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THE ADIABATIC LIMITS OF SIGNATURE OPERATORS FOR Spin^q MANIFOLDS

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Introduction

Let (M, g^M) be a $(2n-1)$-dimensional oriented Riemannian manifold equipped with a Spin^q structure introduced in [11]: $\mathrm{Spin}^q(2n-1) = \mathrm{Spin}(2n-1) \times_{\mathbb{Z}_2} Sp(1)$. Namely, the reduced structure bundle $P_{SO(2n-1)}$ is assumed to have principal $\mathrm{Spin}^q(2n-1)$-, $SO(3)$-bundles $P_{\mathrm{Spin}^q(2n-1)}$, $P_{SO(3)}$ together with a $\mathrm{Spin}^q(2n-1)$-equivariant bundle map

$$\xi^q = (\xi^q_0, \xi^q_1) : P_{\mathrm{Spin}^q(2n-1)} \to P_{SO(2n-1)} \times P_{SO(3)}. \tag{0.1}$$

Using the canonical action of $\mathrm{Spin}^q(2n-1)$ on the quotient $\mathrm{Spin}^q(2n-1)/\,\mathrm{Spin}^c(2n-1) = Sp(1)/U(1) = \mathbb{C}P^1$, we get a $\mathbb{C}P^1$-fibration

$$\pi : Z = P_{\mathrm{Spin}^q(2n-1)} \times_{\mathrm{can}} \frac{\mathrm{Spin}^q(2n-1)}{\mathrm{Spin}^c(2n-1)} \to M, \tag{0.2}$$

whose total space Z is called a (Spin^q-style) twistor space ([12]). We will fix a connection $\alpha_{SO(3)}$ on $P_{SO(3)}$ and take the Levi-Civita connection α^M on $P_{SO(2n-1)}$. Pulling back the product connection $\alpha^M \oplus \alpha_{SO(3)}$ by ξ^q, we obtain a connection on $P_{\mathrm{Spin}^q(2n-1)}$, which induces a splitting of the tangent bundle of Z into horizontal and vertical components

$$TZ = \mathcal{H} \oplus \mathcal{V}, \tag{0.3}$$

through which the given orientation on M and the natural one on the standard fibre $\mathbb{C}P^1$ induce an orientation on Z. Further we take a Riemannian metric $g^{\mathcal{V}}$ on $\mathcal{V}$ associated to the Fubini-Study metric ds^2 of $\mathbb{C}P^1$, and define a metric g^Z_ε $(\varepsilon > 0)$ on Z by

$$g^Z_\varepsilon = \varepsilon^{-1}\pi^* g^M + g^{\mathcal{V}}, \quad \pi^* g^M = g^Z_1 | \mathcal{H}. \tag{0.4}$$

Now let A^Z_ε be the signature operator of the oriented Riemannian manifold (Z, g^Z_ε), i.e., the tangential part of the usual signature operator $(d+\delta)_+$ of the even

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dimensional $(Z \times [0, \infty), g_\varepsilon^Z + dr^2)$. The purpose of the paper is to investigate the limiting behaviors when $\varepsilon \to 0$ of the η-invariants of A_ε^Z and some intrinsically twisted ones. The operation of blowing up the metric $g^Z = \pi^* g^M + g^{\mathcal{V}}$ in the base space direction as in (0.4) is called passing to the adiabatic limit. The idea of extracting some intrinsic values by taking the adiabatic limit is originally due to Witten [15] who relates the adiabatic limit of a certain η-invariant with the so-called global anomaly: refer to [4, 6, 3, 7] for the rigorous treatment and certain extensions. Our argument depends mainly on the general theory of Dai [7]. Essentially because the fibres are all totally geodesic in the case (0.2), our results are fairly neat. Let us state here only the result for the nontwisted A_ε^Z.

We take the Levi-Civita connection ∇^M on TM and denote its curvature 2-form by Ω^M. Further we take the Levi-Civita one ∇^Z associated to g^Z and, by composing the orthogonal projection $P^{\mathcal{V}} : TZ \to \mathcal{V}$, we obtain a connection $\nabla^{\mathcal{V}} = P^{\mathcal{V}} \nabla^Z$ on $\mathcal{V}$, whose curvature 2-form is denoted by $\Omega^{\mathcal{V}}$. Define now the $\hat{\mathbb{L}}$-genus forms associated to the curvatures by

$$(0.5) \qquad \hat{\mathbb{L}}(\Omega^M) = \det{}^{1/2}\left(\frac{\sqrt{-1}\Omega^M/4\pi}{\tanh(\sqrt{-1}\Omega^M/4\pi)}\right),$$

etc. Then we have

Theorem 0.1. *The* (*adiabatic*) *limit of the η-invariant* $\lim_{\varepsilon\to 0}\eta(A_\varepsilon^Z)$ *exists and there is an odd degree form $\tilde{\eta}$ on M such that*

$$(0.6) \qquad \lim_{\varepsilon\to 0}\eta(A_\varepsilon^Z) = 2^{n+1}\int_M \hat{\mathbb{L}}(\Omega^M)\wedge\tilde{\eta},$$

$$(0.7) \qquad d\tilde{\eta} = 2\int_{Z/M}\hat{\mathbb{L}}(\Omega^{\mathcal{V}}),$$

where $\int_{Z/M}$ is the integral over the fibres.

There are some interesting ways of twisting A_ε^Z. We will discuss in §4 the adiabatic limits of the η-invariants of twisted ones.

1. Signature operators

Let us recall the definition of signature operator and some relevant facts related to it ([1], [7, §4.1]).

Let (N, g^N) be a $(2n-1)$-dimensional oriented Riemannian manifold. Denote the $*$-operator, the exterior differential and its formal adjoint by $* = *_N$, $d = d_N$ and $\delta = \delta_N$. Moreover, set $\tau = \tau_N = (\sqrt{-1})^{n+p(p+1)}*$, called the complex $*$-operator, acting on the complex exterior bundle $\wedge^p(N) = \wedge^p(T^*N)$ of degree p. The signature operator of

N is then defined by

$$A^N = \tau(d+\delta), \tag{1.1}$$

which acts on the space $\mathcal{A}(N)$ consisting of C^∞-cross-sections of $\wedge(N) = \oplus \wedge^p(N)$.

The operator is just the tangential part of the signature operator of the even-dimensional oriented Riemannian manifold $(N_+, g^{N_+}) = (N \times [0,\infty), g^N + dr^2)$ in the following sense: Set $\tau_{N_+} = (\sqrt{-1})^{n+p(p-1)} *_{N_+}$ acting on $\wedge^p(N_+)$. Note that $\tau_{N_+}^2 = 1$. Accordingly the signature operators of N_+ are defined by

$$A_\pm^{N_+} = (d_{N_+} + \delta_{N_+})_\pm : \mathcal{A}^\pm(N_+) \to \mathcal{A}^\mp(N_+), \tag{1.2}$$

where we put $\mathcal{A}^\pm(N_+) = \{\phi \in \mathcal{A}(N_+) \mid \tau_{N_+}\phi = \pm\phi\}$. It is well-known that this can be expressed as

$$\begin{aligned} A_\pm^{N_+} &= (\mathrm{ext}_{\partial_r} - \mathrm{int}_{\partial_r})(\partial_r + \tilde{A}_\pm), \\ \tilde{A}_\pm &= (-1)^{n/2+p+1}(\epsilon *_N d_N - d_N *_N) : \mathcal{A}^\pm(N_+) \to \mathcal{A}^\pm(N_+), \end{aligned} \tag{1.3}$$

where $\mathrm{ext}_{\partial_r}$, $\mathrm{int}_{\partial_r}$ are the exterior and interior products of $dr = g^{N_+}(\cdot, \partial_r)$ and the above expression for the tangential part $\tilde{A}_\pm$ is for forms of degree $2p$ ($\epsilon = 1$) or of degree $2p-1$ ($\epsilon = -1$) in the N-direction. Through the identification

$$\mathcal{A}(N) \cong \mathcal{A}^+(N_0) \equiv \mathcal{A}^+(N_+)|r = 0, \quad \phi \leftrightarrow \phi + \tau_{N_+}\phi, \tag{1.4}$$

the operator $\tilde{A}_+$ acting on $\mathcal{A}^+(N_0)$ corresponds now to the signature operator A^N.

Next, let us show that A^N can be seen as a kind of twisted Dirac operator. Take the complex Clifford bundle $\mathbb{C}l(TN)$ the Clifford multiplication of which is denoted by $\circ$, a locally defined Spin structure $\xi : P_{\mathrm{Spin}(2n-1)}(N) \to P_{SO(2n-1)}(N)$ of the reduced structure bundle and the associated locally defined spinor bundle $\mathbf{S}(N) = P_{\mathrm{Spin}(2n-1)}(N) \times_\Delta S_{2n-1}$. Here Δ is the complex spinor representation of $\mathrm{Spin}(2n-1)$ so that $\dim S_{2n-1} = 2^{n-1}$. Note that, since the global existence of a Spin structure is not assumed, N above ought to be replaced by its sufficiently small open subsets. But, to simplify the description, we do not replace so. Similarly, we take those for N_+. Its locally defined spinor bundle $\mathbf{S}(N_+) = P_{\mathrm{Spin}(2n)}(N_+) \times_\Delta S_{2n}$ has a splitting $\mathbf{S}(N_+) = \mathbf{S}^+(N_+) \oplus \mathbf{S}^-(N_+)$ induced from the usual splitting $\Delta = \Delta^+ \oplus \Delta^-$ of the complex spinor representation of $\mathrm{Spin}(2n)$. Set $\mathbb{C}l(TN_0) = \mathbb{C}l(TN_+)|r = 0$ etc. as above. We have natural inclusions $\mathbb{C}l(TN) \subset \mathbb{C}l(TN_0)$ $(TN \ni X \mapsto X \circ \partial_r)$ and $P_{\mathrm{Spin}(2n-1)}(N) \subset P_{\mathrm{Spin}(2n)}(N_0)$, which induce a canonical isomorphism

$$\mathbf{S}(N) \cong \mathbf{S}^+(N_0) \tag{1.5}$$

as $\mathbb{C}l(TN)$-module bundles. Hence we obtain isomorphisms

$$\wedge(N) \cong \wedge^+(N_0) \cong \mathbf{S}^+(N_0) \otimes \mathbf{S}(N_0) \cong \mathbf{S}(N) \otimes \mathbf{S}(N_0) \tag{1.6}$$

as left $\mathbb{C}l(TN)$-module bundles. For $X \in TN \subset \mathbb{C}l(TN)$ the left action $X\circ$ on (1.6) is expressed consistently as $\mathrm{ext}_X - \mathrm{int}_X$, $(\mathrm{ext}_X - \mathrm{int}_X)(\mathrm{ext}_{\partial_r} - \mathrm{int}_{\partial_r})$, $X\circ\partial_r\circ = (X\circ\partial_r\circ) \otimes 1$, $X\circ = (X\circ) \otimes 1$ on each of it. Notice that (1.6) has also the right action $X^*\circ$ of $X \in TN \subset \mathbb{C}l(TN)$ given as $(\mathrm{ext}_X + \mathrm{int}_X)(-1)^p$ (on $\wedge^p(N)$), $-(\mathrm{ext}_X + \mathrm{int}_X)(\mathrm{ext}_{\partial_r} + \mathrm{int}_{\partial_r})$, $(X\circ\partial_r)^*\circ = 1 \otimes (\partial_r\circ X\circ)$, $(X\circ\partial_r)^*\circ = 1 \otimes (\partial_r\circ X\circ)$ on each of it.

Now let us take a positively oriented orthonormal basis $\{e_a\}$ of TN, a connection $\nabla^{\mathbf{S}(N)}$ on $\mathbf{S}(N)$ associated to the the Levi-Civita one on TN, and, moreover, such a connection $\nabla^{\mathbf{S}(N_+)}$ on $\mathbf{S}(N_+)$. This restricts to connections $\nabla^{\mathbf{S}^{(\pm)}(N_0)}$ on $\mathbf{S}^{(\pm)}(N_0)$. We have then a twisted Dirac operator for $\mathbf{S}(N) \otimes \mathbf{S}(N_0)$

$$D^N \otimes \mathbf{S}(N_0) = \sum e_a \circ \left(\nabla^{\mathbf{S}(N)}_{e_a} \otimes 1 + 1 \otimes \nabla^{\mathbf{S}(N_0)}_{e_a}\right). \tag{1.7}$$

It will be now clear that, through (1.6), we may have

$$A^N = D^N \otimes \mathbf{S}(N_0). \tag{1.8}$$

2. Signature operators of Z

Let us take locally defined Spin structures for (M, g^M), $(\mathcal{V}, g^{\mathcal{V}})$, which give naturally a locally defined Spin structure for (Z, g^Z). Accordingly we have locally defined spinor bundles $\mathbf{S}(M)$, $\mathbf{S}(\mathcal{V})$, $\mathbf{S}(Z) = \pi^*\mathbf{S}(M) \otimes \mathbf{S}(\mathcal{V})$. Then, observing (1.6), we have the following canonical identification:

$$\begin{aligned} \mathcal{A}(Z) &\cong \Gamma(\mathbf{S}(Z)) \otimes \Gamma(\mathbf{S}(Z_0)) \\ &\cong \left(\pi^*\Gamma(\mathbf{S}(M)) \otimes \Gamma(\mathbf{S}(\mathcal{V}))\right) \otimes \left(\pi^*\Gamma(\mathbf{S}(M_0)) \otimes \Gamma(\mathbf{S}(\mathcal{V}))\right) \\ &\cong \pi^*\Gamma(\mathbf{S}(M) \otimes \mathbf{S}(M_0)) \otimes \Gamma(\mathbf{S}(\mathcal{V}) \otimes \mathbf{S}(\mathcal{V})) \\ &\cong \pi^*\mathcal{A}(M) \otimes \mathcal{A}(\mathcal{V}). \end{aligned} \tag{2.1}$$

Notice that, given $\phi \in \mathcal{A}(Z)$, its expression in terms of elements of $\pi^*\mathcal{A}(M) \otimes \mathcal{A}(\mathcal{V})$ through (2.1) does not coincide with the one naively gotten through (0.3).

