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MINIMAL DISCREPANCIES OF HYPERSURFACE SINGULARITIES

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0. Introduction

Let Y be a normal, $\mathbb{Q}$-Gorenstein projective variety, and let $f : X \to Y$ be a resolution of singularities. The discrepancy divisor $\Delta = K_X - f^*K_Y = \sum a_j F_j$, where the F_j are the irreducible exceptional divisors for f, plays a key role in the geometry of Y. For example, the singularities allowed on a minimal (resp. on a canonical) model of Y are defined in terms of Δ. Also, effective results for global generation of linear systems on singular threefolds (cf. [2]) depend on an upper bound for certain coefficients of Δ.

There are many difficult conjectures, and several important results (at least in dimension ≤ 3), regarding the discrepancy coefficients of Y (i.e., the coefficients a_j). In this paper we study a special case of the following problem:

Shokurov's conjecture ([10], [6]). *If* $\dim(Y) = n$, *and* $y \in Y$ *is a singular point, then* $\mathrm{md}_y(Y) \leq n - 2$.

The *minimal discrepancy* of Y at y, $\mathrm{md}_y(Y)$, is defined as

$$\mathrm{md}_y(Y) = \inf\{\mathrm{ord}_F(\Delta) \mid f(F) = \{y\};\ \ f : X \to Y \text{ resolution of } Y\}.$$

The following theorem gives an easy way to bound $\mathrm{md}_y(Y)$ for a large class of hypersurface singularities.

Theorem 1. *Assume that the germ* (Y, y) *is analytically equivalent to a hypersurface singularity* $(Y', 0) \subset (\mathbb{A}^{n+1}_{\mathbb{C}}, 0)$, *given by*

$$Y' = \{(y_1, \ldots, y_{n+1}) \mid G(y_1, \ldots, y_{n+1}) = 0\};\ \ G(0, \ldots, 0) = 0.$$

For an n*-tuple* $(a_1, \ldots, a_n)$ *of positive integers, write* $G(t^{a_1}u_1, \ldots, t^{a_n}u_n, t) = t^A\phi(u_1, \ldots, u_n) + t^{A+1}\psi(u_1, \ldots, u_n, t)$ *with* $\phi(u_1, \ldots, u_n) \neq 0$. *Note that* ϕ *is always a polynomial of degree at most* A, *even if* G *is a power series. Assume that* ϕ *has at least one irreducible factor with exponent* 1 *in its factorization.*

Then $\mathrm{md}_y(Y) \leq d$, *where* $d = (a_1 + \cdots + a_n) - A$.

This criterion applies, for example, to hypersurface singularities of multiplicity 2 and rank at least 2 (if the singularity $(Y, 0)$ is defined by $G = 0$, then we define its rank as the rank of the quadratic part of G at 0). It applies also to terminal (and, more generally, *cDV*) singularities in dimension 3. Shokurov's conjecture for terminal threefolds was proved by D. Markushevich [7], using the language of toric geometry, Newton diagrams, admissible weights, etc., and using the fact that the singularities are isolated. The proof we give in this paper (§4) is more elementary, and works for non-isolated singularities as well. For this reason, we can prove Shokurov's conjecture for log-terminal threefolds without using Mori's very difficult results on existence of flips. (The log-terminal case is not covered in [7].)

On the other hand, Shokurov's conjecture is true for non-terminal threefold singularities (and therefore it is true in full generality in dimension 3, by combining this fact with Markushevich's result). But the proof, cf. 2.5 below, which I learned from S. Ishii, uses the existence of a terminal modification — and therefore the existence of flips. From this point of view, the proof is not as satisfying as one might wish; it would certainly be nice to have a complete proof of Shokurov's conjecture in dimension 3 without using the existence of flips. (Several experts have suggested to me that, even in higher dimension, it should be possible to reduce the general case of Shokurov's conjecture to the terminal case, but I don't know how this can be done.)

The paper is organized as follows. In §1 I discuss discrepancy coefficients in general. Everything in this section is well-known to the experts; I wrote it mainly to fix the notation and terminology. I discuss in some detail the invariance of certain definitions under analytic equivalence of germs; I couldn't find a satisfactory reference in the literature. (N. Mohan Kumar pointed out to me that the matter is not completely trivial.) In §2 I discuss minimal discrepancies and prove several reductions of Shokurov's conjecture. In particular, Theorem 7 in 2.5 shows that the conjecture is true for non-terminal threefold singularities. The proof may well be known among the experts, but I couldn't find it in the literature. I am very grateful to S. Ishii for kindly allowing me to include it here. In §3 I prove Theorem 1, and in §4 I carry out the computations for *cDV* threefold singularities. Note that the proof of Theorem 1 is easy; the difficulty rests in applying it to *cDV* singularities.

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1. Generalities about discrepancy coefficients

In this section I recall several definitions and results regarding discrepancy coefficients, cf. [9], [1], [5].

1.1. Let $f : X \to Y$ be a birational morphism of n-dimensional normal projective varieties over $\mathbb{C}$. A prime Weil divisor $F \subset X$ is *f-exceptional* if $\dim f(F) \leq n - 2$. The closed subset $f(F) \subset Y$ is called the *center* of F on Y. More generally, a $\mathbb{Q}$-

Weil divisor $D = \sum a_j F_j$ is f-exceptional if all the irreducible components F_j are f-exceptional. Let $\operatorname{Exc}(f) = \{x \in X \mid f$ is not an isomorphism at $x\}$; then D is f-exceptional if and only if $\operatorname{Supp}(D) \subset \operatorname{Exc}(f)$.

1.2. Choose a canonical divisor K_Y on Y. Assume that Y is $\mathbb{Q}$-Gorenstein, with global index r; i.e., mK_Y is Cartier for some integer $m \geq 1$, and r is the smallest such integer. Then we can define a $\mathbb{Q}$-divisor f^*K_Y on X by $f^*K_Y = (1/r)f^*(rK_Y)$. On the other hand, there is a unique canonical divisor K_X on X such that the $\mathbb{Q}$-divisor $\Delta = K_X - f^*K_Y$ is f-exceptional. (K_X is obtained as follows: let ω be a rational differential n-form on Y_{reg}, the smooth locus of Y; then $f^*\omega$ extends uniquely to a rational form on X, which we still denote by $f^*\omega$. If ω is chosen such that $K_Y = \operatorname{div}_Y(\omega)$, then $K_X = \operatorname{div}_X(f^*\omega)$.) The divisor $\Delta = K_X - f^*K_Y$ is called the *discrepancy divisor* of f. Note that K_Y varies in a linear equivalence class on Y, and correspondingly K_X varies in its own linear equivalence class on X; however, Δ is uniquely determined by f.

