, dt = \frac{\psi_+(\lambda+\mu)/(\lambda+\mu) + \psi_-(\mu)/\mu}{\psi_+(\lambda+\mu) + \psi_-(\mu)} \tag{2.12}$$

for $\lambda > 0$, $\mu > 0$. On the other hand, our assumption on $m_+$ combined with a result of Y. Kasahara ([8]) yields that

$$h_+(\lambda) \left( = \frac{1}{\psi_+(\lambda)} \right) \sim D_\alpha \varphi^{-1}\left(\frac{1}{\lambda}\right), \quad \lambda \to \infty. \tag{2.13}$$

Consequently, $\psi_+(\lambda)$ varies regularly at $\infty$ with exponent $0 < \alpha < 1$ and hence $\psi_+(\lambda + \mu)/(\lambda+\mu) \to 0$ as $\lambda \to \infty$. Therefore, the right-hand side of (2.12) is asymptotically equal to $(1/\psi_+(\lambda))(\psi_-(\mu)/\mu)$ as $\lambda \to \infty$. Thus,

$$\int_0^\infty e^{-\mu t} \{\psi_+(\lambda) E^0[e^{-\lambda \Gamma_+(t)}]\}\, dt \to \int_0^\infty e^{-\mu t} c(t)\, dt$$

as $\lambda \to \infty$. By Lemma 2, this implies

$$\psi_+(\lambda) E^0[e^{-\lambda \Gamma_+(t)}] \to c(t), \quad \lambda \to \infty.$$

Combining this with (2.13), we obtain

$$E^0[e^{-\lambda \Gamma_+(t)}] \sim D_\alpha c(t) \varphi^{-1}\left(\frac{1}{\lambda}\right), \quad \lambda \to \infty.$$

Therefore, by Karamata's Tauberian theorem (cf. Feller [2]), we have

$$P^0(\Gamma_+(t) \le x) \sim \frac{D_\alpha}{\Gamma(1+\alpha)} c(t) \varphi^{-1}(x), \quad x \to 0+$$

which completes the proof of Theorem 1.

REMARK. We shall not go into details but we remark the following fact. Let $X^*(t)$ be a diffusion process corresponding to $\{0, m^*(x)\}$ and let $p^*(t, x, y)$ be the transition density. Then,

$$c(t) = p^*(t, 0, 0).$$

## 3. Large deviations

The following result was obtained by S. Watanabe ([11]).

**Theorem A.** *For* $0 < p < 1$, $0 < \alpha < 1$ *and every fixed* $0 \le x \le 1$,

$$\frac{P^0((1/t)\Gamma_+(t) \le x)}{P^0(Y_{\alpha,p} \le x)} \to 1, \quad t \to \infty \tag{3.1}$$

*if and only if*

$$\psi_\pm(\lambda) \sim \lambda^\alpha L_\pm(\lambda), \quad \lambda \to 0 \tag{3.2}$$

*where* $L_\pm(\lambda)$ *are slowly varying functions at* $\lambda = 0$ *with*

$$\lim_{\lambda \to 0+} \frac{L_-(\lambda)}{L_+(\lambda)} = \frac{1-p}{p}. \tag{3.3}$$

We remark that the latter condition is equivalent to

$$m_\pm(x) \sim x^{1/\alpha - 1} K_\pm(x), \quad x \to \infty \tag{3.4}$$

where $K_\pm(x)$ are slowly varying functions at $x = \infty$ with

$$\lim_{x \to \infty} \frac{K_-(x)}{K_+(x)} = \frac{(1-p)^{1/\alpha}}{p^{1/\alpha}} \tag{3.5}$$

by Y. Kasahara ([8]).

Now it is a natural question to ask how $x = x(t)$ can vary with $t$ in such a way that $x \to 0$ as $t \to \infty$ in order that the relation (3.1) remains true. The answer to this question is as follows.

**Theorem 2.** *Let* $0 < p < 1$ *and* $0 < \alpha < 1$. *Assume that* (3.2) *and* (3.3) *hold, or equivalently,* (3.4) *and* (3.5) *hold. Then* (3.1) *remains true if* $x$ *varies with* $t$ *in such a way that* $x \to 0$, $tx \to \infty$ *and* $L_+(1/t)/L_+(1/tx) \to 1$ *as* $t \to \infty$.

Proof. Let $c(\lambda)$ be a function of $\lambda$ such that $c(\lambda) > 0$, $c(\lambda) \to 0$ and $\lambda/c(\lambda) \to 0$ as $\lambda \to 0$. Using Lemma 1 of the previous section, we have