The first purpose of the section is to express the signature operator A^Z acting on $\mathcal{A}(Z)$ in terms of certain operators acting on $\pi^*\mathcal{A}(M) \otimes \mathcal{A}(\mathcal{V})$ through (2.1).

Let us take a positively oriented local orthonormal frame $(e_1, \ldots, e_{2n-1}) = (e'_1, \ldots, e'_{2n-1})$ of TM, whose lift to $\mathcal{H}$ is dented by the same symbol. Also let $(e_{2n}, e_{2n+1}) = (e''_1, e''_2)$ be such a frame of $\mathcal{V}$. We put $\tau^{\mathcal{V}} = \sqrt{-1}e''_1\circ e''_2$, called the complex volume element of $\mathcal{V}$.

Lemma 2.1. *Through* (2.1), *the actions* $e'_i\circ$, $e''_k\circ$, $e'^*_i\circ$, $e''^*_k\circ$ *on* $\mathcal{A}(Z)$ *correspond respectively to the actions* $e'_i\circ\otimes\tau^{\mathcal{V}}\circ$, $1\otimes e''_k\circ$, $e'^*_i\circ\otimes 1$, $\partial^*_r\otimes(\tau^{\mathcal{V}}\circ e''_k)^*\circ$ *on* $\pi^*\mathcal{A}(M)\otimes\mathcal{A}(\mathcal{V})$.

The lemma is easily shown by rewriting the actions successively according to (2.1). One may be convinced of the comment following (2.1) by the lemma.

Bearing the lemma in mind, we denote by $e'_i\circ$ etc. the actions $e'_i\circ\otimes\tau^{\mathcal{V}}\circ$ etc. on $\pi^*\mathcal{A}(M)\otimes\mathcal{A}(\mathcal{V})$ to simplify the description. Let us then consider the following operators acting on $\pi^*\mathcal{A}(M)\otimes\mathcal{A}(\mathcal{V})$:

$$
\begin{aligned}
\pi^*A^M &= \pi^*A^M\otimes\tau^{\mathcal{V}}\circ = \sum e'_i\circ\left(\pi^*\left(\nabla^{\mathbf{S}(M)}_{e'_i}\otimes 1+1\otimes\nabla^{\mathbf{S}(M_0)}_{e'_i}\right)\otimes 1\right),\\
A^{\mathcal{V}} &= 1\otimes A^{\mathcal{V}} = 1\otimes(d_{\mathcal{V}}+\delta_{\mathcal{V}}) = \sum e''_k\circ\left(1\otimes\left(\nabla^{\mathbf{S}(\mathcal{V})}_{e''_k}\otimes 1+1\otimes\nabla^{\mathbf{S}(\mathcal{V})}_{e''_k}\right)\right),\\
\sum e'_i\circ\nabla^{\mathcal{V}}_{e'_i} &= \sum e'_i\circ\left(1\otimes\nabla^{\mathcal{V}}_{e'_i}\right) = \sum e'_i\circ\left(1\otimes\left(\nabla^{\mathbf{S}(\mathcal{V})}_{e'_i}\otimes 1+1\otimes\nabla^{\mathbf{S}(\mathcal{V})}_{e'_i}\right)\right),\\
c(T) &= \sum_{i\le j} e'_i\circ e'_j\circ c(T)(e'_i,e'_j) = \sum_{i\le j} e'_i\circ e'_j\circ\sum_k g^Z(T(e'_i,e'_j),e''_k)e''_k\circ,\\
\tilde{c}(T) &= \frac{1}{2}\sum g^Z(T(e'_i,e'_j),e''_k)(e''_k\circ e'^*_i\circ e'^*_j\circ+2e'_i\circ e''^*_k\circ e'^*_j\circ),
\end{aligned}
\tag{2.2}
$$

where T is the torsion tensor of the covariant derivative $\nabla^{\oplus}=\pi^*\nabla^M\oplus\nabla^{\mathcal{V}}$ on TZ and $\nabla^{\mathbf{S}(\mathcal{V})}$ is the covariant derivative on $\mathbf{S}(\mathcal{V})$ induced from $\nabla^{\mathcal{V}}$. Here $A^{\mathcal{V}}$ acting on $\mathcal{A}(\mathcal{V})$ can be seen as a family of signature operators along the fibres

$$
A^{\mathcal{V}} = (A^{\mathcal{V}}_x \mid x\in M),\quad A^{\mathcal{V}}_x = d_{\mathcal{V}_{(x)}}+\delta_{\mathcal{V}_{(x)}} = d_{\pi^{-1}(x)}+\delta_{\pi^{-1}(x)}
\tag{2.3}
$$

and, according to the splitting $\mathcal{A}(\mathcal{V})=\mathcal{A}^+(\mathcal{V})\oplus\mathcal{A}^-(\mathcal{V})=(\mathbf{S}^+(\mathcal{V})\otimes\mathbf{S}(\mathcal{V}))\oplus(\mathbf{S}^-(\mathcal{V})\otimes\mathbf{S}(\mathcal{V}))$, it can be expressed as follows:

$$
A^{\mathcal{V}} = \begin{pmatrix} 0 & A^{\mathcal{V}}_- \\ A^{\mathcal{V}}_+ & 0\end{pmatrix},\quad A^{\mathcal{V}}_\pm = (d_{\mathcal{V}}+\delta_{\mathcal{V}})_\pm.
\tag{2.4}
$$

We will now express A^Z acting on $\mathcal{A}(Z)$ in terms of operators (2.2) acting on $\pi^*\mathcal{A}(M)\otimes\mathcal{A}(\mathcal{V})$ through (2.1).

Lemma 2.2 (cf. [7, (4.6)]). $A^Z = \pi^*A^M+\sum e'_i\circ\nabla^{\mathcal{V}}_{e'_i}+A^{\mathcal{V}}-(1/4)c(T)+(1/4)\tilde{c}(T)$

Proof. Set $Q=\nabla^Z-\nabla^{\oplus}$. For horizontal vectors X, Y and a vertical vector U, we have

$$
g^Z(Q(X)U,Y) = -g^Z(Q(X)Y,U) = g^Z(Q(U)X,Y) = \frac{1}{2}g^Z(T(X,Y),U),
\tag{2.5}
$$

and $g^Z(Q(\cdot)\cdot,\cdot)$ vanishes for all other combinations of horizontal and vertical vectors ([12, Lemma 2.1(3)]). Hence, calculating $\nabla^Z_{e_a}=\nabla^{\oplus}_{e_a}+Q(e_a)$, we have

$$
\begin{aligned}
\nabla^Z_{e'_i} &= \pi^*\nabla^M_{e'_i}+\nabla^{\mathcal{V}}_{e'_i}-\frac{1}{2}\sum g^Z(T(e'_i,e'_j),e''_k)(e''_k\otimes f'^j-e'_j\otimes f''^k),\\
\nabla^Z_{e''_k} &= \nabla^{\mathcal{V}}_{e''_k}+\frac{1}{4}\sum g^Z(T(e'_i,e'_j),e''_k)(e'_j\otimes f'^i-e'_i\otimes f'^j).
\end{aligned}
\tag{2.6}
$$

Here $\{f'^i, f''^j\}$ is the dual basis of $\{e'_i, e''_j\}$. These imply

$$\nabla^{\mathbf{S}(Z)}_{e'_i} = \pi^*\nabla^{\mathbf{S}(M)}_{e'_i} + \nabla^{\mathbf{S}(\mathcal{V})}_{e'_i} - \frac{1}{4}\sum g^Z(T(e'_i, e'_j), e''_k)e'_j\circ e''_k\circ,$$
(2.7)
$$\nabla^{\mathbf{S}(Z)}_{e''_k} = \nabla^{\mathbf{S}(\mathcal{V})}_{e''_k} + \frac{1}{8}\sum g^Z(T(e'_i, e'_j), e''_k)e'_i\circ e'_j\circ.$$

Thus we have

$$(2.8)\quad \sum e'_i\circ(\nabla^{\mathbf{S}(Z)}_{e'_i}\otimes 1) + \sum e''_k\circ(\nabla^{\mathbf{S}(Z)}_{e''_k}\otimes 1)$$
$$= \sum e'_i\circ(\pi^*\nabla^{\mathbf{S}(M)}_{e'_i}\otimes 1) + \sum e'_i\circ(\nabla^{\mathbf{S}(\mathcal{V})}_{e'_i}\otimes 1) + \sum e''_k\circ(\nabla^{\mathbf{S}(\mathcal{V})}_{e''_k}\otimes 1) - \frac{1}{4}c(T).$$

Similarly we have

$$(2.9)\quad \sum e'_i\circ(1\otimes\nabla^{\mathbf{S}(Z_0)}_{e'_i}) + \sum e''_k\circ(1\otimes\nabla^{\mathbf{S}(Z_0)}_{e''_k})$$
$$= \sum e'_i\circ(1\otimes\pi^*\nabla^{\mathbf{S}(M_0)}_{e'_i}) + \sum e'_i\circ(1\otimes\nabla^{\mathbf{S}(\mathcal{V})}_{e'_i}) + \sum e''_k\circ(1\otimes\nabla^{\mathbf{S}(\mathcal{V})}_{e''_k})$$
$$+\frac{1}{8}\sum g^Z(T(e'_i, e'_j), e''_k)\{e''_k\circ(1\otimes e'_i\circ e'_j\circ) + 2e'_i\circ(1\otimes e''_k\circ e'_j\circ)\}$$
$$= \sum e'_i\circ(1\otimes\pi^*\nabla^{\mathbf{S}(M_0)}_{e'_i}) + \sum e'_i\circ(1\otimes\nabla^{\mathbf{S}(\mathcal{V})}_{e'_i}) + \sum e''_k\circ(1\otimes\nabla^{\mathbf{S}(\mathcal{V})}_{e''_k}) + \frac{1}{4}\tilde{c}(T).$$

They certainly imply the lemma: see (1.7). Notice that, in our case, the term corresponding to $2^{-1}\langle S(e_i)e_j, f_\alpha\rangle e_i(e_j f_\alpha + e^*_j f^*_\alpha)$ in [7, (4.6)] does not appear. □

Next, we want to find out a similar expression for A^Z_ε. Note that the identification (2.1) depends on the metric g^Z. We need hence to change it into the identification induced by g^Z_ε. It will be obvious then that the result is

$$(2.10)\qquad A^Z_\varepsilon = \varepsilon^{1/2}\left[\pi^*A^M + \sum e'_i\circ\nabla^{\mathcal{V}}_{e'_i}\right] + A^{\mathcal{V}} - \frac{\varepsilon}{4}c(T) + \frac{\varepsilon}{4}\tilde{c}(T).$$

If we pull back the right side which acts on $\pi^*\mathcal{A}(M)\otimes\mathcal{A}(\mathcal{V})$ through (2.1) (induced by g^Z), the resulting operator $\hat{A}^Z_\varepsilon$ does not coincide with the original A^Z_ε. Actually, $\hat{A}^Z_\varepsilon$ is what we obtain by pulling back A^Z_ε through the isomorphism

$$(2.11)\qquad \iota_\varepsilon : \mathcal{A}(Z)\cong\mathcal{A}(Z),\quad \sum c_{\alpha\beta}(f')^\alpha\wedge(f'')^\beta \mapsto \sum c_{\alpha\beta}(\varepsilon^{-1/2}f')^\alpha\wedge(f'')^\beta,$$

where we set $(f')^\alpha = f'^{\alpha_1}\wedge\cdots\wedge f'^{\alpha_i}$ $(\alpha = (\alpha_1,\ldots,\alpha_i))$ etc. Concretely it can be described as $\hat{A}^Z_\varepsilon = \tau_Z(\hat{d}_\varepsilon + \hat{\delta}_\varepsilon)$ with

$$(2.12)\qquad \hat{d}_\varepsilon = \iota^*_\varepsilon d_Z = \varepsilon^{1/2}\pi^*d_M + d_{\mathcal{V}},\quad \hat{\delta}_\varepsilon = \iota^*_\varepsilon\delta_{(Z,g^Z_\varepsilon)} = \varepsilon^{1/2}\pi^*\delta_M + \delta_{\mathcal{V}}.$$

3. Proof of Theorem 0.1

The η-function of A_ε^Z is defined by

$$\eta(A_\varepsilon^Z)(s) = \frac{1}{\Gamma((s+1)/2)} \int_0^\infty t^{(s-1)/2} \operatorname{Tr}\left(A_\varepsilon^Z e^{-t(A_\varepsilon^Z)^2}\right) dt, \quad \operatorname{Re} s \gg 0. \tag{3.1}$$

By analytic continuation to the whole complex plane we obtain a meromorphic function, which is regular at $s = 0$ ([1]). The η-invariant of A_ε^Z is the value at $s = 0$, i.e.,

$$\eta(A_\varepsilon^Z) = \eta(A_\varepsilon^Z)(0) = \frac{1}{\sqrt{\pi}} \int_0^\infty t^{-1/2} \operatorname{Tr}\left(A_\varepsilon^Z e^{-t(A_\varepsilon^Z)^2}\right) dt. \tag{3.2}$$

Note that $\operatorname{Tr}(A_\varepsilon^Z \exp(-t(A_\varepsilon^Z)^2)) = O(t^{1/2})$ as $t \to 0$ ([4, (2.13)]) so that the above integral expression is well-defined.