1.3. Write $\Delta = \sum a_j F_j$; the rational numbers a_j are called *discrepancy coefficients*. Now consider another birational morphism $f' : X' \to Y$ (with X' a normal projective variety of dimension n). $f'^{-1} \circ f$ is a birational map $g : X \cdots\to X'$. Let $F_j \subset X$ be an f-exceptional divisor which intersects the regular locus $\operatorname{Reg}(g)$ of g, and assume that g is an isomorphism at the generic point of F_j; i.e., $\overline{g(F_j \cap \operatorname{Reg}(g))}$ is a *divisor* F'_j on X'. Then F'_j is an f'-exceptional divisor; in fact, F_j and F'_j have the same center on Y, $f(F_j) = f'(F'_j)$. Moreover, if a'_j is the coefficient of F'_j in $\Delta' = K_{X'} - f'^*K_Y$, then $a'_j = a_j$. In other words, for every exceptional divisor F_j, the discrepancy coefficient and the center on Y depend only on the discrete valuation of the rational function field $\mathbb{C}(Y)$ determined by F_j.

1.4. Let v be any divisorial discrete valuation of $\mathbb{C}(Y)$; that is, v is associated to some divisor $F^0 \subset X^0$ for some birational morphism $f^0 : X^0 \to Y$. Then, by Hironaka's embedded resolution of singularities, if we start with *any* birational morphism $f : X \to Y$ as before, we can find a suitable f^0 with X^0 smooth, $\operatorname{Exc}(f^0)$ a divisor with normal crossings, and X^0 obtained from X by a finite sequence of blowing-ups along smooth centers. $f^0(F^0) \subset Y$ depends only on v; this closed subset is called the *center of v on Y*. v is *Y-exceptional* if this center has dimension at most $n - 2$, and in this case v has a well-defined discrepancy coefficient with respect to Y.

1.5. Let $f : X \to Y$ be as before, and let $F_j \subset X$ be f-exceptional. The computation of the discrepancy coefficient a_j is local on X; i.e., we may replace Y with an open neighborhood of the generic point of $f(F_j)$, and X with an open neighborhood of the generic point of F_j. From this point of view, the projectivity requirement is irrelevant. In particular, we may consider discrepancy coefficients for *germs* (Y, y)

of algebraic varieties; one such coefficient is associated to each Y-exceptional discrete valuation of $\mathbb{C}(Y)$ whose center on Y contains y.

Moreover, the requirement that X be normal is also irrelevant in some situations; for example, if F_j is a Cartier divisor on X (or at least on some open subset $U \subset X$ with $F_j \cap U \neq \emptyset$), then the generic point of F_j has a *nonsingular* open neighborhood in X, and we may replace X with this neighborhood if we are interested only in the discrepancy coefficient of F_j.

1.6. Definition. A projective variety Y as before (i.e. normal, $\mathbb{Q}$-Gorenstein, n-dimensional) has *only terminal* (*canonical, log-terminal, log-canonical*) *singularities* if all discrepancy coefficients of Y are > 0 (resp. ≥ 0, > -1, ≥ -1). Similarly, Y is terminal (canonical, etc.) at a point y, or the germ (Y, y) is terminal (etc.), if all discrepancy coefficients of divisorial discrete valuations with center containing y are > 0 (resp. ≥ 0, etc.)

Proposition 2 ([1, Proposition 6.5]). *Let $f : X \to Y$ be a proper birational morphism, with X smooth and* $\operatorname{Exc}(f)$ *a divisor with only normal crossings. Let $\Delta = K_X - f^*K_Y = \sum a_j F_j$, and let $\alpha = \min\{a_j\}$.*

If $-1 \leq \alpha \leq 1$, then all *the discrepancy coefficients of Y are $\geq \alpha$ (even for those divisorial discrete valuations of $\mathbb{C}(Y)$ which are Y-exceptional but do not correspond to divisors on X). And if $\alpha \geq 1$, then all the discrepancy coefficients of Y are ≥ 1.*

In particular, to check whether Y (or a germ (Y, y)) is terminal (etc.), it suffices to examine the discrepancy coefficients of a single log-resolution f as above.

We reproduce the proof here (cf. [1]) for the reader's convenience, since the same computation will be used again in 1.7 and in Definition 2.1.

Proof. As explained in 1.4, it suffices to consider a single blowing-up of X along a smooth center $Z \subset X$. Let $h : X' \to X$ be this blowing-up, $F'_j = h^{-1}F_j$ (proper transform), and F' = the exceptional divisor of h. Let $r = \operatorname{codim}_X(Z) \geq 2$. Since $\operatorname{Exc}(f) = \cup F_j$ has only normal crossings, Z is contained in at most r of the divisors F_j; say $Z \subset F_1, \ldots, F_s$, $s \leq r$. Let $f' = f \circ h : X' \to Y$, and $\Delta' = K_{X'} - f'^*K_Y$; then

$$\begin{aligned} \Delta' &= K_{X'} - h^* f^* K_Y = K_{X'} - h^*(K_X - \Delta) \\ &= K_{X'} - h^* K_X + h^*\left(\sum a_j F_j\right) = (r-1)F' + \sum a_j F'_j + \left(\sum_{j=1}^{s} a_j\right) F'; \end{aligned}$$

the discrepancy coefficient of F' is therefore $a' = (r-1) + (\sum_1^s a_j)$. If $\alpha \leq 0$, we have $\sum_1^s a_j \geq s\alpha \geq r\alpha$ (because $s \leq r$ and $\alpha \leq 0$), and therefore $a' \geq (r-1) + r\alpha \geq \alpha$ (because $r > 1$ and $\alpha \geq -1$). If $0 < \alpha \leq 1$, then we get $a' \geq r - 1 \geq 1 \geq \alpha$, and if $\alpha \geq 1$ we get at least $a' \geq 1$. $\square$

REMARKS. 1. The condition $\alpha \leq 1$ can always be achieved for a suitable f, as follows: let $f : X \to Y$ be any log-resolution; choose a smooth subvariety $T \subset X$ of codimension 2, such that $T \not\subseteq \mathrm{Exc}(f)$; and replace f with $f \circ g$, where $g : \widetilde{X} \to X$ is the blowing-up of X along T. The computation used in the proof of the proposition shows that the exceptional divisor of g has discrepancy coefficient 1 relative to Y.
2. If $\alpha < -1$, then the infimum of all discrepancy coefficients relative to Y is $-\infty$, cf. [1, Claim 6.3]. We prove a more precise statement in §2, Lemma 4.

