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A VARIATION ON THE GLAUBERMAN CORRESPONDENCE

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1. Introduction

Suppose that G is a finite p-solvable group, where p is a prime. Let $\mathrm{IBr}(G)$ be the set of irreducible Brauer characters of G, and let $\mathrm{IBr}_{p'}(G)$ be those $\varphi \in \mathrm{IBr}(G)$ of degree not divisible by p.

The Glauberman correspondence, in the important case where a p-group acts on a p'-group, can be viewed as a natural correspondence between $\mathrm{IBr}_{p'}(G)$ and $\mathrm{IBr}(\mathbf{N}_G(P))$, where $P \in \mathrm{Syl}_p(G)$ and G is a group with a normal p-complement. Our point in this note is to show that it is not necessary to assume that G has a normal p-complement: it suffices to assume that $\mathbf{N}_G(P)$ does.

Theorem A. *Suppose that G is p-solvable, and let $P \in \mathrm{Syl}_p(G)$. Assume that $\mathbf{N}_G(P)$ has a normal p-complement. Then for every $\varphi \in \mathrm{IBr}_{p'}(G)$, there is a unique $\varphi^* \in \mathrm{IBr}(\mathbf{N}_G(P))$ such that*

$$\varphi_{\mathbf{N}_G(P)} = e\varphi^* + p\Delta\,,$$

where e is not divisible by p and Δ is some Brauer character of $\mathbf{N}_G(P)$ or zero. Also, the map $\mathrm{IBr}_{p'}(G) \to \mathrm{IBr}(\mathbf{N}_G(P))$ given by $\varphi \mapsto \varphi^*$ is a bijection. On the other hand, if $\tau \in \mathrm{IBr}(G)$ has degree divisible by p, then

$$\tau_{\mathbf{N}_G(P)} = p\Xi\,,$$

where Ξ is some Brauer character of $\mathbf{N}_G(P)$.

Even in the case where $\mathbf{N}_G(P) = P$, Theorem A above tells us something nontrivial (although well-known): a Sylow p-subgroup P of a p-solvable group G is self-normalizing, if and only if all nontrivial irreducible Brauer characters of G have degree divisible by p.

The condition of $\mathbf{N}_G(P)$ having a normal p-complement is natural enough that can be read off from the character table of G (whenever G is p-solvable).

Theorem B. *Suppose that G is p-solvable and let $P \in \mathrm{Syl}_p(G)$. Then $\mathbf{N}_G(P)$ has a normal p-complement iff the number of p-regular classes of G of size not divis-*

ible by p is the number of irreducible Brauer characters of G of degree not divisible by p.

Theorem B is already false for $G = A_5$ and $p = 2$. In this case, G has only one irreducible Brauer character of odd degree and only one 2-regular class of odd size. However the Sylow 2-normalizer of G does not have a normal 2-complement.

2. Proofs

We begin with a lemma.

Lemma 2.1. *Suppose that G is a p-solvable and let P be a Sylow p-subgroup of G. Suppose that $N \triangleleft G$ and that $\theta \in \mathrm{IBr}(N)$ is P-invariant of p'-degree. Then there exists $\varphi \in \mathrm{IBr}(G)$ of p'-degree lying over θ.*

Proof. We argue by induction on $|G : N|$. If $N = G$, we let $\varphi = \theta$ and the proof of the lemma follows. Now, let M/N be a chief factor of G. If M/N is a p-group, then θ is M-invariant since $M \subseteq NP$. By Green's Theorem (8.11) of [3], there exists a unique $\eta \in \mathrm{IBr}(M)$ lying over θ. Furthermore, η extends θ. In particular, η has p'-degree and by uniqueness is P-invariant. Now, $|G : M| < |G : N|$ and by induction there is some $\varphi \in \mathrm{IBr}(G)$ of p'-degree lying over η. Then φ lies over θ and the proof of the lemma is complete. Suppose now that M/N is a p'-group. In this case, all irreducible constituents of θ^M have p'-degree by Theorem (8.30) of [3]. Now, P acts on the irreducible constituents of the Brauer character θ^M. Since this character has p'-degree, necessarily it follows that P fixes some irreducible constituent $\xi \in \mathrm{IBr}(M)$ of θ^M. Now, ξ lies over θ (by Corollary (8.7) of [3]) and the proof of the lemma follows by induction (as in the previous case). □

Proof of Theorem A. Let $N = \mathbf{O}_{p'}(G)$ and let $C = \mathbf{C}_N(P)$. If $N = G$, then there is nothing to prove. We claim that $\mathbf{N}_G(P) = P \times C$. Write $M = \mathbf{N}_G(P)$. By hypothesis, we know that $M = P \times K$. Hence, $K = \mathbf{O}_{p'}(M) \subseteq N$, by a well-known group theoretical fact. Hence, the claim easily follows.