We begin with investigating the limiting behavior of $\operatorname{Tr}(A_\varepsilon^Z \exp(-t(A_\varepsilon^Z)^2))$ when $\varepsilon \to 0$. Let us take a $\mathbb{Z}_2$-graded infinite dimensional vector bundle $H^\infty = H_+^\infty \oplus H_-^\infty$ over M defined by

$$H_{\pm,x}^\infty = \mathcal{A}^\pm(\mathcal{V})|\pi^{-1}(x) \tag{3.3}$$

at each $x \in M$. The obvious functorial isomorphism

$$\mathcal{A}(\mathcal{V}) \cong \Gamma(H^\infty), \quad \psi \leftrightarrow \tilde{\psi}, \quad \tilde{\psi}(x) = (\pi^{-1}(x) \ni z \mapsto \psi(z)) \tag{3.4}$$

induces a fibrewise hermitian metric on H^∞ as $(\tilde{\psi}_1, \tilde{\psi}_2)_x = \int_{\pi^{-1}(x)} \psi_1 \wedge *_{\mathcal{V}} \overline{\psi_2}$. Now, observing the expression (2.10) for A_ε^Z, we will take a unitary superconnection on H^∞

$$B_t = B_{[1]} + t^{1/2} B_{[0]} + t^{-1/2} B_{[2]} = \tilde{\nabla}^{\mathcal{V}} + t^{1/2} A^{\mathcal{V}} - \frac{1}{4t^{1/2}} \hat{c}(T), \tag{3.5}$$

where we set $\tilde{\nabla}^{\mathcal{V}}_{e_i'} \tilde{\psi} = (\nabla^{\mathcal{V}}_{e_i'} \psi)\tilde{}$ and $\hat{c}(T) = \sum_{i \le j} f'^i \wedge f'^j \wedge \otimes c(T)(e_i', e_j')$ (a $\mathbb{C}l(\mathcal{V})$-valued 2-form on M).

Let us explain here the origin of (3.5): see [3, 4]. Consider an obvious functorial isomorphism

$$\pi^* \mathcal{A}(M) \otimes \mathcal{A}(\mathcal{V}) \cong \Gamma(\wedge(M) \otimes H^\infty), \quad \pi^*\phi \otimes \psi \leftrightarrow \phi \otimes \tilde{\psi} \tag{3.6}$$

and take a superconnection

$$\mathbf{B}_t = \nabla^M \otimes 1 + 1 \otimes B_t \tag{3.7}$$

on the right side. Through (2.1) and (3.6) this may be seen as a superconnection on $\wedge(Z)$. $A_\varepsilon^Z - \varepsilon \tilde{c}(T)/4$ is then just the quantization of $\varepsilon^{1/2} \mathbf{B}_{1/\varepsilon}$, i.e., what is obtained by replacing the wedge products $f'^i\wedge$, $f'^i\wedge f'^j\wedge$, etc. by the Clifford products $e'_i\circ$, $e'_i\circ e'_j\circ$, etc. The superconnection (3.5) was taken so that it fits such a framework.

If $i\neq 1$, then we have

$$B_{[i]}\in\Gamma(M,\wedge^i(M)\otimes\mathrm{End}\,(H^\infty)). \tag{3.8}$$

Further, we may think of the curvature B_t^2 as being (the operator given by) an element of $\Gamma(M,\wedge(M)\otimes\mathrm{End}\,(H^\infty))$ so that the heat operator $e^{-B_t^2}$ can be regarded also as (the operator given by) an element of it ([2, Proposition 1.38]),

$$e^{-B_t^2}\in\Gamma(M,\wedge(M)\otimes\mathrm{End}\,(H^\infty)). \tag{3.9}$$

We set $\mathrm{End}^j(H^\infty)=\mathrm{Hom}\,(H^\infty_\pm,H^\infty_\pm)$ (if $j=0$), $\mathrm{Hom}\,(H^\infty_\pm,H^\infty_\mp)$ (if $j=1$), and say that the elements of $\Gamma(M,\wedge^i(M)\otimes\mathrm{End}^j(H^\infty))$ are of total degree $i+j$. Then $B_{[i]}$ $(i\neq 1)$ is of odd total degree. Further B_t^2 is of even total degree and so is the heat operator $e^{-B_t^2}$. The fibrewise supertrace

$$\hat{\eta}(t)=\mathrm{str}\left[\left(A^{\mathcal{V}}+\frac{\hat{c}(T)}{4t}\right)e^{-B_t^2}\right] \tag{3.10}$$

on H^∞ is thus an odd degree form on M. We denote by $[\hat{\eta}(t)]_{2j-1}$ its homogeneous component of degree $2j-1$, and set

$$\tilde{\eta}(t)=\sum\frac{1}{(2\pi\sqrt{-1})^j}\,[\hat{\eta}(t)]_{2j-1}\,. \tag{3.11}$$

Then, in the same way as the proof of [3, (4.40)], we can prove

Lemma 3.1. *We have uniform convergence as* $\varepsilon\to 0$

$$\mathrm{Tr}\,(A^Z_\varepsilon e^{-t(A^Z_\varepsilon)^2})=2^n\sqrt{\pi}\int_M\hat{\mathbb{L}}(\Omega^M)\wedge\tilde{\eta}(t)\;+\;O(\varepsilon^{1/2}(1+t^N)) \tag{3.12}$$

for some $N>0$.

Proof. A^M may be seen as a twisted Dirac operator $D^M\otimes\mathbf{S}(M_0)$: see (1.7). Hence it is easily shown in the same way as the proof of [3, (4.40)] that the left side of (3.12) can be expanded when $\varepsilon\to 0$ as follows:

$$\frac{\sqrt{\pi}}{(2\pi\sqrt{-1})^n}\int_M\hat{\mathbb{A}}(2\pi\sqrt{-1}\Omega^M)\,\mathrm{Tr}(\exp(-\Omega^{\mathbf{S}(M_0)}))\wedge\hat{\eta}(t)\;+\;O(\varepsilon^{1/2}(1+t^N)), \tag{3.13}$$

where $\hat{\mathbb{A}}(2\pi\sqrt{-1}\Omega^M)$ is the (renormalized) $\hat{\mathbb{A}}$-genus form associated to Ω^M, i.e., $\hat{\mathbb{A}}(2\pi\sqrt{-1}\Omega^M)=\det^{1/2}\left((\Omega^M/2)/\sinh(\Omega^M/2)\right)$, and $\Omega^{\mathbf{S}(M_0)}$ is the curvature of $\nabla^{\mathbf{S}(M_0)}$.

Notice that the left side of (3.12) contains the term $\varepsilon\tilde{c}(T)/4$, which, however, does not contribute to the first term in (3.13). This is because, in the Getzler's transformation (a rescaling method) which induces the first term, we only rescale the left Clifford variables but not the right ones. The first term is then equal to

$$\begin{aligned}&\frac{\sqrt{\pi}}{(2\pi\sqrt{-1})^n}\int_M 2^n\hat{\mathbb{L}}(2\pi\sqrt{-1}\Omega^M)\wedge\hat{\eta}(t)\\ =&2^n\sqrt{\pi}\int_M\sum_{2i+j=n}\frac{1}{(2\pi\sqrt{-1})^{2i}}\hat{\mathbb{L}}(2\pi\sqrt{-1}\Omega^M)\wedge\frac{1}{(2\pi\sqrt{-1})^j}[\hat{\eta}(t)]_{2j-1}\\ =&2^n\sqrt{\pi}\int_M\hat{\mathbb{L}}(\Omega^M)\wedge\tilde{\eta}(t).\end{aligned}$$

□

Next, let us consider the family of signature operators along the fibres given in (2.4). It is easily verified by the fact $\mathrm{Ker}(A^{\mathcal{V}}|H_x^\infty)\cong\mathrm{Ker}(d_{\mathbb{C}P^1}+\delta_{\mathbb{C}P^1})\cong H^*_{DR}(\mathbb{C}P^1)$ that its index bundle

$$\mathrm{Ind}\,A^{\mathcal{V}}=\coprod_{x\in M}\mathrm{Ker}(A^{\mathcal{V}}|H_x^\infty)=\mathrm{Ind}\,A_+^{\mathcal{V}}\oplus\mathrm{Ind}\,A_-^{\mathcal{V}} \tag{3.14}$$

is a trivial two dimensional $\mathbb{Z}_2$-graded vector bundle over M with canonical cross-sections $(1\pm\tau_{\mathcal{V}}1)\breve{}\in\Gamma(\mathrm{Ind}\,A_\pm^{\mathcal{V}})$, where $\tau_{\mathcal{V}}$ is the family of complex $*$-operators along the fibres. The orthogonal projection of $\tilde{\nabla}^{\mathcal{V}}$ to the subbundle $\mathrm{Ind}\,A^{\mathcal{V}}$ gives its connection ∇^{Ind}, which obviously equals just the exterior differential on M. Now let us take a twisted signature operator

$$A^M\otimes\mathrm{Ind}\,A^{\mathcal{V}}=\sum e_i'\circ\left(\nabla^M_{e_i'}\otimes 1+1\otimes\nabla^{\mathrm{Ind}}_{e_i'}\right) \tag{3.15}$$

acting on $\mathcal{A}(M)\otimes\Gamma(\mathrm{Ind}\,A^{\mathcal{V}})$.

Lemma 3.2. $\eta(A^M\otimes\mathrm{Ind}\,A^{\mathcal{V}})=0$

Proof. Obviously we have an identification

$$\mathcal{A}(M)\otimes\Gamma(\mathrm{Ind}\,A^{\mathcal{V}})\cong\mathcal{A}(M)\oplus\mathcal{A}(M) \tag{3.16}$$

given by $\phi_+\otimes(1+\tau_{\mathcal{V}}1)\breve{}+\phi_-\otimes(1-\tau_{\mathcal{V}}1)\breve{}\leftrightarrow(\phi_+,\phi_-)$. Since the action $e_i'\circ$ on the left side means $e_i'\circ\otimes\tau^{\mathcal{V}}\circ$ (refer to Lemma 2.1), we may identify accordingly

$$A^M\otimes\mathrm{Ind}\,A^{\mathcal{V}}=A^M\oplus(-A^M). \tag{3.17}$$

Hence we have $\eta(A^M\otimes\mathrm{Ind}\,A^{\mathcal{V}})=\eta(A^M)-\eta(A^M)=0$. □

Now, applying [7, Corollary 4.1] to A^Z_ε, we will investigate the limit of $\eta(A^Z_\varepsilon)$. Since we have $\operatorname{Ker} A^Z_\varepsilon \cong \operatorname{Ker}(d_Z + \delta_{(Z,g^Z_\varepsilon)}) \cong H^*_{DR}(Z)$, the dimension of $\operatorname{Ker} A^Z_\varepsilon$ is constant. Hence $\lim_{\varepsilon\to 0}\eta(A^M_\varepsilon)$ really exists (refer to the argument following (3.21)) and we have

$$
\begin{aligned}
(3.18)\quad \lim_{\varepsilon\to 0}\eta(A^Z_\varepsilon) &= \lim_{\varepsilon\to 0}\frac{1}{\sqrt{\pi}}\int_0^\infty t^{-1/2}\operatorname{Tr}\left(A^Z_\varepsilon e^{-t(A^Z_\varepsilon)^2}\right)dt \\
&= \frac{1}{\sqrt{\pi}}\int_0^\infty t^{-1/2}\lim_{\varepsilon\to 0}\operatorname{Tr}\left(A^Z_\varepsilon e^{-t(A^Z_\varepsilon)^2}\right)dt + \eta(A^M\otimes \operatorname{Ind} A^{\mathcal{V}}) + \lim_{\varepsilon\to 0}\sum\nolimits_{(0)}\operatorname{sgn}\lambda_\varepsilon \\
&= 2^{n+1}\int_M \hat{\mathbb{L}}(\Omega^M)\wedge\tilde{\eta} + \lim_{\varepsilon\to 0}\sum\nolimits_{(0)}\operatorname{sgn}\lambda_\varepsilon,
\end{aligned}
$$

where the last equality is due to Lemmata 3.1 and 3.2 with

$$
(3.19)\qquad \tilde{\eta} = \int_0^\infty \tilde{\eta}(t)\frac{dt}{2t^{1/2}},
$$

which is convergent because of $\tilde{\eta}(t) = O(t^{-1})$ $(t\to\infty)$, $O(1)$ $(t\to 0)$ ([2, Theorems 9.23 and 10.32(1)]). Hence, to finish the proof of (0.6), it suffices to show that $\lim_{\varepsilon\to 0}\sum_{(0)}\operatorname{sgn}\lambda_\varepsilon$ vanishes.