In general, the infimum of all discrepancy coefficients is called the (*total*) *discrepancy* of Y, notation: discrep(Y). Thus discrep(Y) = $-\infty$ if Y is not log-canonical; if Y *is* log-canonical, then $-1 \leq$ discrep(Y) ≤ 1, and discrep(Y) can be calculated by examining a single resolution of singularities $f : X \to Y$ as in the proposition.

We may also define the total discrepancy at a given point y: discrep(Y, y) is the infimum of all discrepancy coefficients of exceptional divisor whose center on Y contains y.
3. If $\alpha \geq 0$, the proof shows that every Y-exceptional discrete valuation of $\mathbb{C}(Y)$, other than those associated to the exceptional divisors of f, has discrepancy coefficient ≥ 1.

1.7. Let (Y, y) be an algebraic germ, as before, and let (Y^{an}, y) be the corresponding analytic germ; note that Y normal and irreducible $\implies$ Y^{an} normal and irreducible. Also, Y $\mathbb{Q}$-Gorenstein $\implies$ Y^{an} $\mathbb{Q}$-Gorenstein. The theory of discrepancy divisors, discrepancy coefficients, terminal singularities, etc., can be developed in parallel in the category of germs of Moishezon analytic spaces; the results discussed so far are identical in the two categories.

An interesting question arises when we try to compare the discrepancy coefficients for (Y, y) and (Y^{an}, y). For example, is it true that (Y, y) is terminal if and only if (Y^{an}, y) is terminal? (If this is true, then "terminal" depends only on the analytic equivalence class of an algebraic germ.) In general, the field of meromorphic functions of Y^{an}, $\mathcal{M}(Y^{an})$, has many divisorial discrete valuations which vanish identically on the rational function field, $\mathbb{C}(Y)$; therefore the question is non-trivial.

The answer is given by the following observation:

Proposition 3. *Let $f : X \to (Y, y)$ be a proper birational morphism with X smooth and* Exc(f) *a divisor with normal crossings. Let $\{F_j\}_{j \in J}$ be the f-exceptional divisors on X, and let $\Delta = \sum a_j F_j$.*

Then the set of all *discrepancy coefficients of (Y, y) is completely determined by the following combinatorial data*:
(1) *The finite set J;*
(2) *The rational numbers a_j (one for each $j \in J$); and*
(3) *For each subset $I \subset J$, the logical value of $\bigcap_{j \in I} F_j \neq \emptyset$* (TRUE or FALSE).

This observation (and its proof below) is valid in the algebraic as well as in the analytic case. In particular, the set of all "algebraic" and the set of all "analytic" discrepancy coefficients of (Y, y) coincide: we may start with the same algebraic resolution $f : X \to (Y, y)$ in the analytic category, as $f^{an} : X^{an} \to (Y^{an}, y)$; then the initial combinatorial data for f^{an} is the same as for f.

Proof. Let v be a Y-exceptional discrete valuation of $\mathbb{C}(Y)$ with center containing y. By [3, Main Theorem II], there exists a finite succession of blowing-ups $f_i : X_{i+1} \to X_i$ along $Z_i \subset X_i$, where $0 \leq i < N$ and $X_0 = X$, with the following properties:

(i) v corresponds to a divisor on X_N;

(ii) Z_i is smooth and irreducible; and

(iii) If $E_0 = \mathrm{Exc}(f)$, and $E_{i+1} = f_i^{-1}(E_i)_{\mathrm{red}} \cup f_i^{-1}(Z_i)_{\mathrm{red}}$, $0 \leq i < N$, then E_i has only normal crossings with Z_i.

(Recall what this means, from [3, Definition 2]: at each point $x \in Z_i$ there is a regular system of parameters of $\mathcal{O}_{X_i,x}$, say $(z_1, \ldots, z_n)$, such that each component of E_i which passes through x has ideal in $\mathcal{O}_{X_i,x}$ generated by one of the z_j, and the ideal of Z_i in $\mathcal{O}_{X_i,x}$ is generated by some of the z_j.)

Let $f_1 : X_1 \to X$ be the blowing-up along a smooth irreducible subvariety $Z \subset X$, of codimension $r \geq 2$, such that $\mathrm{Exc}(f) = \cup_{j \in J} F_j$ has only normal crossings with Z. Say $Z \subset F_j$ if and only if $j \in \{j_1, \ldots, j_s\}$; $s \leq r$.

Considering $g = f \circ f_1 : X_1 \to Y$, we get a new element j' added to J, $J_1 = J \cup \{j'\}$, where j' corresponds to the exceptional divisor F' of f_1. The corresponding number is $a_{j'} = (r-1) + (a_{j_1} + \cdots + a_{j_s})$. Since the F_j have only normal crossings with Z, the "intersection data" for J_1 is completely determined by the data for J, plus the following combinatorial data for Z:

(4) For each $I \subset J$, the non-negative integer $d_I = \dim(Z \cap [\bigcap_{j \in I} F_j])$.

(Note that this collection of data contains, in particular, the codimension r of Z, in the form $d_\emptyset = n - r$, and also the information about which F_j's contain Z, in the form $Z \subset F_j \Leftrightarrow d_{\{j\}} = n - r$.)

Finally, which such functions $\{d_I\}_{I \subset J}$ are possible is completely determined by the "intersection data" for J. Since every discrepancy coefficient of Y is obtained after a finite number of such elementary operations on the combinatorial data (corresponding to a succession of blowing-ups along smooth centers), the result follows by induction. $\square$

2. Minimal discrepancies and Shokurov's conjecture

2.1. Definition. Let (Y, y) be an algebraic or analytic germ (as always, we assume it is normal, $\mathbb{Q}$-Gorenstein, n-dimensional). The *minimal discrepancy of Y at y*, $\mathrm{md}_y(Y)$, is the infimum of all discrepancy coefficients of divisorial discrete valuations of $\mathbb{C}(Y)$, resp. $\mathcal{M}(Y^{an})$, whose center on Y is y.

Lemma 4. *If* (Y, y) *is not log-canonical at* y*, then* $\mathrm{md}_y(Y) = -\infty$.