Let $\varphi \in \mathrm{IBr}_{p'}(G)$. We claim that φ_N has a unique irreducible P-invariant constituent $\theta \in \mathrm{Irr}(N)$. Let $\nu \in \mathrm{Irr}(N)$ be an irreducible constituent of φ_N. Since φ has p'-degree it follows that the inertia group of ν in G has p'-index (by the Clifford correspondence, Theorem (8.9) of [3]). Hence, some conjugate θ of ν has stabilizer T containing P. Therefore θ is P-invariant. Suppose that $\mu \in \mathrm{Irr}(N)$ is some other P-invariant irreducible constituent of φ_N. Then $\mu = \theta^g$, by Clifford's theorem. Now, we have that P and $P^{g^{-1}}$ are inside T. Therefore, $P^{tg} = P$ for some $t \in T$, and we deduce that μ and θ are M-conjugate. However $M = CP$, and therefore $\mu = \theta$, as claimed.

Now, let $\theta \in \mathrm{Irr}(N)$ be P-invariant. We claim that there is a unique $\varphi \in \mathrm{IBr}_{p'}(G)$

over θ. By Lemma (2.1), we see that there is some $\varphi \in \mathrm{IBr}(G)$ of p'-degree lying over θ. We prove that φ is unique by induction on $|G|$. By hypothesis, we have that $MN/N \subseteq PN/N \subseteq \mathbf{O}^{p'}(G/N)$. Hence, by the Frattini argument, we have that $\mathbf{O}^{p'}(G/N) = G/N$. So let $K/N = \mathbf{O}^{p}(G/N) < G/N$, and let $L/N = \mathbf{O}^{p'}(K/N)$. Write $U = LP$. Hence, $G = KU$ and $K \cap U = L$. Since $M = CP \subseteq NP$, we have that $M \subseteq U$. In particular, $\mathbf{C}_{K/L}(P) = 1$. Now, $N = \mathbf{O}_{p'}(L)$ since $L \lhd G$. Since U/L is a p-group, it follows that $N = \mathbf{O}_{p'}(U)$. If $U = G$, then $K = L = N$ and we have that $G = NP$. In this case, φ is unique by Green's Theorem (8.11) of [3]. Hence, we may assume that U is proper in G. By induction, there is a unique $\eta \in \mathrm{IBr}_{p'}(U)$ lying over θ. Suppose now that $\delta \in \mathrm{IBr}_{p'}(G)$ also lies over θ and has p'-degree. Now, δ_U has a p'-degree irreducible constituent ξ. Also, ξ_N has a P-invariant constituent (by the second paragraph, for instance). Since by the second paragraph, δ_N has a unique P-invariant irreducible constituent, we deduce that ξ_N contains θ. By induction, we have that $\eta = \xi$. By the same reason, φ_U contains η. Now, by using repeatedly Corollary (8.22) of [3], we have that δ_K and φ_K are P-invariant irreducible Brauer characters of K lying over η_L. Now, let δ_1, $\varphi_1 \in B_{p'}(K)$ and $\eta_1 \in B_{p'}(L)$ be the canonical Isaacs liftings of δ_K, φ_K and η_L, respectively (see Corollary (10.3) of [1]). By uniqueness, we have that these three characters are P-invariant. Also, by Corollary (7.5) and Corollary (10.3) of [1], it easily follows that δ_1 and φ_1 lie over η_1. By Problem (13.10) of [2], we have that $\delta_1 = \varphi_1$. Hence $\varphi_K = \delta_K$. By Theorem (8.11) of [3], we have that $\varphi = \delta$, and the claim is proven.

Now, given $\varphi \in \mathrm{IBr}_{p'}(G)$, we have that φ_N has a unique P-invariant irreducible constituent $\theta \in \mathrm{Irr}(N)$, and that θ and φ uniquely determine one each other. In particular, we have proven that

$$|\mathrm{IBr}_{p'}(G)| = |\mathrm{Irr}_P(N)|\,,$$

where, as usual, $\mathrm{Irr}_P(N)$ denotes the irreducible P-invariant characters of N. Let Ω be the set of G-conjugates of θ. Hence, P acts on Ω fixing only θ, and we may write

$$\varphi_N = d\left(\theta + \sum_{\mathcal{O}}\left(\sum_{\eta\in\mathcal{O}}\eta\right)\right),$$

where $\mathcal{O}$ runs over the different P-orbits not equal $\{\theta\}$. Also, since $\varphi(1)$ is not divisible by p, we have that d is not divisible by p. Now, since $C = C_N(P)$, notice that $\eta_C = (\eta^x)_C$ for $x \in P$ and $\eta \in \mathrm{Irr}(N)$. Therefore we may write $\varphi_C = d\theta_C + p\Psi$, where Ψ is some character of C or zero. Now, by Theorem (13.14) of [2], we have that $\theta_C = e\theta^* + p\Delta$, where $\theta^* \in \mathrm{Irr}(C)$ is the Glauberman correspondent of θ, p does not divide e and Δ is a character of C or zero. Since $\mathbf{N}_G(P) = P \times C$, and the irreducible Brauer characters of $\mathbf{N}_G(P)$ are naturally identifiable with the irreducible characters of C, the first part of the theorem easily follows. Now, since

$$|\mathrm{IBr}_{p'}(G)| = |\mathrm{Irr}_P(N)| = |\mathrm{Irr}(C)| = |\mathrm{Irr}(\mathbf{N}_G(P)/P)| = |\mathrm{IBr}(\mathbf{N}_G(P))|\,,$$

(where the equality $|\mathrm{Irr}_P(N)| = |\mathrm{Irr}(C)|$ follows from the Glauberman correspondence) to prove that the map $\varphi \mapsto \varphi^*$ is bijective, it suffices to show that * is one to one. Assume that $\varphi^* = \delta^*$, where $\varphi, \delta \in \mathrm{IBr}_{p'}(G)$. By how our map is constructed and using that the Glauberman correspondence is one to one, we easily deduce that φ and δ lie over the same P-invariant irreducible character of N. Hence, by the third paragraph of this proof, we have that $\varphi = \delta$, as required.