Let us begin with the explanation of the (finite) summation $\sum_{(0)}\operatorname{sgn}\lambda_\varepsilon$ ([7, Theorem 1.5]). The eigenvalues of A^Z_ε are analytic with respect to $\varepsilon^{1/2}$ on $\varepsilon > 0$. Namely, there exist (infinitely many) functions λ_ε analytic with respect to $\varepsilon^{1/2}$ on $\varepsilon > 0$ such that the spectrum of A^Z_ε, $\operatorname{Spec}(A^Z_\varepsilon)$, equals $\{\lambda_\varepsilon\}$ for each $\varepsilon > 0$. Moreover, λ_ε with $\lim_{\varepsilon\to 0}\lambda_\varepsilon = 0$ may be extended to a function smooth (may not analytic) with respect to $\varepsilon^{1/2}$ up to $\varepsilon^{1/2} = 0$ so that it has an asymptotic expansion when $\varepsilon\to 0$

$$
(3.20)\qquad \lambda_\varepsilon \sim c_1(\lambda)\varepsilon^{1/2} + c_2(\lambda)(\varepsilon^{1/2})^2 + c_3(\lambda)(\varepsilon^{1/2})^3 + \cdots .
$$

The coefficient $c_1(\lambda)$ is an eigenvalue of $A^M\otimes\operatorname{Ind}A^{\mathcal{V}}$ and the map $\lambda_\varepsilon\mapsto c_1(\lambda)$ defines a bijective correspondence between $\{\lambda_\varepsilon \mid \lim_{\varepsilon\to 0}\lambda_\varepsilon = 0\}$ and $\operatorname{Spec}(A^M\otimes\operatorname{Ind}A^{\mathcal{V}})$ (with multiplicity). Now, taking only λ_ε corresponding to the zero-eigenvalues of $A^M\otimes \operatorname{Ind}A^{\mathcal{V}}$, we set $\operatorname{sgn}\lambda_\varepsilon = 1$ (if $\lambda_\varepsilon > 0$), $= -1$ (if $\lambda_\varepsilon < 0$), or $= 0$ (if $\lambda_\varepsilon = 0$), which sum up to $\sum_{(0)}\operatorname{sgn}\lambda_\varepsilon$. Put

$$
(3.21)\qquad \rho(\varepsilon) = \sum\nolimits_{(0)}\operatorname{sgn}\lambda_\varepsilon .
$$

Since $\dim\operatorname{Ker}A^Z_\varepsilon$ is constant and λ_ε are analytic, each λ_ε in (3.21) is nowhere zero or identically zero so that $\rho(\varepsilon)$ is constant, which certifies the existence of $\lim_{\varepsilon\to 0}\eta(A^Z_\varepsilon)$. We denote the constant by ρ, i.e.,

$$
(3.22)\qquad \rho = \lim_{\varepsilon\to 0}\rho(\varepsilon) = \lim_{\varepsilon\to 0}\sum\nolimits_{(0)}\operatorname{sgn}\lambda_\varepsilon .
$$

Before studying ρ, we want to make an important comment. It will be clear that, in the argument above and below, we may replace A_ε^Z by $\hat{A}_\varepsilon^Z$ given in (2.12) and, furthermore, the study becomes quite manageable by replacing so. Actually, for example, as explained in the comment preceding (2.11), though the inner product of $\mathcal{A}(Z)$ on which A_ε^Z acts varies according to ε, the inner product of $\mathcal{A}(Z)$ on which $\hat{A}_\varepsilon^Z$ acts does not vary fortunately. In the following we proceed with the argument using $\hat{A}_\varepsilon^Z$, so that $\{\lambda_\varepsilon\} = \mathrm{Spec}(\hat{A}_\varepsilon^Z)$ and, more important, the metric of Z is g^Z (not g_ε^Z).

Now we denote by $E(\lambda_\varepsilon)$ the eigenspace associated to λ_ε (of $\hat{A}_\varepsilon^Z$) in (3.21). Take $r \geq 2$ and define $E_r(\varepsilon)$ to be the direct sum of $E(\lambda_\varepsilon)$ corresponding to λ_ε with $c_i(\lambda) = 0$ for all $i \leq r-1$ in the expansion (3.20). It is proved in [7, Proposition 4.2] that $E_r(\varepsilon)$, $\varepsilon^{1/2} > 0$, is a family of finite dimensional vector space that depends smoothly on $\varepsilon^{1/2}$ down to $\varepsilon^{1/2} = 0$, that is, there exist smooth forms $\varphi_{1,\varepsilon}, \ldots, \varphi_{k,\varepsilon}$ ($\varepsilon^{1/2} \geq 0$) which are orthonormal (with respect to g^Z) to each other and generate $E_r(\varepsilon)$ for each $\varepsilon^{1/2} > 0$. Hence the limit

$$E_r = \lim_{\varepsilon \to 0} E_r(\varepsilon) \tag{3.23}$$

is a well-defined finite dimensional vector space. Further the following map is well-defined (see the proof of [7, Theorem 0.2]):

$$d_r : E_r \to E_r, \quad \phi = \lim_{\varepsilon \to 0} \phi_\varepsilon \mapsto d_r\phi = \lim_{\varepsilon \to 0} \varepsilon^{-r/2} \hat{d}_\varepsilon \phi_\varepsilon, \tag{3.24}$$

where $\hat{d}_\varepsilon$ is given in (2.12). We may state now an important lemma.

Lemma 3.3 (Dai [7, Theorem 0.2]).
(1) $\{(E_r, d_r)\}_{r \geq 2}$ *forms a spectral sequence.*
(2) *The Dai's spectral sequence* $\{(E_r, d_r)\}_{r \geq 2}$ *is isomorphic to the Leray spectral sequence* $\{(\overline{E}_r, \overline{d}_r)\}_{r \geq 2}$ *of the fibration* (0.2) *through the canonical map* $E_r \ni \phi \mapsto [\phi] \in \overline{E}_r$.

By investigating the Leray spectral sequence we have

Lemma 3.4. *The Dai's spectral sequence degenerates at $r = 2$ in our case.*

Proof. It is obvious that the local coefficient system $H_{DR}^*(\mathcal{V})$ consisting of de Rham cohomology rings $H_{DR}^*(\pi^{-1}(x))$ (with coefficients in $\mathbb{C}$) is trivial. The general theory implies then that the Leray spectral sequence degenerates at $r = 2$. ☐

Now we can prove

Lemma 3.5. $\rho = 0$

Proof. The lemma follows directly from Lemma 3.4 and the Dai's topological formula ([7, §4.3]) for ρ, i.e., set $E_r^{(n)} = \lim_{\varepsilon\to 0} E_r(\varepsilon) \cap \hat{\delta}_\varepsilon \mathcal{A}^{n+1}(Z)$, then we have

$$\rho = \sum_{r\geq 2} \rho_r = 2\sum_{r\geq 2} \operatorname{sgn}\left(E_r^{(n)} \times E_r^{(n)} \to \mathbb{C}\right) \tag{3.25}$$

where the pairing is given by $(\phi, \psi) \mapsto \int_Z \phi \wedge \overline{d_r\psi}$ (if $2n+1 = 4m-1$), $(\phi, \psi) \mapsto \sqrt{-1}\int_Z \phi\wedge\overline{d_r\psi}$ (if $2n+1 = 4m+1$). (The proof of the formula in the case $2n+1 = 4m+1$ is similar.) □

(3.18) and Lemma 3.5 imply now (0.6). Next we will show (0.7) for (3.19).

Lemma 3.6 (cf. [2, Corollary 9.22 and Theorem 10.23], etc.).

$$\lim_{t\to\infty} \operatorname{str}\left(e^{-B_t^2}\right) = 0, \tag{3.26}$$

$$\lim_{t\to 0} \operatorname{str}\left(e^{-B_t^2}\right) = (\pi\sqrt{-1})^{-1}\int_{Z/M} \hat{\mathbb{L}}(2\pi\sqrt{-1}\Omega^{\mathcal{V}}). \tag{3.27}$$

Proof. Since the connection ∇^{Ind} on $\operatorname{Ind} A^{\mathcal{V}}$ is trivial, its curvature Ω^{Ind} vanishes. Hence [2, Corollary 9.22] implies

$$\lim_{t\to\infty} \operatorname{str}\left(e^{-B_t^2}\right) = \operatorname{str}\left(\exp(-\Omega^{\mathrm{Ind}})\right) = 1 - 1 = 0. \tag{3.28}$$

On the other hand, if we denote by $\Omega^{\mathbf{S}(\mathcal{V})}$ the curvature of $\nabla^{\mathbf{S}(\mathcal{V})}$, [2, Theorem 10.23] implies

$$\lim_{t\to 0} \operatorname{str}\left(e^{-B_t^2}\right) = (2\pi\sqrt{-1})^{-1}\int_{Z/M} \hat{\mathbb{A}}(2\pi\sqrt{-1}\Omega^{\mathcal{V}})\operatorname{Tr}(\exp(-\Omega^{\mathbf{S}(\mathcal{V})})), \tag{3.29}$$

which obviously equals the right side of (3.27). □

Proof of (0.7) for (3.19). The transgression formula ([2]) for the superconnection B_t says

$$\frac{\partial}{\partial t}\operatorname{str}\left(e^{-B_t^2}\right) = -d\,\frac{\hat{\eta}(t)}{2t^{1/2}}. \tag{3.30}$$

Hence, by putting $\hat{\eta} = \int_0^\infty \hat{\eta}(t)/(2t^{1/2})dt$, we have

$$d\hat{\eta} = \lim_{t\to 0}\operatorname{str}\left(e^{-B_t^2}\right) - \lim_{t\to\infty}\operatorname{str}\left(e^{-B_t^2}\right) = (\pi\sqrt{-1})^{-1}\int_{Z/M}\hat{\mathbb{L}}(2\pi\sqrt{-1}\Omega^{\mathcal{V}}). \tag{3.31}$$

The second equality is due to Lemma 3.6. Thus we obtain (0.7). □

4. Some twisted signature operators

(Z, g^Z) has a natural Spin^c structure ([12])

$$\xi^c = (\xi^c_0, \xi^c_1) : P_{\mathrm{Spin}^c(2n+1)}(Z) \to P_{SO(2n+1)}(Z) \times P_{U(1)}(Z), \tag{4.1}$$

which is constructed as follows: The map $P_{\mathrm{Spin}^q(2n-1)} \to Z,\ p_x \mapsto [p_x, [1]]$, obviously has a structure of principal Spin$^c(2n-1)$-bundle, whose total space is denoted by $P_{\mathrm{Spin}^c(2n-1)}(Z)$. This gives a Spin^c structure of $\pi^* P_{SO(2n-1)}$, which is isomorphic by π to the reduced structure bundle of $(\mathcal{H}, g^Z|\mathcal{H})$,

$$\xi^c : P_{\mathrm{Spin}^c(2n-1)}(Z) \to \pi^* P_{SO(2n-1)} \times P^{\mathcal{H}}_{U(1)}(Z). \tag{4.2}$$

On the other hand, the reduced structure bundle $P^{\mathcal{V}}_{SO(2)}(Z)$ of $(\mathcal{V}, g^{\mathcal{V}})$ has a canonical Spin^c structure ([10, Example D.6])

$$\xi^c : P^{\mathcal{V}}_{\mathrm{Spin}^c(2)}(Z) \to P^{\mathcal{V}}_{SO(2)} \times P^{\mathcal{V}}_{U(1)}(Z). \tag{4.3}$$

Here $P^{\mathcal{V}}_{U(1)}(Z)$ is the set of unitary frames with respect to the hermitian complex bundle structure $(ds^{\mathcal{V}}, J^{\mathcal{V}})$ induced from the canonical $(ds^2, J^{\mathbb{C}P^1})$ of $\mathbb{C}P^1$. We may regard Spin$^c(2n-1)$ and Spin$^c(2)$ as subgroups of Spin$^c(2n+1)$ through the inclusions $\mathbb{R}^{2n-1}, \mathbb{R}^2 \hookrightarrow \mathbb{R}^{2n+1}$ and define a group homomorphism mult : Spin$^c(2n-1) \times$ Spin$^c(2) \to$ Spin$^c(2n+1)$ by multiplication in Spin$^c(2n+1)$. Then we set

$$P_{\mathrm{Spin}^c(2n+1)}(Z) = \left(P_{\mathrm{Spin}^c(2n-1)}(Z) \times P^{\mathcal{V}}_{\mathrm{Spin}^c(2)}(Z)\right) \times_{\mathrm{mult}} \mathrm{Spin}^c(2n+1). \tag{4.4}$$

We have hence $P_{U(1)}(Z) = P^{\mathcal{H}}_{U(1)}(Z) \otimes P^{\mathcal{V}}_{U(1)}(Z)$.