Proof. Let $f : X \to (Y, y)$ be a resolution of singularities with $\mathrm{Exc}(f)$ a divisor with normal crossings. We may also assume that $f^{-1}(y)$ is a union of (f-exceptional) divisors. Let $F_j \subset X$ be an f-exceptional divisor with $y \in f(F_j)$ and discrepancy coefficient $a_j < -1$. Since $f^{-1}(y)$ is a union of f-exceptional divisors, and F_j meets $f^{-1}(y)$, there is at least one exceptional divisor F_i with $f(F_i) = \{y\}$ and $F_i \cap F_j \neq \emptyset$. We may assume that F_i and F_j are distinct (if $F_j \subset f^{-1}(y)$ and it is the only component of the fiber, we may blow up X at a point of F_j; then take the exceptional divisor of this blowing-up in place of F_i, and the proper transform of F_j in place of F_j). Set $Z = F_i \cap F_j$; then Z is a smooth subvariety of codimension 2 in X, and is not contained in any other exceptional divisor. Let a_i be the discrepancy coefficient of F_i.

Let $g : \widetilde{X} \to X$ be the blowing-up of X along Z. Let F' be its exceptional divisor, with discrepancy coefficient a' relative to Y. Then $a' = 1 + a_j + a_i$ (see the proof of Proposition 2 in §1). Moreover, F' has center $\{y\}$ on Y, and intersects the proper transform F_j' of F_j on $\tilde{X}$ (which has discrepancy coefficient $a_j' = a_j$ relative to Y).

Note that $a_j < -1 \implies a' < a_i$. In fact, since all the discrepancy coefficients of (Y, y) are integer multiples of $1/r$ (if r is the index of K_Y at y), we see that $a' \leq a_i - 1/r$. Therefore the proof may be completed by induction. □

2.2. Recall the statement of Shokurov's conjecture from the Introduction. The lemma we have just proved shows that the conjecture is true for non-log-canonical singularities.

Shokurov's conjecture is vacuously true for curves (there are no singular normal points in dimension 1). It is also true in dimension 2: if (Y, y) is a normal singularity and $f : X \to (Y, y)$ is the *minimal* desingularization, then *all* the coefficients of $\Delta = K_X - f^*K_Y$ are ≤ 0.

In dimension $n \geq 3$, let (Y, y) be an *isolated* singularity. Then there exists a resolution of singularities $f : X \to (Y, y)$ with $\mathrm{Exc}(f)$ a divisor with normal crossings and such that $\mathrm{Exc}(f) = f^{-1}(y)$; see [3, Main Theorem I in the strong form, p. 132]. If $\alpha = \mathrm{discrep}(Y, y) < 1$ (see 1.6, Remark 2), then Proposition 2 shows that some divisor F_j in $\mathrm{Exc}(f)$ has discrepancy coefficient equal to α. Then $\mathrm{md}_y(Y) = \alpha$, for $f(F_j) = \{y\}$, and in particular Shokurov's conjecture is true in this case. In other words, for *isolated* singularities (in any dimension) there only remains to prove Shokurov's conjecture when $\mathrm{discrep}(Y, y) = 1$. (For singularities with index one, the last condition is equivalent to "terminal".)

2.3. The following lemma shows that the conjecture can be reduced to the case of singularities of index one:

Lemma 5. *Let $\varphi : Y' \to Y$ be a finite morphism of normal, $\mathbb{Q}$-Gorenstein varieties. Assume that φ is étale in codimension one. Let y' be a point of Y', and $y = \varphi(y')$.*

Then $\mathrm{md}_y(Y) \le \mathrm{md}_{y'}(Y')$.

In particular, if (Y, y) has index r, then there exists a $\varphi : Y' \to Y$ as in the lemma, with Y' having index one (the "index-one cover", cf. [1, Definition 6.8]). Thus it would suffice to prove Shokurov's conjecture for singularities of index one.

Proof (cf. [2, proof of Lemma 2.2]). Let $f' : X' \to Y'$ be a resolution of singularities of Y' such that $\mathrm{md}_{y'}(Y') = \mathrm{ord}_{F'}(\Delta') \ge -1$, where $\Delta' = K_{X'} - f'^* K_{Y'}$ and F' is a divisor on X' with $f'(F') = \{y'\}$. (If $\mathrm{md}_{y'}(Y') = -\infty$, let $\alpha < -1$ be a rational number, and choose f', F' such that $f'(F') = \{y'\}$ and $\mathrm{ord}_{F'}(\Delta') < \alpha$.)

Let $f : X \to Y$ be a resolution of singularities of Y. By blowing up X, then X', if necessary, we may assume that $\psi = f^{-1} \circ \varphi \circ f' : X' \to X$ is a morphism and that $\psi(F')$ is a divisor $F \subset X$. Let $\Delta = K_X - f^* K_Y$ and $a = \mathrm{ord}_F(\Delta)$.

Let t be the ramification index of ψ along F'. Then:

$$
\begin{aligned}
K_{X'} &= \psi^* K_X + (t-1)F' + \text{ other terms} \\
&= \psi^*(f^* K_Y + aF + \text{ other terms }) + (t-1)F' + \text{ other terms} \\
&= \psi^* f^* K_Y + (ta + t - 1)F' + \text{ other terms} \\
&= f'^* \varphi^* K_Y + (ta + t - 1)F' + \text{ other terms} \\
&= f'^* K_{Y'} + (ta + t - 1)F' + \text{ other terms}
\end{aligned}
$$

(note that $\varphi^* K_Y = K_{Y'}$, because φ is étale in codimension one). Therefore $\mathrm{ord}_{F'}(\Delta') = ta + t - 1$.

If $\mathrm{ord}_{F'}(\Delta') \ge -1$, we get $\mathrm{ord}_F(\Delta) = a \le \mathrm{ord}_{F'}(\Delta') = \mathrm{md}_{y'}(Y')$; indeed, $t \ge 1$, and therefore $ta \le ta + (t-1)(1 + \mathrm{ord}_{F'}(\Delta')) = t\,\mathrm{ord}_{F'}(\Delta')$.

If $\mathrm{ord}_{F'}(\Delta') < \alpha < -1$, then $\mathrm{ord}_F(\Delta) = a \le (1/t)\,\mathrm{ord}_{F'}(\Delta') < (1/t)\alpha$, with $1 \le t \le \deg(\varphi)$ and α an arbitrarily negative rational number.

Since $f(F) = \{y\}$, the lemma is proved. □

2.4. Now we show that $\mathrm{md}_y(Y)$ is an analytic invariant. In fact, we show that the set of all discrepancy coefficients for divisors with center $\{y\}$ on Y is the same in the algebraic and in the analytic category.