Suppose now that $\tau \in \mathrm{IBr}(G)$ has degree divisible by p. We distinguish two cases. Suppose first that τ_N contains a P-invariant irreducible constituent $\theta \in \mathrm{Irr}(N)$. Let T be the inertia group of θ in G, and let $\mu \in \mathrm{IBr}(T \mid \theta)$ be the Clifford correspondent of τ over θ (Theorem (8.9) of [3]). Since $|G : T|$ is not divisible by p, we conclude that p divides $\mu(1)$ since $\tau(1) = |G : T|\mu(1)$. Now, since $\mu_N = d\theta$ and p does not divide $\theta(1)$, we conclude that p divides d. Since d is the multiplicity of θ in τ_N (again, by Theorem (8.9) of [3]), by Clifford's theorem, we deduce that $\tau_C = p\Xi$, for some ordinary character Ξ of C. In this case, the last part of the theorem follows. Finally, suppose that τ_N does not contain any P-invariant irreducible constituent. In this case, we may write

$$\varphi_N = d\left(\sum_{\mathcal{O}}\left(\sum_{\eta\in\mathcal{O}}\eta\right)\right),$$

where $\mathcal{O}$ runs over the different P-orbits on the action of P on the irreducible constituents of τ_N. Since elements in the same P-orbit have the same restriction to P, the proof of the theorem is completed. $\square$

To prove Theorem B, we use the following notation. We denote by $\mathrm{cl}(G)$ the set of conjugacy classes of G. Also, $\mathrm{cl}(G^0)$ is the set of conjugacy classes of p-regular elements of G, and $\mathrm{cl}(G^0 \mid P)$ is the set of p-regular classes of G with defect group P.

Proof of Theorem B. First, we prove that in a group G with a normal Sylow p-subgroup P, we have that G has a normal p-complement iff

$$|\mathrm{cl}(G/P)| = |\mathrm{cl}(G^0 \mid P)|.$$

Let K be a p-complement of G. If $K \lhd G$, then $G = P \times K$, and $|\mathrm{cl}(G/P)| = |\mathrm{cl}(K)|$. Also, if $x \in G$ is p-regular, then $x \in K$ and $P \subseteq \mathbf{C}_G(x)$. So

$$|\mathrm{cl}(G^0 \mid P)| = |\mathrm{cl}(G^0)| = |\mathrm{cl}(K)|,$$

and one direction is proven. Conversely, assume now that

$$|\mathrm{cl}(G/P)| = |\mathrm{cl}(G^0 \mid P)|.$$

Hence, we have that

$$|\mathrm{cl}(K)| = |\mathrm{cl}(G^0 \mid P)| \le |\mathrm{cl}(G^0)| \le |\mathrm{cl}(K)|,$$

and we conclude that all p-regular classes of G have defect group P. Hence, we have that $K \subseteq \mathbf{C}_G(P)$, and the claim is proven.

Since G is p-solvable, it is well known that

$$|\mathrm{IBr}_{p'}(G)| = |\mathrm{Irr}(\mathbf{N}_G(P)/P)|\,.$$

(This follows, for instance, from Corollary (1.16) of [4], Lemma (5.4) and Corollary (10.3) of [1]). Now, by Lemma (4.16) of [3], it follows that

$$|\mathrm{cl}(G^0 \mid P)| = |\mathrm{cl}(\mathbf{N}_G(P^0) \mid P)|\,.$$

Hence

$$|\mathrm{IBr}_{p'}(G)| = |\mathrm{cl}(G^0 \mid P)|$$

iff

$$|\mathrm{cl}(\mathbf{N}_G(P)/P)| = |\mathrm{cl}(\mathbf{N}_G(P^0) \mid P)|$$

which happens iff $\mathbf{N}_G(P)$ has a normal p-complement, by the first paragraph. $\square$

Of course, the numbers $|\mathrm{IBr}_{p'}(G)|$ and $|\mathrm{cl}(G^0 \mid P)|$ can be read off from the character table of G, whenever G is p-solvable. Higman's theorem (8.21) of [2], allows us to distinguish if an element $x \in G$ is p-regular. In this case, the class of x has defect group a Sylow p-subgroup of G iff $|\mathbf{C}_G(x)|$ is divisible by $|G|_p$. On the other hand, Corollary (10.4) of [3], allows to construct the Brauer character table of G from its ordinary one, and we can easily count how many irreducible Brauer characters of G have degree not divisible by p.

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