Further (Z, g^Z) admits remarkably a canonical Spin structure ([13])

$$\xi : P_{\mathrm{Spin}(2n+1)}(Z) \to P_{SO(2n+1)}(Z), \tag{4.5}$$

which is uniquely determined (if it exists) by the condition that there exists an isomorphism

$$P_{\mathrm{Spin}^c(2n+1)}(Z) \cong P_{\mathrm{Spin}(2n+1)}(Z) \times_{\mathrm{can}} \mathrm{Spin}^c(2n+1). \tag{4.6}$$

Thus (4.1) is just a trivial Spin^c structure induced from the canonical action of Spin$(2n+1)$ on Spin$^c(2n+1)$, so that we have $P_{U(1)}(Z) \cong U(1)_Z$ (trivial).

It will suffice to manage only (4.5) from the standpoint of studying only topological invariants such as indices, but η-invariant is not such a invariant, nor is its adiabatic limit. Actually it is not permitted, in Theorem 0.1, to replace $\hat{\mathbb{L}}(\Omega^M)$, $\hat{\mathbb{L}}(\Omega^{\mathcal{V}})$ by their cohomology classes $\hat{\mathbb{L}}(M)$, $\hat{\mathbb{L}}(\mathcal{V})$. Accordingly, if there exist some nontrivial intrinsic connections on $P_{U(1)}(Z)$ which is trivial, then certain intrinsic properties of Z will be reflected naturally in the signature operators twisted by such connections. In particular in connection with Theorem 0.1, it will be of interest to investigate the limits of their adiabatic versions.

Let us introduce now such an interesting connection on it. $P_{\mathrm{Spin}^c(2n-1)}(Z) \equiv P_{\mathrm{Spin}^q(2n-1)}$ can be embedded as a subbundle into $\pi^* P_{\mathrm{Spin}^q(2n-1)}$ by the map $p_x \mapsto ([p_x, [1]], p_x)$: see [12, Lemma 1.3]. Accordingly $P^{\mathcal{H}}_{U(1)}(Z) = P_{\mathrm{Spin}^c(2n-1)}(Z) \times_{\xi_1^c} U(1)$ can be embedded as a subbundle into $\pi^* P_{SO(3)} = \pi^* P_{\mathrm{Spin}^q(2n-1)} \times_{\xi_1^q} SO(3)$ naturally, where ξ_1^q is the adjoint action of $Sp(1)$ on $\mathfrak{sp}(1) = \mathbb{R}^3$. $U(1) = SO(2)$ is then canonically reductive in $SO(3)$, i.e., there is a natural splitting $\mathfrak{so}(3) = \mathfrak{u}(1) \oplus \mathfrak{m}$ with $\mathrm{Ad}(U(1))\mathfrak{m} \subset \mathfrak{m}$. Hence, the $\mathfrak{u}(1)$-component of $\pi^* \alpha_{SO(3)}$ restricted to the subbundle gives its connection $\alpha^{\mathcal{H}}_{U(1)}$. On the other hand, since the Ehresmann connection $\alpha^{\mathcal{V}}$ associated to $\nabla^{\mathcal{V}}$ is unitary with respect to $(ds^{\mathcal{V}}, J^{\mathcal{V}})$ ([12, Lemma 2.1 (4)]), it induces a connection $\alpha^{\mathcal{V}}_{U(1)}$ on $P^{\mathcal{V}}_{U(1)}(Z)$. Thus we obtain a connection

$$\alpha_{U(1)} = \alpha^{\mathcal{H}}_{U(1)} \otimes 1 + 1 \otimes \alpha^{\mathcal{V}}_{U(1)} \tag{4.7}$$

on $P_{U(1)}(Z) = P^{\mathcal{H}}_{U(1)}(Z) \otimes P^{\mathcal{V}}_{U(1)}(Z)$.

Let us twist $A^Z_{(\varepsilon)}$ using $\alpha_{U(1)}$. Using the standard representation $r_C : U(1) \to GL_{\mathbb{C}}(\mathbb{C}) \equiv GL_{\mathbb{C}}(C)$ we define a spinor bundle

$$\mathbf{S}^c(Z) = P_{\mathrm{Spin}^c(2n+1)}(Z) \times_{\Delta\otimes r_C} S^c_{2n+1}, \quad S^c_{2n+1} = S_{2n+1} \otimes C, \tag{4.8}$$

to which we attach a covariant derivative $\nabla^{\mathbf{S}^c(Z)}$ associated to the connection $\alpha^{Z,c} = \xi^{c*}(\alpha^Z \oplus \alpha_{U(1)})$. Since $P_{\mathrm{Spin}^c(2n+1)}(Z) \times_{r_C} C = P_{\mathrm{Spin}(2n+1)}(Z) \times_{\mathrm{can}} (\mathrm{Spin}^c(2n+1) \times_{r_C} C) = \mathbb{C}_Z$ (trivial), we have

$$\mathbf{S}^c(Z) \cong \mathbf{S}(Z) \otimes \mathbb{C}_Z \cong \mathbf{S}(Z) \equiv P_{\mathrm{Spin}(2n+1)}(Z) \times_\Delta S_{2n+1}. \tag{4.9}$$

$\mathbf{S}^c(Z)$ with $\nabla^{\mathbf{S}^c(Z)}$ may be thus regarded as $\mathbf{S}(Z)$ with a twisted covariant derivative $\nabla^{\mathbf{S}(Z),c}$. In this manner, we obtain a twisted signature operator

$$\begin{aligned} A^{Z,c} = D^{Z,c} \otimes \mathbf{S}(Z_0) &= \sum e_a \circ \left(\nabla^{\mathbf{S}(Z),c}_{e_a} \otimes 1 + 1 \otimes \nabla^{\mathbf{S}(Z_0)}_{e_a} \right) \\ &= \sum e_a \circ \left(\nabla^{\mathbf{S}^c(Z)}_{e_a} \otimes 1 + 1 \otimes \nabla^{\mathbf{S}(Z_0)}_{e_a} \right) \end{aligned} \tag{4.10}$$

on

$$\wedge(Z) \cong \mathbf{S}(Z) \otimes \mathbf{S}(Z_0) \cong \mathbf{S}^c(Z) \otimes \mathbf{S}(Z_0). \tag{4.11}$$

In the following we want to investigate the limit of η-invariant of its adiabatic version $A^{Z,c}_\varepsilon$. But we face a problem here. Namely, the dimension of kernel of the twisted one may vary according to ε. I hope it does not. (If it does not, then the following conclusion can be more strengthened.) But I am not sure so far. We will lay down a scheme which relieves us from such a concern. To do so, let us assume for a moment that the dimension varies really. Precisely, we assume that, for any $\varepsilon_0 > 0$, it is not constant on the interval $(0, \varepsilon_0)$. Take eigenvalues λ^c_ε of $A^{Z,c}_\varepsilon$ and set $\rho^c(\varepsilon) = \sum_{(0)} \operatorname{sgn} \lambda^c_\varepsilon$ as in (3.21). then $\rho^c(\varepsilon)$ has no limit when $\varepsilon \to 0$, nor has $\eta(A^{Z,c}_\varepsilon)$ consequently. Thus we cannot investigate $\lim_{\varepsilon\to 0} \eta(A^{Z,c}_\varepsilon)$ naively. Here it is wise to perceive the fact that $\rho^c(\varepsilon) + \dim \operatorname{Ker} A^{Z,c}_\varepsilon$ is constant modulo $2\mathbb{Z}$. Accordingly we will set

$$\bar{\eta}(A^{Z,c}_\varepsilon) = \frac{1}{2}\left\{\dim \operatorname{Ker} A^{Z,c}_\varepsilon + \eta(A^{Z,c}_\varepsilon)\right\}, \tag{4.12}$$

called the reduced η-invariant, and investigate its limit in $\mathbb{R}/\mathbb{Z}$. It will be obvious that this limit (mod $\mathbb{Z}$) may exist even if the limit of $\rho^c(\varepsilon)$ does not.

Let us take hermitian complex line bundles

$$\mathbf{L}_{\mathcal{H}} = P^{\mathcal{H}}_{U(1)}(Z) \times_{rc} C, \quad \mathbf{L}_{\mathcal{V}} = P^{\mathcal{V}}_{U(1)}(Z) \times_{rc} C \tag{4.13}$$

with covariant derivatives $\nabla^{\mathbf{L}_{\mathcal{H}}}$, $\nabla^{\mathbf{L}_{\mathcal{V}}}$ induced from $\alpha^{\mathcal{H}}_{U(1)}$, $\alpha^{\mathcal{V}}_{U(1)}$. We denote their curvatures by $\Omega^{\mathcal{H}}$, $\Omega^{\mathcal{V}}$, which take values in $\mathfrak{u}(1)$. (Note that we use the same symbol $\Omega^{\mathcal{V}}$ to denote the curvatures of $\nabla^{\mathbf{L}_{\mathcal{V}}}$ and $\nabla^{\mathcal{V}}$ because the latter covariant derivative is the underlying real one of the former.) Define then their first Chern forms by $c_1(\Omega^{\mathcal{H}}) = \operatorname{tr}(\sqrt{-1}\Omega^{\mathcal{H}}/2\pi) = \sqrt{-1}\Omega^{\mathcal{H}}/2\pi$ etc. We have now

Theorem 4.1. *The* (*adiabatic*) *limit* $\lim_{\varepsilon\to 0} \bar{\eta}(A^{Z,c}_\varepsilon)$ *exists in* $\mathbb{R}/\mathbb{Z}$ *and there is an odd degree form* $\tilde{\eta}^l$ *on* M *such that*

$$\lim_{\varepsilon\to 0} \bar{\eta}(A^{Z,c}_\varepsilon) \equiv 2^n \int_M \hat{\mathbb{L}}(\Omega^M) \wedge \tilde{\eta}^l \pmod{\mathbb{Z}}, \tag{4.14}$$

$$d\tilde{\eta}^l = 2\int_{Z/M} \hat{\mathbb{L}}(\Omega^{\mathcal{V}}) \wedge \exp\left(\frac{1}{2}c_1(\Omega^{\mathcal{V}}) + \frac{1}{2}c_1(\Omega^{\mathcal{H}})\right). \tag{4.15}$$

Proof. First of all, let us attach the standard fibre metric to the trivial line bundle $\mathbb{C}_Z$. Then we have a natural hermitian line bundle isomorphism $\mathbf{L}_{\mathcal{H}} \otimes \mathbf{L}_{\mathcal{V}} \cong \mathbb{C}_Z$ and, hence,

$$\mathbf{L}_{\mathcal{H}} \cong \mathbf{L}^*_{\mathcal{V}}. \tag{4.16}$$

Furthermore, it follows from [13, Proposition 3.3] that the covariant derivatives $\nabla^{\mathbf{L}_{\mathcal{H}}}$, $\nabla^{\mathbf{L}^*_{\mathcal{V}}}$ (induced from $\nabla^{\mathbf{L}_{\mathcal{V}}}$) restricted to the fibres coincide with each other through (4.16), i.e.,

$$\nabla^{\mathbf{L}_{\mathcal{H}}}|\pi^{-1}(x) = \nabla^{\mathbf{L}^*_{\mathcal{V}}}|\pi^{-1}(x). \tag{4.17}$$

Let us take locally defined spinor bundles $\mathbf{S}(M)$ and $\mathbf{S}(\mathcal{V})$ with $\mathbf{S}(Z) = \pi^*\mathbf{S}(M) \otimes \mathbf{S}(\mathcal{V})$

and, moreover, locally defined vector bundles

$$l_{\mathcal{H}} = P_{\mathrm{Spin}^c(2n-1)}(Z) \times_{rc} C, \quad l_{\mathcal{V}} = P^{\mathcal{V}}_{\mathrm{Spin}^c(2)}(Z) \times_{rc} C \tag{4.18}$$

with $\mathbf{S}^c(\mathcal{H}) \equiv P_{\mathrm{Spin}^c(2n-1)}(Z) \times_{\Delta\otimes rc} S^c = \pi^*\mathbf{S}(M) \otimes l_{\mathcal{H}}$ and $\mathbf{S}^c(\mathcal{V}) \equiv P^{\mathcal{V}}_{\mathrm{Spin}^c(2)}(Z) \times_{\Delta\otimes rc} S^c = \mathbf{S}(\mathcal{V}) \otimes l_{\mathcal{V}}$. Then it will be clear that there exist global canonical hermitian line bundle isomorphisms

$$l_{\mathcal{H}} \otimes l_{\mathcal{H}} \cong \mathbf{L}_{\mathcal{H}}, \quad l_{\mathcal{V}} \otimes l_{\mathcal{V}} \cong \mathbf{L}_{\mathcal{V}}, \quad l \equiv l_{\mathcal{H}} \otimes l_{\mathcal{V}} \cong \mathbb{C}_Z, \quad l_{\mathcal{H}} \cong l^*_{\mathcal{V}}. \tag{4.19}$$

The connections on $P_{\mathrm{Spin}^c(2n-1)}(Z)$ and $P^{\mathcal{V}}_{\mathrm{Spin}^c(2)}(Z)$ induce unitary covariant derivatives $\nabla^{l_{\mathcal{H}}}$, $\nabla^{l_{\mathcal{V}}}$ on $l_{\mathcal{H}}$, $l_{\mathcal{V}}$. Through (4.19), we can identify as follows:

$$\begin{gathered} \nabla^{l_{\mathcal{H}}} \otimes 1 + 1 \otimes \nabla^{l_{\mathcal{H}}} = \nabla^{\mathbf{L}_{\mathcal{H}}}, \quad \nabla^{l_{\mathcal{V}}} \otimes 1 + 1 \otimes \nabla^{l_{\mathcal{V}}} = \nabla^{\mathbf{L}_{\mathcal{V}}}, \\ \nabla^{l_{\mathcal{H}}}|\pi^{-1}(x) = \nabla^{l^*_{\mathcal{V}}}|\pi^{-1}(x). \end{gathered} \tag{4.20}$$

The third one is due to (4.17). We will end the preparation here: refer to [13] for more information.