Proposition 6. *Let $f : X \to (Y, y)$ be a resolution of singularities, as in* Proposition 3. *Then the set of all discrepancy coefficients for divisors with center $\{y\}$ on Y is completely determined by the combinatorial data* (1), (2), (3) *in* Proposition 3, *plus*: (3+) *For each $j \in J$, the logical value of "$f(F_j) = \{y\}$"* (TRUE or FALSE).

Proof. Let $f_1 : X_1 \to X$ be the blowing-up of a smooth subvariety $Z \subset X$, as in the proof of Proposition 3. Put $g = f \circ f_1$, and let F' be the exceptional divisor of f_1. Then $[g(F') = \{y\}] \Leftrightarrow [f(F_j) = \{y\}$ for at least one of the F_j's containing $Z]$. Indeed, if $Z \subset F_j$ and $f(F_j) = \{y\}$, then $g(F') = f(Z) \subset \{y\}$, so that in fact $g(F') = \{y\}$. Conversely, $g(F') = \{y\} \implies Z \subset f^{-1}(y)$. As Z is irreducible and $f^{-1}(y)$ is a union of divisors F_j with $f(F_j) = \{y\}$, Z must be contained in at least one such F_j.

Therefore the "extended" combinatorial data for g (including the information in (3+)) can be obtained from the "extended" combinatorial data for f. The conclusion follows by induction. □

2.5. Finally, we show that Shokurov's conjecture is true for non-terminal threefold singularities.

Theorem 7. *Let* (Y, y) *be a non-terminal three-dimensional singularity* (*normal and* $\mathbb{Q}$*-Gorenstein as always*)*. Then* $md_y(Y) \leq 1$.

Proof (S. Ishii). By [8, Theorem (0.3.12), (i)], there exists a projective birational morphism $f : X \to Y$ such that X has only ($\mathbb{Q}$-factorial) terminal singularities and K_X is f-semiample. (X, or f, is called a *$\mathbb{Q}$-factorial terminal modification* of Y.)

Write $K_X - f^*K_Y = \sum a_j F_j$; then $a_j \leq 0$, $\forall j$, since K_X is f-nef. Since (Y, y) is not terminal, f is not an isomorphism above y. And since y is normal and f is birational, this means that $f^{-1}(y)$ contains at least one integral curve C.

Let $g : X' \to X$ be the blowing-up of X along C, with exceptional divisor F, and put $h = f \circ g : X' \to Y$. Note that X' may be non-normal; but even then, since terminal threefold singularities are isolated and C is integral, both X and C are smooth at the generic point of C. Therefore X' and F are smooth at the generic point of F, and the discrepancy coefficient of F with respect to X is 1.

Then $K_{X'} - h^*K_Y = F + \sum a_j g^* F_j$; since $a_j \leq 0$ for all j and the coefficient of F in each g^*F_j is non-negative, we see that the discrepancy coefficient of F with respect to Y is ≤ 1. As $h(F) = f(C) = \{y\}$, we get $\mathrm{md}_y(Y) \leq 1$, as stated. □

REMARK. Putting together Theorem 7 and the main theorem in [7] (or our computations in §4), we see that Shokurov's conjecture is true in dimension 3.

REMARK. In several conference and seminar talks I gave on the results contained in this paper, I was repeatedly asked to comment on the following passage from the Utah seminar [6, Remark 17.1.3] (slightly modified and simplified):

> ... Assume that [Shokurov's] conjecture [in *any* dimension] fails for $y \in Y$. Then Y is terminal. Thus if a list of terminal singularities is known, the conjecture can be verified. [...] For $\dim Y = 3$ it was checked by Markushevich. [*Then an incorrect bibliographical reference is given. The correct one is* [7], *which was pub-*

lished several years after *the Utah seminar.*]

These same claims have been circulated among the experts for some time. For example, J. Kollár repeated them at the Santa Cruz Summer Institute in 1995.

Here is my take on these claims. First, if the conjecture fails for $y \in Y$, I don't see why Y must be terminal – unless the existence of a terminal modification of Y is known, as is the case in dimension 3. (And even then, if the conjecture is false for terminal singularities, then it may be false for non-terminal singularities as well; in that case the existence of a terminal modification wouldn't solve the problem.) I believe that any questions related to this point should be addressed to the author of that passage, not to me.

And second, even if a list of terminal singularities were known, I don't know how Shokurov's conjecture could be verified – even, say, for hypersurface singularities in dimension at least 4, and even if explicit equations were known. I am not aware of any general method for calculating the minimal discrepancy even when the equation is given. The computations in dimension 3, both in this paper and in [7], are ad hoc – in a sense, we just got lucky here. Again, I believe that any further questions should be addressed to the author of the passage quoted above, who certainly knows much more about these things than me.

3. Proof of Theorem 1

3.1. Recall the statement of Theorem 1 from the Introduction. By 2.4, we may assume that Y is the hypersurface $G = 0$ in $\mathbb{A}^{n+1}$, with $y = 0$. For convenience, denote $\mathbb{A}^{n+1}$ by V; thus $Y \subset V$. Let $U = \mathbb{A}^{n+1}$; write the coordinates in V as $(y_1, \ldots, y_{n+1})$, and the coordinates in U as $(u_1, \ldots, u_n, t)$.

Let $f : U \to V$ be the birational morphism defined by $y_{n+1} = t; y_i = t^{a_i}u_i, i = 1, \ldots, n$. Let $E \subset U$ be the hyperplane $(t = 0)$; then $\mathrm{Exc}(f) = E$. (Of course, f is just one affine patch in a weighted blowing-up of V at the origin, but this observation plays no role in the proof.)

3.2. Let $\overline{Y} \subset U$ be the proper transform of Y by f, $\overline{f} : \overline{Y} \to Y$ the restriction of f to $\overline{Y}$, and $\overline{E} = E|_{\overline{Y}}$ (as a Cartier divisor). By hypothesis, $f^*Y = \overline{Y} + AE$, and $\overline{E}$ has equation $\phi(u_1, \ldots, u_n) = 0$ in $E \cong \mathbb{A}^n$. Since ϕ has at least one irreducible factor with exponent 1, $\overline{E}$ has at least one irreducible component with multiplicity one: $\overline{E} = F_1 + \cdots$. As explained in 1.5, since $\overline{Y}$ is smooth in a neighborhood of the generic point of F_1, and F_1 is the exceptional divisor which will display the desired discrepancy coefficient, we need not worry about the normality of $\overline{Y}$.