$$\begin{aligned}
&\frac{\psi_+(\lambda/c(\lambda))}{\psi_-(\lambda)} \cdot \int_0^\infty e^{-\mu u}\,du \int_0^\infty e^{-sv}\,dP^0\left(\lambda\Gamma_+\left(\frac{u}{\lambda}\right) \le v \cdot c(\lambda)\right) \\
&= \frac{\psi_+(\lambda/c(\lambda))}{\psi_-(\lambda)} \int_0^\infty \lambda e^{-\mu\lambda u}\,du \int_0^\infty e^{-(s\lambda/c(\lambda))v}\,dP^0(\Gamma_+(t) \le v) \\
&= \frac{\psi_+(\lambda/c(\lambda))}{\psi_-(\lambda)} \cdot \lambda \cdot \frac{\frac{\psi_+(s\lambda/c(\lambda)+\mu\lambda)}{s\lambda/c(\lambda)+\mu\lambda} + \psi_-(\mu\lambda)/\mu\lambda}{\psi_+\left(s\lambda/c(\lambda) + \mu\lambda\right) + \psi_-(\mu\lambda)} \\
&= \frac{c(\lambda)/(s + \mu c(\lambda)) \cdot \frac{\psi_+(s\lambda/c(\lambda)+\mu\lambda)}{\psi_-(\lambda)} + \psi_-(\mu\lambda)/\mu\psi_-(\lambda)}{\frac{\psi_+(s\lambda/c(\lambda)+\mu\lambda)}{\psi_+(\lambda/c(\lambda))} + \frac{\psi_-(\mu\lambda)}{\psi_+(\lambda/c(\lambda))}} \\
&\to \frac{0 + (1/\mu)\cdot\mu^\alpha}{s^\alpha + 0} = \frac{\mu^{\alpha-1}}{s^\alpha} = \frac{1}{s^\alpha}\int_0^\infty e^{-\mu u}\frac{u^{-\alpha}}{\Gamma(1-\alpha)}\,du, \quad \lambda \to 0.
\end{aligned}$$

By Lemma 2 of Section 2, this implies

$$\frac{\psi_+(\lambda/c(\lambda))}{\psi_-(\lambda)} \cdot \int_0^\infty e^{-sv}\,dP^0\left(\lambda\Gamma_+\left(\frac{u}{\lambda}\right) \le v \cdot c(\lambda)\right) \to \frac{u^{-\alpha}}{\Gamma(1-\alpha)} \cdot \frac{1}{s^\alpha}, \quad \lambda \to 0.$$

Applying again the continuity theorem for Laplace transforms, we have

$$\frac{\psi_+(\lambda/c(\lambda))}{\psi_-(\lambda)} \cdot P^0\left(\lambda\Gamma_+\left(\frac{u}{\lambda}\right) \le v \cdot c(\lambda)\right) \to \frac{u^{-\alpha}v^\alpha}{\Gamma(1-\alpha)\Gamma(1+\alpha)}, \quad \lambda \to 0.$$

Hence we obtain

$$P^0\left(\lambda\Gamma_+\left(\frac{u}{\lambda}\right) \le v \cdot c(\lambda)\right) \sim \frac{1}{\Gamma(1-\alpha)\Gamma(1+\alpha)}\left(\frac{v}{u}\right)^\alpha \frac{\psi_-(\lambda)}{\psi_+(\lambda/c(\lambda))}, \quad \lambda \to 0.$$

Assume further that $L_+(\lambda)/L_+(\lambda/c(\lambda)) \to 1$ as $\lambda \to 0$. Then, setting $u = v = 1$ in this formula, we have

$$P^0\left(\lambda\Gamma_+\left(\frac{1}{\lambda}\right) \le c(\lambda)\right) \sim \frac{1}{\Gamma(1-\alpha)\Gamma(1+\alpha)}\frac{1-p}{p}(c(\lambda))^\alpha, \quad \lambda \to 0.$$

This completes the proof if we set $t = 1/\lambda$ and $x = c(\lambda)$. □

ACKNOWLEDGEMENTS. We are grateful to the referee for careful reading the manuscript and suggesting several improvements.

## References

[1] M.T. Barlow, J.W. Pitman, and M. Yor: *Une extension multidimentionnelle de la loi de l'arc sinus*, Séminaire de prob. XXIII, ed J. Azéma, P.A. Meyer et M. Yor, Lecture Notes in Mathematics, **1372** (1989), 294–314.
[2] W. Feller: An introduction to probability theory and its applications, II, Wiley, 1971.
[3] N. Ikeda and S. Watanabe: Stochastic differential equations and diffusion processes, North-Holland/Kodansha, 1989.
[4] K. Itô: Stochastic processes, II, Iwanami (in Japanese), 1957.
[5] K. Itô: Lectures on stochastic processes, Tata institute of fundamental research, Bombay, 1968.
[6] K. Itô and H.P. McKean Jr.: Diffusion processes and their sample paths, Springer, Berlin-Heidelberg-New York, 1965.
[7] I. Karatzas and S.E. Shreve: Brownian motion and stochastic calculus, Springer, Berlin-Heidelberg-New York, 1991.
[8] Y. Kasahara: *Spectral theory of generalized second order differential operators and its applications to Markov processes*, Japan J. Math. **1** (1975), 67–84.
[9] S. Kotani and S. Watanabe: *Krein's spectral theory of strings and generalized diffusion processes*, Functional analysis in Markov processes, ed. M. Fukushima, Lecture Notes in Mathematics, **923** (1982), 235–259.
[10] J. Lamperti: *An occupation time theorem for a class of stochastic processes*, Trans. Amer. Math. Soc. **88** (1958), 380–387.
[11] S. Watanabe: *Generalized arc-sine laws for one-dimensional diffusion processes and random walks*; Proc. Sympos. Pure Math. **57**, Amer. Math. Soc. 1995, 157–172.

Yuji Kasahara
Department of Information Sciences
Ochanomizu University
2-1-1 Otsuka Bunkyo-ku Tokyo 112-8610, Japan
e-mail: kasahara@math.tsukuba.ac.jp

Yuko Yano
Graduate School of Integrated Sciences
Ochanomizu University
2-1-1 Otsuka Bunkyo-ku Tokyo 112-8610, Japan