Now, as is easily understood, to prove the theorem, it suffices to modify the proof of Theorem 0.1 a little bit. Namely, we have only to twist A^Z_ε using the trivial line bundle $l = l_{\mathcal{H}} \otimes l_{\mathcal{V}}$ with nontrivial covariant derivative $\nabla^l = \nabla^{l_{\mathcal{H}}} \otimes 1 + 1 \otimes \nabla^{l_{\mathcal{V}}}$.

Let us take now a covariant derivative $\nabla^{\mathcal{V},l} = \nabla^{\mathcal{V}} \otimes 1 + 1 \otimes \nabla^l$ acting on $\mathcal{A}(\mathcal{V}) = \mathcal{A}(\mathcal{V} \otimes l)$, which induces a covariant derivative $\nabla^{\mathbf{S}(\mathcal{V}),l}$ on $\mathbf{S}(\mathcal{V})$. We define then an operator

$$A^{\mathcal{V},l} = \sum e''_k \circ \left(1 \otimes \left(\nabla^{\mathbf{S}(\mathcal{V}),l}_{e''_k} \otimes 1 + 1 \otimes \nabla^{\mathbf{S}(\mathcal{V})}_{e''_k}\right)\right) \tag{4.21}$$

acting on $\pi^*\mathcal{A}(M) \otimes \mathcal{A}(\mathcal{V})$ as in (2.2). It will be obvious now that we have the expression

$$A^{Z,c}_\varepsilon = \varepsilon^{1/2}\left[\pi^* A^M + \sum e'_i \circ \nabla^{\mathcal{V},l}_{e'_i}\right] + A^{\mathcal{V},l} - \frac{\varepsilon}{4}c(T) + \frac{\varepsilon}{4}\tilde{c}(T), \tag{4.22}$$

corresponding to (2.10). Following the comment around (3.7) we take a unitary superconnection

$$B^l_t = \tilde{\nabla}^{\mathcal{V},l} + t^{1/2} A^{\mathcal{V},l} - \frac{1}{4t^{1/2}}\hat{c}(T) \tag{4.23}$$

on the infinite dimensional hermitian vector bundle $H^{\infty,l} = H^\infty$. Similarly to (3.10) and (3.11) we define $\hat{\eta}^l(t)$ and $\tilde{\eta}^l(t)$. Then we have uniform convergence as $\varepsilon \to 0$

$$\mathrm{Tr}(A^{Z,c}_\varepsilon e^{-t(A^{Z,c}_\varepsilon)^2}) = 2^n\sqrt{\pi}\int_M \hat{\mathbb{L}}(\Omega^M) \wedge \tilde{\eta}^l(t) + O(\varepsilon^{1/2}(1 + t^N)) \tag{4.24}$$

for some $N > 0$ as in Lemma 3.1. Moreover, since the third equality in (4.20) implies $A^{\mathcal{V},l}|H_x^{\infty,l} = A^{\mathcal{V}}|H_x^{\infty}$, Lemma 3.2 asserts

$$\eta(A^M \otimes \operatorname{Ind} A^{\mathcal{V},l}) = 0. \tag{4.25}$$

Thus, similarly to (3.18), we have

$$\lim_{\varepsilon\to 0} \overline{\eta}(A_\varepsilon^{Z,c}) = 2^n \int_M \hat{\mathbb{L}}(\Omega^M) \wedge \tilde{\eta}^l \tag{4.26}$$
$$+ \lim_{\varepsilon\to 0} \frac{1}{2} \left\{ \dim \operatorname{Ker}(A^M \otimes \operatorname{Ind} A^{\mathcal{V},l}) + \rho^c(\varepsilon) + \dim \operatorname{Ker} A_\varepsilon^{Z,c} \right\},$$

where we set

$$\tilde{\eta}^l = \int_0^\infty \tilde{\eta}^l(t) \frac{dt}{2t^{1/2}}. \tag{4.27}$$

Precisely stating, (4.26) acquires a meaning only when the limit in the right side exists (in some sense) and, as explained before, the limit may not exist in $\mathbb{R}$. However, since the term $2^{-1}\{\dim \operatorname{Ker}(A^M \otimes \operatorname{Ind} A^{\mathcal{V},l}) + \rho^c(\varepsilon) + \dim \operatorname{Ker} A_\varepsilon^{Z,c}\}$ is obviously an integer, the limit exists in $\mathbb{R}/\mathbb{Z}$. Accordingly the limit $\lim_{\varepsilon\to 0} \overline{\eta}(A_\varepsilon^{Z,c})$ exits in $\mathbb{R}/\mathbb{Z}$ and equals the first term in the right side of (4.26). That is, (4.14) was proved. Next it will be obvious that (4.15) follows from the similar formulas as in Lemma 3.6, i.e.,

$$\lim_{t\to\infty} \operatorname{str}\left(e^{-(B_t^l)^2}\right) = 0, \tag{4.28}$$

$$\lim_{t\to 0} \operatorname{str}\left(e^{-(B_t^l)^2}\right) = \tag{4.29}$$
$$(\pi\sqrt{-1})^{-1} \int_{Z/M} \hat{\mathbb{L}}(2\pi\sqrt{-1}\Omega^{\mathcal{V}}) \wedge \exp\left(\frac{1}{2} c_1(2\pi\sqrt{-1}(\Omega^{\mathcal{V}} + \Omega^{\mathcal{H}}))\right).$$

(4.28) is just a reformation of (3.26) because $\operatorname{Ind} A^{\mathcal{V},l}$ and $\operatorname{Ind} A^{\mathcal{V}}$ coincide with each other including hermitian metrics. By noticing that the curvature of ∇^l is equal to $(\Omega^{\mathcal{V}} + \Omega^{\mathcal{H}})/2$, (4.29) is shown similarly to (3.27). □

Second, let us define $\mathbf{S}^c(Z_0)\,(= \mathbf{S}(Z_0))$ with $\nabla^{\mathbf{S}^c(Z_0)}\,(= \nabla^{\mathbf{S}(Z_0),c})$ similarly to (4.8) and take a twisted signature operator

$$A^{Z,cc} = D^{Z,c} \otimes \mathbf{S}^c(Z_0) = \sum e_a \circ \left(\nabla_{e_a}^{\mathbf{S}(Z),c} \otimes 1 + 1 \otimes \nabla_{e_a}^{\mathbf{S}(Z_0),c}\right) \tag{4.30}$$
$$= \sum e_a \circ \left(\nabla_{e_a}^{\mathbf{S}^c(Z)} \otimes 1 + 1 \otimes \nabla_{e_a}^{\mathbf{S}^c(Z_0)}\right)$$

on

$$\wedge(Z) \cong \mathbf{S}(Z) \otimes \mathbf{S}(Z_0) \cong \mathbf{S}^c(Z) \otimes \mathbf{S}^c(Z_0). \tag{4.31}$$

In contrast to $A^{Z,c}$ which is A^Z twisted with $l = l_{\mathcal{H}} \otimes l_{\mathcal{V}}$, this is gotten by twisting A^Z with $\mathbf{L} = \mathbf{L}_{\mathcal{H}} \otimes \mathbf{L}_{\mathcal{V}}$. The proof of the following is quite similar to the proof of Theorem 4.1.

Theorem 4.2. *The (adiabatic) limit* $\lim_{\varepsilon \to 0} \overline{\eta}(A_\varepsilon^{Z,cc})$ *exists in* $\mathbb{R}/\mathbb{Z}$ *and there is an odd degree form* $\tilde{\eta}^{\mathbf{L}}$ *on* M *such that*

$$\lim_{\varepsilon \to 0} \overline{\eta}(A_\varepsilon^{Z,cc}) \equiv 2^n \int_M \hat{\mathbb{L}}(\Omega^M) \wedge \tilde{\eta}^{\mathbf{L}} \quad (\mathrm{mod}\ \mathbb{Z}), \tag{4.32}$$

$$d\tilde{\eta}^{\mathbf{L}} = 2 \int_{Z/M} \hat{\mathbb{L}}(\Omega^{\mathcal{V}}) \wedge \exp\left(c_1(\Omega^{\mathcal{V}}) + c_1(\Omega^{\mathcal{H}})\right). \tag{4.33}$$

Third, let us discuss an extraordinary twisting. We put $\mathrm{Spin}^{qc}(2n+1) = \mathrm{Spin}(2n+1) \times_{\mathbb{Z}_2} Sp(1) \times_{\mathbb{Z}_2} U(1)$, which has a double covering homomorphism

$$\xi^{qc} : \mathrm{Spin}^{qc}(2n+1) \to SO(2n+1) \times SO(3) \times U(1) \tag{4.34}$$

with $\xi^{qc} = (\xi_0^{qc}, \xi_1^{qc}, \xi_2^{qc}) = (\xi, \xi_1^q, \xi_1^c)$. Note that the Lie group $Sp(1) \times_{\mathbb{Z}_2} U(1) = SU(2) \times_{\mathbb{Z}_2} U(1)$ plays a significant role in the Weinberg-Salam theory which unifies weak and electromagnetic interactions. The twistor space (Z, g^Z) has a twisted Spin structure, say, a Spin^{qc} structure,

$$\xi^{qc} : P_{\mathrm{Spin}^{qc}(2n+1)}(Z) \to P_{SO(2n+1)}(Z) \times \pi^* P_{SO(3)} \times P_{U(1)}^{\mathcal{V}}(Z) \tag{4.35}$$

with $P_{\mathrm{Spin}^{qc}(2n+1)}(Z) = \left(\pi^* P_{\mathrm{Spin}^q(2n-1)} \times P^{\mathcal{V}}_{\mathrm{Spin}^c(2)}(Z)\right) \times_{\mathrm{mult}} \mathrm{Spin}^{qc}(2n+1)$: see [12, §5]. This produces a spinor bundle

$$\mathbf{S}^{qc}(Z) = P_{\mathrm{Spin}^{qc}(2n+1)}(Z) \times_{\Delta \otimes r_H \otimes r_C} S^{qc}_{2n+1}, \quad S^{qc}_{2n+1} = S_{2n+1} \otimes H \otimes C. \tag{4.36}$$

Here r_H is the standard complex representation of $Sp(1)$ given by left multiplication on $\mathbb{H}$, $r_H : Sp(1) \to (GL_{\mathbb{H}}(\mathbb{H}) \hookrightarrow) GL_{\mathbb{C}}(\mathbb{C}^2) \equiv GL_{\mathbb{C}}(H)$. The connection $\alpha^Z \oplus \pi^* \alpha_{SO(3)} \oplus \alpha^{\mathcal{V}}_{U(1)}$ on the right side of (4.35) induces a covariant derivative $\nabla^{\mathbf{S}(Z),qc}$ on (4.36). We obtain thus a twisted signature operator

$$A^{Z,qc} = D^{Z,qc} \otimes \mathbf{S}(Z_0) = \sum e_a \circ \left(\nabla^{\mathbf{S}(Z),qc}_{e_a} \otimes 1 + 1 \otimes \nabla^{\mathbf{S}(Z_0)}_{e_a}\right) \tag{4.37}$$

on $\mathbf{S}^{qc}(Z) \otimes \mathbf{S}(Z_0)$. Let us take a locally defined vector bundle

$$\mathbf{H} = P_{\mathrm{Spin}^q(2n-1)} \times_{r_H} H \tag{4.38}$$

with $\mathbf{S}^q(M) \equiv P_{\mathrm{Spin}^q(2n-1)} \times_{\Delta \otimes r_H} S \otimes H = \mathbf{S}(M) \otimes \mathbf{H}$. (Notice that $\mathbf{S}(M)$ was taken in the proof of Theorem 4.1 and has been fixed.) Then $\pi^* \mathbf{H} \otimes l_{\mathcal{V}}$ is globally defined and we have

$$\wedge(Z) \otimes \pi^* \mathbf{H} \otimes l_{\mathcal{V}} \cong \mathbf{S}^{qc}(Z) \otimes \mathbf{S}(Z_0). \tag{4.39}$$

Accordingly (4.37) is just A^Z twisted with $\pi^*\mathbf{H} \otimes l_{\mathcal{V}}$ having covariant derivative $\pi^*\nabla^{\mathbf{H}} \otimes 1 + 1 \otimes \nabla^{l_{\mathcal{V}}}$. Here $\nabla^{\mathbf{H}}$ is a covariant derivative on $\mathbf{H}$ induced from $\alpha_{SO(3)}$. In the following we want to investigate the limiting behavior of the reduced η-invariant of its adiabatic version $A^{Z,qc}_{\varepsilon}$.