3.3. Take $\omega = dy_1 \wedge \cdots \wedge dy_{n+1}$ on V; then $f^*\omega = t^{a_1+\cdots+a_n} du_1 \wedge \cdots \wedge du_n \wedge dt$ on U, so that $K_U - f^*K_V = (a_1 + \cdots + a_n)E$.

The adjunction formula gives $K_Y = K_V + Y|_Y$ and $K_{\overline{Y}} = K_U + \overline{Y}|_{\overline{Y}}$. Therefore we have:

$$\begin{aligned} K_{\overline{Y}} - \overline{f}^* K_Y &= (K_U + \overline{Y})|_{\overline{Y}} - \overline{f}^*(K_V + Y|_Y) \\ &= (K_U - f^* K_V + \overline{Y} - f^* Y)|_{\overline{Y}} \\ &= ((a_1 + \cdots + a_n)E - AE)|_{\overline{Y}} \\ &= d\overline{E} = dF_1 + \cdots . \end{aligned}$$

(Recall that $d = (a_1 + \cdots + a_n) - A$.)

Thus the discrepancy coefficient of $F_1 \subset \overline{Y}$ with respect to Y is equal to d. Since $\overline{f}(F_1) = \{y\}$, Theorem 1 is proved. $\square$

3.4. Example. Let (Y, y) be a singular germ of multiplicity 2; that is, Y is a hypersurface in $\mathbb{A}^{n+1}$ given by an equation $G = 0$; $y = 0$; G and all its first-order partial derivatives at 0 are equal to zero, and some second-order partial derivative of G at 0 is non-zero.

If $(Y, 0)$ has rank at least 2, then $\mathrm{md}_y(Y) \leq n - 2$ (as predicted by Shokurov's conjecture). Indeed, consider the usual blowing-up of V at 0; that is, take $a_1 = \cdots = a_n = 1$. The hypothesis means that $A = 2$, and — after a linear change of coordinates, if necessary — $\phi(u_1, \ldots, u_n) = u_1^2 + \cdots + u_r^2$, where $r \geq 2$ is the rank of the singularity. Thus $d = (a_1 + \cdots + a_n) - A = n - 2$ in this case, and ϕ is irreducible (if $r \geq 3$), resp. a product of two distinct irreducible factors, if $r = 2$.

4. Minimal discrepancies of log-terminal threefold singularities

Let (Y, y) be a three-dimensional log-terminal singularity. In this section we will show that $\mathrm{md}_y(Y) \leq 1$, without using the existence of a terminal modification.

4.1. As shown in 2.3, we may assume that (Y, y) has index one. Then (Y, y) is canonical (the index-one cover of a log-terminal singularity is again log-terminal, by Proposition 2, and therefore canonical).

In this case, M. Reid [9, Theorem 2.2] proved that either (Y, y) is a *cDV* point (see below), or there exists a proper birational morphism $f : Y' \to Y$ with $f^* K_Y = K_{Y'}$ and $f^{-1}(y)$ containing at least one prime divisor of Y'. Of course, in the latter case we have $\mathrm{md}_y(Y) = 0$. There only remains to consider the case when (Y, y) is a compound Du Val (*cDV*) point; that is, (Y, y) is analytically equivalent to a hypersurface singularity at the origin $0 \in \mathbb{A}^4$, with equation $G = 0$,

$$G(y_1, y_2, y_3, t) = f(y_1, y_2, y_3) + tg(y_1, y_2, y_3, t),$$

where $f(y_1, y_2, y_3) = 0$ defines a Du Val singularity (rational double point) of a surface at $0 \in \mathbb{A}^3$.

To simplify notation, we write $\mathbf{y}$ for y_1, y_2, y_3 and $\mathbf{u}$ for u_1, u_2, u_3.

By 2.4, we may assume that (Y, y) *is* the hypersurface $(G = 0) \subset \mathbb{A}^4$, with $y = 0$. By Theorem 1, it suffices to find $a_1, a_2, a_3 \geq 1$ such that

$$G(t^{a_1}u_1, t^{a_2}u_2, t^{a_3}u_3, t) = t^A\phi(\mathbf{u}) + t^{A+1}\psi(\mathbf{u}, t)$$

with $\phi(\mathbf{u}) \neq 0$, $(a_1 + a_2 + a_3) - A = 1$, and ϕ having at least one irreducible factor with exponent one in its prime decomposition.

4.2. We will do a case-by-case analysis, according to the type of singularity; $(Y, 0)$ is of type cA_n, cD_n, or cE_n, if the surface singularity $f(\mathbf{y}) = 0 \subset \mathbb{A}^3$ is of type A_n, D_n, or E_n.

In each case, $f(\mathbf{y})$ is completely known. $g(\mathbf{y}, t)$, on the other hand, is not. Of course, we have $g(0, 0, 0, 0) = 0$, or else $(Y, 0)$ would be a smooth point. We will not make any other assumptions about g.

Write $g = g_1 + g_2 + \cdots$, where g_i is a homogeneous form of degree i, and $o_k(g) = g_k + g_{k+1} + \cdots$ $(k \geq 1)$.

A note on terminology: we distinguish between *form* and *polynomial*; for instance, a quadratic polynomial is the sum of a quadratic form, a linear form, and a constant term. We say that a polynomial (or a form) *contains* a certain monomial if the coefficient of the monomial in that polynomial is non-zero. We say that a monomial *contains* y_1 if that monomial is divisible by y_1.

4.3. Case $\mathbf{cA_n}$: $f(\mathbf{y}) = y_1^2 + y_2^2 + y_3^{n+1}$ $(n \geq 1)$.

Then $(Y, 0)$ is a singularity of multiplicity 2 and rank at least 2. This case is therefore covered by Example 3.4.

4.4. Case $\mathbf{cD_n}$: $f(\mathbf{y}) = y_1^2 + y_2^2 y_3 + y_3^{n-1}$ $(n \geq 4)$.

If $g_1(\mathbf{y}, t) \neq 0$, then the quadratic part of $G = f + tg$ is $y_1^2 + tg_1(\mathbf{y}, t)$. If this quadratic part has rank at least 2, then the conclusion follows from Example 3.4. If it has rank 1, i.e. if $y_1^2 + tg_1(\mathbf{y}, t)$ is the square of a linear form, then a linear change of variable, $y_1' = y_1 + \alpha y_2 + \beta y_3 + \gamma t$, transforms the equation $G = 0$ into a similar one with $g_1(\mathbf{y}, t) = 0$.