The locally defined vector bundles $TM \otimes \mathbf{H}$ and $\mathcal{V} \otimes l_{\mathcal{V}}$ have covariant derivatives $\nabla^{M,\mathbf{H}} = \nabla^M \otimes 1 + 1 \otimes \nabla^{\mathbf{H}}$ and $\nabla^{\mathcal{V},l_{\mathcal{V}}} = \nabla^{\mathcal{V}} \otimes 1 + 1 \otimes \nabla^{l_{\mathcal{V}}}$. These induce covariant derivatives $\nabla^{\mathbf{S}(M),\mathbf{H}}$, $\nabla^{\mathbf{S}(\mathcal{V}),l_{\mathcal{V}}}$ on the locally defined spinor bundles $\mathbf{S}(M)\otimes\mathbf{H}$, $\mathbf{S}(\mathcal{V})\otimes l_{\mathcal{V}}$. Let us consider now twisted signature operators

$$
\begin{aligned}
A^{M,\mathbf{H}} &= \sum e'_i \circ \left(\nabla^{\mathbf{S}(M),\mathbf{H}}_{e'_i} \otimes 1 + 1 \otimes \nabla^{\mathbf{S}(M_0)}_{e'_i}\right) = A^M \otimes \mathbf{H}, \\
A^{\mathcal{V},l_{\mathcal{V}}} &= \sum e''_k \circ \left(\nabla^{\mathbf{S}(\mathcal{V}),l_{\mathcal{V}}}_{e''_k} \otimes 1 + 1 \otimes \nabla^{\mathbf{S}(\mathcal{V})}_{e''_k}\right) = A^{\mathcal{V}} \otimes l_{\mathcal{V}}
\end{aligned}
\tag{4.40}
$$

acting on $\mathcal{A}(M)\otimes\Gamma(\mathbf{H}) = \Gamma(\mathbf{S}(M)\otimes\mathbf{H})\otimes\Gamma(\mathbf{S}(M_0))$, $\mathcal{A}(\mathcal{V})\otimes\Gamma(l_{\mathcal{V}}) = \Gamma(\mathbf{S}(\mathcal{V})\otimes l_{\mathcal{V}})\otimes\Gamma(\mathbf{S}(\mathcal{V}))$. Regarding these as operators acting on $\pi^*(\mathcal{A}(M)\otimes \Gamma(\mathbf{H})) \otimes \mathcal{A}(\mathcal{V}) \otimes \Gamma(l_{\mathcal{V}})$, we obtain a similar expression as in (2.10),

$$
A^{Z,qc}_{\varepsilon} = \varepsilon^{1/2}\left[\pi^* A^{M,\mathbf{H}} + \sum e'_i \circ \nabla^{\mathcal{V},l_{\mathcal{V}}}_{e'_i}\right] + A^{\mathcal{V},l_{\mathcal{V}}} - \frac{\varepsilon}{4} c(T) + \frac{\varepsilon}{4}\tilde{c}(T). \tag{4.41}
$$

Let us define a locally defined infinite dimensional hermitian vector bundle $H^{\infty,l_{\mathcal{V}}} = H^{\infty,l_{\mathcal{V}}}_+ \oplus H^{\infty,l_{\mathcal{V}}}_-$ over M by $H^{\infty,l_{\mathcal{V}}}_{\pm,(x)} = \mathcal{A}^{\pm}(\mathcal{V}) \otimes \Gamma(l_{\mathcal{V}})|\pi^{-1}(x)$. Again following the comment around (3.7), we will take its unitary superconnection

$$
B^{l_{\mathcal{V}}}_t = \tilde{\nabla}^{\mathcal{V},l_{\mathcal{V}}} + t^{1/2} A^{\mathcal{V},l_{\mathcal{V}}} - \frac{1}{4t^{1/2}}\hat{c}(T). \tag{4.42}
$$

Similarly to (3.10) and (3.11), we define $\hat{\eta}^{l_{\mathcal{V}}}(t)$ and $\tilde{\eta}^{l_{\mathcal{V}}}(t)$. Since $\operatorname{End} H^{\infty,l_{\mathcal{V}}}$ is globally defined, these odd degree forms on M are globally defined. We have now uniform convergence as $\varepsilon \to 0$

$$
\operatorname{Tr}(A^{Z,qc}_{\varepsilon} e^{-t(A^{Z,qc}_{\varepsilon})^2}) = 2^n \sqrt{\pi} \int_M \hat{\mathbb{L}}(\Omega^M) \wedge \operatorname{ch}(\Omega^{\mathbf{H}}) \wedge \tilde{\eta}^{l_{\mathcal{V}}}(t) + O(\varepsilon^{1/2}(1 + t^N)) \tag{4.43}
$$

for some $N > 0$ as in Lemma 3.1. Here $\Omega^{\mathbf{H}}$ is the curvature 2-form of $\nabla^{\mathbf{H}}$ and $\operatorname{ch}(\Omega^{\mathbf{H}})$ is the (globally defined) Chern character form, i.e., $\operatorname{ch}(\Omega^{\mathbf{H}}) = \operatorname{tr}\exp(\sqrt{-1}\Omega^{\mathbf{H}}/2\pi)$. Denote by $\Omega_{SO(3)}$ the curvature 2-form of $\alpha_{SO(3)}$ and take its first Pontryagin form $p_1(\Omega_{SO(3)}) = -c_2(\Omega_{SO(3)})$. Then it follows from [12, (4.9)] that we have

$$
\operatorname{ch}(\Omega^{\mathbf{H}}) = 2\cosh\left(\frac{1}{2} p_1(\Omega_{SO(3)})^{1/2}\right). \tag{4.44}
$$

We will next regard $A^{\mathcal{V},l_{\mathcal{V}}}$ as a family of twisted signature operators along the fibres and investigate its index bundle $\operatorname{Ind} A^{\mathcal{V},l_{\mathcal{V}}}$. The bundle $l_{\mathcal{V}}$ restricted to each fibre

$\pi^{-1}(x)$ has a holomorphic line bundle structure induced from the structure of $\mathbb{C}P^1$. Let us denote the spaces of their holomorphic cross-sections by $Hol(l_{\mathcal{V}}|\pi^{-1}(x))$, which together form a (locally defined) vector bundle $Hol(l_{\mathcal{V}})$.

Lemma 4.3. *We have* $\operatorname{rank} Hol(l_{\mathcal{V}}) = 2$ *and*

$$\operatorname{Ind} A_+^{\mathcal{V},l_{\mathcal{V}}} = (1 + \tau_{\mathcal{V}} 1)\breve{} \otimes Hol(l_{\mathcal{V}}), \quad \operatorname{Ind} A_-^{\mathcal{V},l_{\mathcal{V}}} = 0.$$

Proof. $\mathbf{L}_{\mathcal{V}}|\pi^{-1}(x)$ is canonically isomorphic to the canonical tangent bundle (or the holomorphic tangent bundle) $K^*_{\mathbb{C}P^1} = T\mathbb{C}P^1 = \wedge^{1,0}(T_{\mathbb{C}}\mathbb{C}P^1)$ and $l_{\mathcal{V}}|\pi^{-1}(x)$ is canonically isomorphic to the hyperplane bundle $H_{\mathbb{C}P^1}$. Moreover, the covariant derivative $\nabla^{l_{\mathcal{V}}}$ restricted to the fibre $\pi^{-1}(x)$ can be identified with the covariant derivative associated to ds^2 through the isomorphism: see [13]. Accordingly let us take the covariant exterior differential $\mathcal{D}$ and its formal adjoint $\mathcal{D}^*$ acting on $\Gamma(\wedge(\mathbb{C}P^1)\otimes H_{\mathbb{C}P^1})$ and consider a twisted signature operator

$$(4.45)\qquad (\mathcal{D} + \mathcal{D}^*)_\pm : \Gamma(\wedge^\pm(\mathbb{C}P^1)\otimes H_{\mathbb{C}P^1}) \to \Gamma(\wedge^\mp(\mathbb{C}P^1)\otimes H_{\mathbb{C}P^1}).$$

From the above, it will suffice to investigate its kernel. We have decompositions

$$(4.46)\qquad \begin{aligned} \wedge^+(\mathbb{C}P^1) \otimes H_{\mathbb{C}P^1} &= ((1 + \tau_{\mathbb{C}P^1} 1) \otimes H_{\mathbb{C}P^1}) \oplus (K_{\mathbb{C}P^1} \otimes H_{\mathbb{C}P^1}), \\ \wedge^-(\mathbb{C}P^1) \otimes H_{\mathbb{C}P^1} &= ((1 - \tau_{\mathbb{C}P^1} 1) \otimes H_{\mathbb{C}P^1}) \oplus (\overline{K}_{\mathbb{C}P^1} \otimes H_{\mathbb{C}P^1}). \end{aligned}$$

We want to show

$$(4.47)\qquad \operatorname{Ker}(\mathcal{D} + \mathcal{D}^*)_+ = (1 + \tau_{\mathbb{C}P^1} 1) \otimes Hol(H_{\mathbb{C}P^1}), \quad \operatorname{Ker}(\mathcal{D} + \mathcal{D}^*)_- = 0,$$

where $Hol(H_{\mathbb{C}P^1})$ is the space of holomorphic cross-sections of $H_{\mathbb{C}P^1}$. To show these let us take canonical holomorphic local cross-sections u_ℓ of the universal bundle $H^*_{\mathbb{C}P^1}$ over local coordinate neighborhoods $(W_\ell = \{[z_0, z_1] \mid z_\ell \neq 0\}, w_\ell(= z_1/z_0\,(\ell = 0), = z_0/z_1\,(\ell = 1)))$, i.e., $u_\ell(w_\ell) = (w_0, (1, w_0))\,(\ell = 0)$, $(w_1, (w_1, 1))\,(\ell = 1)$. An element $\phi = \phi_0 + \phi_1$ of $\Gamma(\wedge^+(\mathbb{C}P^1) \otimes H_{\mathbb{C}P^1})$ can be expressed on $(W, w) = (W_\ell, w_\ell)$ as $\phi_0 = f(1+\tau_{\mathbb{C}P^1} 1)\otimes u^*$, $\phi_1 = g dw \otimes u^*$, where $u^* = u^*_\ell$ is the dual of u_ℓ. Then it is easily shown that ϕ belongs to $\operatorname{Ker}(\mathcal{D}+\mathcal{D}^*)_+$ if and only if $\mathcal{D}''(f \otimes u^*) = \mathcal{D}''(g dw \otimes u^*) = \mathcal{D}'^*(g dw \otimes u^*) = 0$. Here we use the usual decompositions $\mathcal{D} = \mathcal{D}' + \mathcal{D}''$ $(\mathcal{D}'' = \bar{\partial})$ and $\mathcal{D}^* = \mathcal{D}'^* + \mathcal{D}''^*$. $\mathcal{D}''(f \otimes u^*) = 0$ means that ϕ_0 belongs to $(1 + \tau_{\mathbb{C}P^1} 1) \otimes Hol(H_{\mathbb{C}P^1})$, and $\mathcal{D}''(g dw \otimes u^*) = 0$ means that ϕ_1 belongs to the space $H^0(\mathbb{C}P^1, \Omega^1(H_{\mathbb{C}P^1}))$ of holomorphic 1-forms with coefficients in $H_{\mathbb{C}P^1}$. By the Kodaira vanishing theorem the space $H^0(\mathbb{C}P^1, \Omega^1(H_{\mathbb{C}P^1})) = H^0(\mathbb{C}P^1, \mathcal{O}(H^*_{\mathbb{C}P^1}))$ equals $\{0\}$. Thus we obtain the first equality in (4.47). Next, an element $\phi = \phi_0 + \phi_1$ of $\Gamma(\wedge^-(\mathbb{C}P^1) \otimes H_{\mathbb{C}P^1})$ can be expressed on $(W, w) = (W_\ell, w_\ell)$ as $\phi_0 = f(1 - \tau_{\mathbb{C}P^1} 1) \otimes u^*$, $\phi_1 = g d\bar{w} \otimes u^*$. Then ϕ belongs to $\operatorname{Ker}(\mathcal{D} + \mathcal{D}^*)_-$ if and only if $\mathcal{D}'(f \otimes u^*) = \mathcal{D}(g d\bar{w} \otimes u^*) =$

$\mathcal{D}^*(gd\overline{w} \otimes u^*) = 0$. $\mathcal{D}'(f \otimes u^*) = 0$ means that $f\tau_{\mathbb{C}P^1}1 \otimes u^*$ belongs to $\operatorname{Ker}\mathcal{D}'' \cap \operatorname{Ker}\mathcal{D}''^*(\cong H^1(\mathbb{C}P^1, \Omega^1(H_{\mathbb{C}P^1})) = 0)$, and $\mathcal{D}(gd\overline{w} \otimes u^*) = \mathcal{D}^*(gd\overline{w} \otimes u^*) = 0$ means that ϕ_1 belongs to $\operatorname{Ker}\mathcal{D}'' \cap \operatorname{Ker}\mathcal{D}''^*(\cong H^1(\mathbb{C}P^1, \mathcal{O}(H_{\mathbb{C}P^1})) \cong H^0(\mathbb{C}P^1, \Omega^1(H^*_{\mathbb{C}P^1})) \cong H^0(\mathbb{C}P^1, \mathcal{O}(3H^*_{\mathbb{C}P^1})) = 0)$. Thus we get the second equality in (4.47). Finally, as is well known ([9]), we have $\dim Hol(H_{\mathbb{C}P^1}) = 2$. □

Now we take a locally defined hermitian infinite dimensional vector bundle $l^\infty_{\mathcal{V}}$ over M defined by $l^\infty_{\mathcal{V},(x)} = \Gamma(l_{\mathcal{V}}|\pi^{-1}(x))$. This has a unitary connection $\tilde{\nabla}^{l^\infty_{\mathcal{V}}}$ defined by $\tilde{\nabla}^{l^\infty_{\mathcal{V}}}_{e'_i}\tilde{\psi} = (\nabla^{l_{\mathcal{V}}}_{e'_i}\psi)\tilde{}$ as in (3.5). Its orthogonal projection to the subbundle $Hol(l_{\mathcal{V}})$ gives its connection $\nabla^{Hol(l_{\mathcal{V}})}$, whose curvature 2-form is denoted by $\Omega^{Hol(l_{\mathcal{V}})}$. $\mathbf{H} \otimes Hol(l_{\mathcal{V}})$ is a globally defined vector bundle over M with connection $\nabla^{\mathbf{H}} \otimes 1 + 1 \otimes \nabla^{Hol(l_{\mathcal{V}})}$. Twisting A^M with it we obtain a twisted signature operator

$$A^M \otimes \mathbf{H} \otimes Hol(l_{\mathcal{V}}) = A^{M,\mathbf{H}} \otimes Hol(l_{\mathcal{V}}). \tag{4.48}$$

Then we have

Theorem 4.4. *The (adiabatic) limit* $\lim_{\varepsilon\to 0}\overline{\eta}(A^{Z,qc}_\varepsilon)$ *exists in* $\mathbb{R}/\mathbb{Z}$ *and there is an odd degree form* $\tilde{\eta}^{l_{\mathcal{V}}}$ *on* M *such that*

$$\lim_{\varepsilon\to 0}\overline{\eta}(A^{Z,qc}_\varepsilon) \equiv 2^n \int_M \hat{\mathbb{L}}(\Omega^M) \wedge \operatorname{ch}(\Omega^{\mathbf{H}}) \wedge \tilde{\eta}^{l_{\mathcal{V}}} + \overline{\eta}(A^M \otimes \mathbf{H} \otimes Hol(l_{\mathcal{V}})) \pmod{\mathbb{Z}}, \tag{4.49}$$

$$d\tilde{\eta}^{l_{\mathcal{V}}} = 2\int_{Z/M} \hat{\mathbb{L}}(\Omega^{\mathcal{V}}) \wedge \exp\left(\frac{1}{2}c_1(\Omega^{\mathcal{V}})\right) - \operatorname{ch}(\Omega^{Hol(l_{\mathcal{V}})}). \tag{4.50}$$

Proof. Since $\tilde{\nabla}^{\mathcal{V},l_{\mathcal{V}}}((1+\tau_{\mathcal{V}}1)\tilde{} \otimes \tilde{\psi}) = (1+\tau_{\mathcal{V}}1)\tilde{} \otimes \tilde{\nabla}^{l^\infty_{\mathcal{V}}}\tilde{\psi}$, we have

$$\wedge(M) \otimes \mathbf{H} \otimes \operatorname{Ind} A^{\mathcal{V},l_{\mathcal{V}}}_+ \cong \wedge(M) \otimes \mathbf{H} \otimes Hol(l_{\mathcal{V}}) \tag{4.51}$$

by deleting $(1+\tau_{\mathcal{V}}1)\tilde{}$ as in (3.16). Through it we have $A^{M,\mathbf{H}} \otimes \operatorname{Ind} A^{\mathcal{V},l_{\mathcal{V}}}_+ = A^{M,\mathbf{H}} \otimes Hol(l_{\mathcal{V}})$. Thus, by considering the same formula as in (3.18) or (4.26), we obtain (4.49) with

$$\tilde{\eta}^{l_{\mathcal{V}}} = \int_0^\infty \tilde{\eta}^{l_{\mathcal{V}}}(t)\frac{dt}{2t^{1/2}}. \tag{4.52}$$

Next let us investigate $\operatorname{str}(\exp(-(B^{l_{\mathcal{V}}}_t)^2))$. The orthogonal projection of $\tilde{\nabla}^{\mathcal{V},l_{\mathcal{V}}}$ to the subbundle $\operatorname{Ind} A^{\mathcal{V},l_{\mathcal{V}}}_+$ gives its covariant derivative, whose curvature 2-form is equal to $\Omega^{Hol(l_{\mathcal{V}})}$. Hence, [2, Corollary 9.22] implies

$$\lim_{\varepsilon\to\infty}\operatorname{str}\left(e^{-(B^{l_{\mathcal{V}}}_t)^2}\right) = \operatorname{tr}\left(\exp(-\Omega^{Hol(l_{\mathcal{V}})})\right) = \operatorname{ch}(2\pi\sqrt{-1}\Omega^{Hol(l_{\mathcal{V}})}). \tag{4.53}$$

On the other hand, since the curvature 2-form of $\nabla^{l_{\mathcal{V}}}$ equals $\Omega^{\mathcal{V}}/2$, we can show

$$(4.54)\quad \lim_{t\to 0}\operatorname{str}\left(e^{-(B_t^{l_{\mathcal{V}}})^2}\right) = (\pi\sqrt{-1})^{-1}\int_{Z/M}\hat{\mathbb{L}}(2\pi\sqrt{-1}\Omega^{\mathcal{V}})\wedge\exp\left(\frac{1}{2}c_1(2\pi\sqrt{-1}\Omega^{\mathcal{V}})\right)$$

in the same way as in the proof of (3.27). Thus we get (4.50). □

Fourth, we consider a bundle

$$(4.55)\qquad \mathbf{S}^{qc}(Z)\otimes\mathbf{S}^{qc}(Z_0)\ \cong\ \wedge(Z)\otimes\pi^*(\mathbf{H}\otimes\mathbf{H})\otimes\mathbf{L}_{\mathcal{V}}$$

and a twisted signature operator on it

$$(4.56)\qquad A^{Z,qcqc} = D^{Z,qc}\otimes\mathbf{S}^{qc}(Z_0) = \sum e_a\circ\left(\nabla_{e_a}^{\mathbf{S}(Z),qc}\otimes 1 + 1\otimes\nabla_{e_a}^{\mathbf{S}(Z_0),qc}\right).$$

We want to investigate the limiting behavior of the reduced η-invariant of its adiabatic version $A_\varepsilon^{Z,qcqc}$. $\mathbf{H}\otimes\mathbf{H}$ with $\nabla^{\mathbf{H}\otimes\mathbf{H}}$ can be decomposed into symmetric and anti-symmetric parts

$$(4.57)\qquad \mathbf{H}\otimes\mathbf{H} = S\mathbf{H}\oplus\wedge\mathbf{H},\quad \nabla^{\mathbf{H}\otimes\mathbf{H}} = \nabla^{S\mathbf{H}}\oplus\nabla^{\wedge\mathbf{H}}.$$

Let us consider the standard representation $r_E : SO(3)\to GL_{\mathbb{C}}(\mathbb{C}^3)\equiv GL_{\mathbb{C}}(E)$ and take a vector bundle over M

$$(4.58)\qquad \mathbf{E} = P_{SO(3)}\times_{r_E}\mathbf{E}$$

with covariant derivative $\nabla^{\mathbf{E}}$ associated to $\alpha_{SO(3)}$. Then canonically we have

$$(4.59)\qquad \begin{aligned} &S\mathbf{H}\cong\mathbf{E},\quad \nabla^{S\mathbf{H}} = \nabla^{\mathbf{E}},\\ &\wedge\mathbf{H}\cong\mathbb{C}_M,\quad \nabla^{\wedge\mathbf{H}} = d_M.\end{aligned}$$

Accordingly we have the decomposition

$$(4.60)\qquad A^{Z,qcqc} = A^{Z,\mathbf{E}\mathcal{V}}\oplus A^{Z,\mathcal{V}} = (A^Z\otimes\pi^*\mathbf{E}\otimes\mathbf{L}_{\mathcal{V}})\oplus(A^Z\otimes\mathbf{L}_{\mathcal{V}}).$$

It suffices now to investigate their reduced η-invariants respectively.

Similarly to $Hol(l_{\mathcal{V}})$ we take a bundle $Hol(\mathbf{L}_{\mathcal{V}})$. This is obviously globally defined and its rank equals 3. Further, similarly to $(l_{\mathcal{V}}^\infty, \tilde{\nabla}^{l_{\mathcal{V}}^\infty})$ we define $(\mathbf{L}_{\mathcal{V}}^\infty, \tilde{\nabla}^{\mathbf{L}_{\mathcal{V}}^\infty})$, and, in the same way as before, take a unitary covariant derivative $\nabla^{Hol(\mathbf{L}_{\mathcal{V}})}$ on $Hol(\mathbf{L}_{\mathcal{V}})$, whose curvature 2-form is denoted by $\Omega^{Hol(\mathbf{L}_{\mathcal{V}})}$. Then we have

Theorem 4.5. *The (adiabatic) limits* $\lim_{\varepsilon\to 0}\bar{\eta}(A_\varepsilon^{Z,\mathbf{E}\mathcal{V}})$ *and* $\lim_{\varepsilon\to 0}\bar{\eta}(A_\varepsilon^{Z,\mathcal{V}})$ *exist in* $\mathbb{R}/\mathbb{Z}$ *and there is an odd degree form* $\tilde{\eta}^{\mathbf{L}_{\mathcal{V}}}$ *on* M *such that*

$$\lim_{\varepsilon\to 0}\bar\eta(A_\varepsilon^{Z,\mathbf{E}\mathcal{V}}) \equiv 2^n\int_M \hat{\mathbb{L}}(\Omega^M)\wedge \mathrm{ch}(\Omega_{SO(3)})\wedge\tilde\eta^{\mathbf{L}_\mathcal{V}} + \bar\eta(A^M\otimes\mathbf{E}) + \bar\eta(A^M\otimes\mathbf{E}\otimes Hol(\mathbf{L}_\mathcal{V})) \pmod{\mathbb{Z}}, \tag{4.61}$$

$$\lim_{\varepsilon\to 0}\bar\eta(A_\varepsilon^{Z,\mathcal{V}}) \equiv 2^n\int_M \hat{\mathbb{L}}(\Omega^M)\wedge\tilde\eta^{\mathbf{L}_\mathcal{V}} + \bar\eta(A^M) + \bar\eta(A^M\otimes Hol(\mathbf{L}_\mathcal{V})) \pmod{\mathbb{Z}}, \tag{4.62}$$

$$d\tilde\eta^{\mathbf{L}_\mathcal{V}} = 2\int_{Z/M}\hat{\mathbb{L}}(\Omega^\mathcal{V})\wedge\exp\left(\frac{1}{2}c_1(\Omega^\mathcal{V})\right) - \mathrm{ch}\,(\Omega^{Hol(\mathbf{L}_\mathcal{V})}) - 1. \tag{4.63}$$

Proof. Let us take a twisted signature operator

$$A^{\mathcal{V},\mathbf{L}_\mathcal{V}} = \sum e_k''\circ\left(\nabla^{\mathbf{S}(\mathcal{V}),\mathbf{L}_\mathcal{V}}_{e_k''}\otimes 1 + 1\otimes\nabla^{\mathbf{S}(\mathcal{V})}_{e_k''}\right) = A^\mathcal{V}\otimes\mathbf{L}_\mathcal{V} \tag{4.64}$$

acting on $\mathcal{A}(\mathcal{V})\otimes\Gamma(\mathbf{L}_\mathcal{V}) = \Gamma(\mathbf{S}(\mathcal{V})\otimes\mathbf{L}_\mathcal{V})\otimes\Gamma(\mathbf{S}(\mathcal{V}))$ and investigate the index bundle $\mathrm{Ind}\,A^{\mathcal{V},\mathbf{L}_\mathcal{V}}$. Similarly to $Hol(\mathbf{L}_\mathcal{V})$ we define $Hol(\mathbf{L}_\mathcal{V}^*\otimes\mathbf{L}_\mathcal{V})$. Then, in the same way as in the proof of Lemma 4.3, we can show

$$\begin{aligned}&\mathrm{Ind}\,A_+^{\mathcal{V},\mathbf{L}_\mathcal{V}} = ((1+\tau_\mathcal{V}1)\check{}\otimes Hol(\mathbf{L}_\mathcal{V}))\oplus Hol(\mathbf{L}_\mathcal{V}^*\otimes\mathbf{L}_\mathcal{V}) \cong Hol(\mathbf{L}_\mathcal{V})\oplus\mathbb{C}_M,\\ &\mathrm{Ind}\,A_-^{\mathcal{V},\mathbf{L}_\mathcal{V}} = 0.\end{aligned} \tag{4.65}$$

The covariant derivative on $\mathrm{Ind}\,A_+^{\mathcal{V},\mathbf{L}_\mathcal{V}}$ induced from $\tilde\nabla^{\mathcal{V},\mathbf{L}_\mathcal{V}}$ is then equal to $\nabla^{Hol(\mathbf{L}_\mathcal{V})}\oplus d_M$. Thus, similarly to Theorem 4.4 we obtain the theorem. □

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