So we need to consider only the case $g_1 = 0$. Note that a similar argument applies to singularities of type cE_n.

Assume that $g_1 = 0$. Then put $a_1 = 2$, $a_2 = a_3 = 1$; that is, put $y_1 = t^2u_1$, $y_2 = tu_2$, $y_3 = tu_3$. We have:

$$G(t^2u_1, tu_2, tu_3, t) = t^3\phi(\mathbf{u}) + t^4\psi(\mathbf{u}, t),$$

where $\phi(\mathbf{u}) = u_2^2u_3 + \delta_{n,4}u_3^3 +$ [terms of degree ≤ 2 in the u_j]; $\delta_{n,4} = 1$ if $n = 4$, otherwise $\delta_{n,4} = 0$. (The terms of lower degree come from $tg_2(t^2u_1, tu_2, tu_3, t)$, with g_2 — the quadratic component of $g(\mathbf{y}, t)$. Note that not *all* the terms in tg_2 contribute to

$\phi(\mathbf{u})$: as $y_1 = t^2u_1$, the monomials in $g_2(\mathbf{y}, t)$ which contain y_1 give rise to monomials containing t to the fourth or higher power.)

The proof in this case is complete, for $(a_1 + a_2 + a_3) - A = (2 + 1 + 1) - 3 = 1$, and ϕ has at least one irreducible factor with exponent one (otherwise ϕ would have to be the cube of a linear polynomial in $\mathbf{u}$; that linear polynomial would have to contain u_2, because ϕ contains $u_2^2u_3$, and then ϕ, being the cube of that linear polynomial, would contain u_2^3, which is not the case).

4.5. Case $\mathbf{cE_6}$: $f(\mathbf{y}) = y_1^2 + y_2^3 + y_3^4$.

As in 4.4, we may assume that the linear part $g_1(\mathbf{y}, t)$ of $g(\mathbf{y}, t)$ is equal to zero.

In g_2 (the quadratic part of g), separate the monomials which contain y_1 from those that don't: $g_2(\mathbf{y}, t) = y_1L(\mathbf{y}, t) + Q(y_2, y_3, t)$, where L is a linear form and Q is a quadratic form.

Put $y_1 = t^2u_1$, $y_2 = tu_2$, $y_3 = tu_3$; then

$$G(t^2u_1, tu_2, tu_3, t) = t^3\phi(\mathbf{u}) + t^4\psi(\mathbf{u}, t),$$

with $\phi(\mathbf{u}) = u_2^3 + Q(u_2, u_3, 1)$.

If ϕ is not the cube of a linear polynomial, then we complete the proof just as in 4.4. However, in this case it might be that ϕ *is* a perfect cube. If this is so, then $y_2^3 + tQ(y_2, y_3, t)$ is the cube of a linear form in y_2, y_3, t. A linear change of variable, $y_2' = y_2 + \alpha y_3 + \beta t$, reduces the proof to the case $Q = 0$. This argument is valid also in the cases cE_7 and cE_8, discussed below.

There only remains to consider the case $g_2(\mathbf{y}, t) = y_1L(\mathbf{y}, t)$, where L is a linear form (possibly zero). In this case put $a_1 = a_2 = 2, a_3 = 1$, i.e. $y_1 = t^2u_1, y_2 = t^2u_2, y_3 = tu_3$. Then:

$$G(\mathbf{y}, t) = y_1^2 + y_2^3 + y_3^4 + t[y_1L(\mathbf{y}, t) + o_3(g)], \quad \text{and}$$
$$G(t^2u_1, t^2u_2, tu_3, t) = t^4\phi(\mathbf{u}) + t^5\psi(\mathbf{u}, t),$$

where $\phi(\mathbf{u}) = u_1^2 + u_3^4 +$ [terms of degree ≤ 3 in the u_j], and the expression in brackets does not contain u_1^2 (so that u_1^2 doesn't cancel out from ϕ).

Note that $\deg(\phi) = 4$, and ϕ cannot be the square of a quadratic polynomial in the u_j (if it were, then ϕ would contain the mixed product $u_1u_3^2$, because it contains u_1^2 and u_3^4 but no u_1^4; the monomial $u_1u_3^2$ could only arise from a monomial $t(y_1y_3^2t^k)$ of $tg(\mathbf{y}, t)$, $k \geq 0$; but $t(t^2u_1)(tu_3)^2t^k = t^{5+k}u_1u_3^2$, so $u_1u_3^2$ cannot be a monomial of ϕ). Therefore ϕ has an irreducible factor with exponent one, and the conclusion follows — note that $(a_1 + a_2 + a_3) - A = (2 + 2 + 1) - 4 = 1$.

4.6. Case cE$_7$: $f(\mathbf{y}) = y_1^2 + y_2^3 + y_2 y_3^3$.

We may again assume that $g_1 = 0$, as in 4.4, and that $g_2 = y_1 L(\mathbf{y}, t)$ with L a linear form (possibly zero), as in 4.5.

Write $L(\mathbf{y}, t) = L_1(y_1, y_2) + L_2(y_3, t)$, and $g_3(\mathbf{y}, t) = C_1(\mathbf{y}, t) + C_2(y_3, t)$, where C_1 and C_2 are cubic forms such that every monomial of C_1 contains y_1 or y_2.

Put $a_1 = a_2 = 2, a_3 = 1$; then

$$\begin{aligned} G(\mathbf{y}, t) &= y_1^2 + y_2^3 + y_2 y_3^3 + t[y_1 L_1(y_1, y_2) + y_1 L_2(y_3, t) \\ &\qquad + C_1(\mathbf{y}, t) + C_2(y_3, t) + o_4(g)], \quad \text{and} \\ G(t^2 u_1, t^2 u_2, t u_3, t) &= t^4 \phi(\mathbf{u}) + t^5 \psi(\mathbf{u}, t), \end{aligned}$$

where $\phi(\mathbf{u}) = u_1^2 + u_1 L_2(u_3, 1) + C_2(u_3, 1)$.

Note that $(a_1 + a_2 + a_3) - A = (2 + 2 + 1) - 4 = 1$.

If ϕ has degree 3 (i.e. if $C_2(y_3, t)$ contains y_3^3), then ϕ has an irreducible factor with exponent one, because ϕ cannot be a perfect cube (it contains u_1^2 but no u_1^3); in this case the proof is complete.

Otherwise ϕ has degree 2 (for it contains u_1^2). Then either ϕ has an irreducible factor with exponent one (and then the proof is complete), or else ϕ is the square of a linear polynomial. In the latter case, $y_1^2 + t y_1 L_2(y_3, t) + t C_2(y_3, t)$ is a perfect square. The (non-linear) change of variable $y_1' = y_1 + (1/2) t L_2(y_3, t)$ transforms the equation $G = 0$ into a similar one with $L_2 = C_2 = 0$. Therefore we may assume that G has the form:

$$G(\mathbf{y}, t) = y_1^2 + y_2^3 + y_2 y_3^3 + t[y_1 L_1(y_1, y_2) + C_1(\mathbf{y}, t) + o_4(g)],$$

where L_1 is a linear form, and C_1 is a cubic form such that every monomial of C_1 contains y_1 or y_2. (The same argument carries over unchanged to the last case, cE_8.)

Now put $a_1 = 3, a_2 = 2, a_3 = 1$; that is, $y_1 = t^3 u_1, y_2 = t^2 u_2, y_3 = t u_3$. We have:

$$G(t^3 u_1, t^2 u_2, t u_3, t) = t^5 \phi(\mathbf{u}) + t^6 \psi(\mathbf{u}, t),$$

where $\phi(\mathbf{u}) = u_2 u_3^3 + u_2 p(u_3) + q(u_3)$; $u_2 p(u_3)$ corresponds to the monomials of $C_1(\mathbf{y}, t)$ of the form $y_2 y_3^k t^{2-k}, k = 0, 1, 2$ (all other monomials of C_1 produce at least a sixth power of t; recall that all monomials of the cubic form C_1 contain y_1 or y_2), and $q(u_3)$ corresponds to the monomials of g_4 of the form $y_3^k t^{4-k}, k = 0, \ldots, 4$. Note that ϕ has degree exactly one as a polynomial in u_2, and therefore ϕ cannot be the square of another polynomial in the u_j. Since ϕ has (total) degree 4, it must have an irreducible factor with exponent one. As $(a_1 + a_2 + a_3) - A = (3 + 2 + 1) - 5 = 1$, the proof is complete in this case.

4.7. Case cE$_8$: $f(\mathbf{y}) = y_1^2 + y_2^3 + y_3^5$.
As in the previous case, we may assume that

$$G(\mathbf{y}, t) = y_1^2 + y_2^3 + y_3^5 + t[y_1 L_1(y_1, y_2) + C_1(\mathbf{y}, t) + o_4(g)],$$

where L_1 is a linear form and C_1 is a cubic form such that every monomial of C_1 contains y_1 or y_2.

If we take $a_1 = 3, a_2 = 2, a_3 = 1$, i.e. $y_1 = t^3 u_1, y_2 = t^2 u_2, y_3 = t u_3$, we get

$$G(t^3 u_1, t^2 u_2, t u_3, t) = t^5 \phi(\mathbf{u}) + t^6 \psi(\mathbf{u}, t)$$

with $\phi(\mathbf{u}) = u_3^5 + u_2 p(u_3) + q(u_3)$, where $p(u_3)$ and $q(u_3)$ are exactly as in the previous case.

$(a_1 + a_2 + a_3) - A = (3 + 2 + 1) - 5 = 1$; so the proof is complete if ϕ has an irreducible factor with exponent one.

Since $\deg(\phi) = 5$, if ϕ does not have an irreducible factor with exponent one then $\phi = M^2 N^3$ for two linear polynomials $M = M(u_2, u_3)$ and $N = N(u_2, u_3)$ (possibly equal). In particular, if this is the case then ϕ cannot have degree exactly one as a polynomial in u_2, and therefore $p(u_3) = 0$; i.e., $C_1(\mathbf{y}, t)$ contains no monomials of the form $y_2 y_3^k t^{2-k}$ ($k = 0, 1, 2$). As every monomial of C_1 contains y_1 or y_2, this means that every monomial of C_1 actually contains y_1 or y_2^2.

Now $\phi(\mathbf{u}) = u_3^5 + q(u_3)$ (q of degree at most four) $= M^2(u_3) N^3(u_3)$. If we write $g_4(\mathbf{y}, t) = F_1(\mathbf{y}, t) + F_2(y_3, t)$, with F_1, F_2 forms of degree 4 such that every monomial of F_1 contains y_1 or y_2, then $q(u_3) = F_2(u_3, 1)$. $u_3^5 + q(u_3) = M^2(u_3) N^3(u_3)$ means that $y_3^5 + F_2(y_3, t) = \tilde{M}^2(y_3, t) \tilde{N}^3(y_3, t)$, with $\tilde{M}, \tilde{N}$ linear forms in y_3, t ($M(u_3) = \tilde{M}(u_3, 1)$, etc.) A linear change of variable $y_3' = y_3 + \alpha t$ reduces G to the form

$$G(\mathbf{y}, t) = y_1^2 + y_2^3 + y_3^3 (y_3 + at)^2 + t[y_1 L(y_1, y_2) + C_1(\mathbf{y}, t) + F_1(\mathbf{y}, t) + o_5(g)],$$

where $a \in \mathbb{C}$ (possibly $a = 0$), every monomial of the cubic form C_1 contains y_1 or y_2^2, and every monomial of the quartic form F_1 contains y_1 or y_2.

Put $a_1 = 3, a_2 = a_3 = 2$; that is, $y_1 = t^3 u_1, y_2 = t^2 u_2, y_3 = t^2 u_3$. Then

$$G(t^3 u_1, t^2 u_2, t^2 u_3, t) = t^6 \phi(\mathbf{u}) + t^7 \psi(\mathbf{u}, t)$$

with $\phi(\mathbf{u}) = u_1^2 + u_2^3 +$ [terms of degree at most 2 in the u_j], and u_1^2 is not among the terms inside the brackets. Therefore ϕ has an irreducible factor with exponent one; as $(a_1 + a_2 + a_3) - A = (3 + 2 + 2) - 6 = 1$, the proof is complete in all cases. □

4.8. Remarks. 1. If (Y, y) is a terminal threefold singularity of index one, then $\mathrm{md}_y(Y) = 1$. On the other hand, Kawamata [4] proved that the minimal discrepancy of a terminal threefold singularity of index $r \geq 2$ is $1/r$.
2. Our computations in §4 seem related to those in [7], except for the fact that Markushevich uses the toric language. At first glance, it looks like his proof works

only if the singularity is isolated; however, this is needed only to reduce the equation $G = 0$ to various standard forms, and — as our elementary computations in 4.3-4.7 show — this can be done without assuming the singularity is isolated.

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