LINEAR PROJECTIONS AND SUCCESSIVE MINIMA

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Abstract. Let X be an arithmetic surface, and let L be a line bundle on X. Choose a metric h on the lattice Λ of sections of L over X. When the degree of the generic fiber of X is large enough, we get lower bounds for the successive minima of (Λ, h) in terms of the normalized height of X. The proof uses an effective version (due to C. Voisin) of a theorem of Segre on linear projections and Morrison's proof that smooth projective curves of high degree are Chow semistable.

In arithmetic geometry, cohomology groups are not vector spaces as in classical algebraic geometry but rather are Euclidean lattices. As a consequence, to understand these groups we need to evaluate not only their rank but also their successive minima, which are fundamental invariants in the geometry of numbers. The goal of this article is to perform this task for line bundles on projective curves.

Let K be a number field, let $\mathcal{O}_K$ be its ring of integers, and let E be a projective $\mathcal{O}_K$-module of finite rank N. We endow $E \otimes_{\mathbb{Z}} \mathbb{C}$ with a Hermitian metric h, and we let $\mu_1, \ldots, \mu_N$ be the logarithm of the successive minima of (E, h). Assume that $X_K \subset \mathbb{P}(E_K^\vee)$ is a smooth geometrically irreducible curve of genus $g > 0$. We shall find a lower bound for the numbers μ_i, $g + 8 \le i \le N - 3$, in terms of a normalized height of X_K and the average of the μ_i (Theorem 2). This result is a complement to [13, Theorem 4], which gives a lower bound for μ_1.

The method of proof is a variant of [13, Theorem 4]. It relies upon Morrison's [11] proof of the fact that X_K is Chow semistable. We use a filtration $V_1 = E_K \supset V_2 \supset \cdots \supset V_N$ of the vector space E_K. But, contrary to [13], this filtration is chosen such that, for suitable values of i, the projection $\mathbb{P}(V_i^\vee) \cdots \to \mathbb{P}(V_{i+1}^\vee)$ does not change the degree of the image of X_K. That such a choice is possible follows from a result of Voisin, namely, an effective version of a theorem of Segre on linear projections of complex projective

curves (Theorem 1). I thank C. Voisin for proving that result and for helpful discussions. I am also grateful to the referee for useful corrections.

§1. Linear projections of projective curves

Let $C \subset \mathbb{P}^n$ be an integral projective curve over $\mathbb{C}$, and let d be its degree. Assume that C is not contained in some hyperplane, that $d \geq 3$, and that $n \geq 3$.

THEOREM 1 (Voisin). *There exists an integer $A(d)$ and a finite set Σ of points in $(\mathbb{P}^n - C)(\mathbb{C})$, of order at most $A(d)$, such that, for every point $P \in \mathbb{P}^n(\mathbb{C}) - \Sigma \cup C(\mathbb{C})$, the linear projection $\mathbb{P}^n \cdots \to \mathbb{P}^{n-1}$ of center P maps C birationally onto its image.*

Proof. The existence of a finite set Σ with the property above is a special case of a theorem of Segre (see [5]). The order of Σ can be bounded as follows by a function of d.

If $n > 3$, a generic linear projection into $\mathbb{P}^3$ will map C isomorphically onto its image [10], and the exceptional set $\Sigma \subset \mathbb{P}^n$ will map bijectively onto the exceptional set in $\mathbb{P}^3$. Therefore we can assume that $n = 3$.

When the projection with center $P \in \mathbb{P}^3(\mathbb{C})$ is not birational from the curve C to its image $C' \subset \mathbb{P}^2$, we have $d' = \deg(C') \leq d/2$; hence $d' \leq d - 2$, and P is the vertex of a cone K with base C' containing C. So we have to bound the number of such cones.

Let N be the dimension of the kernel of the restriction map

$$\alpha : H^0\big(\mathbb{P}^3, \mathcal{O}(d')\big) \to H^0\big(C, \mathcal{O}(d')\big).$$

Clearly, N is bounded as a function of d, and any $f \in \ker(\alpha)$ is a homogeneous polynomial of degree d' that vanishes on C.

Let $Z \subset \mathbb{P}^3(\mathbb{C}) \times \mathbb{P}^{N-1}(\mathbb{C})$ be the set of pairs (P, f) such that f is the equation of a cone K of vertex P. If $p_1 : \mathbb{P}^3 \times \mathbb{P}^{N-1} \to \mathbb{P}^3$ is the first projection, we have to bound the order of $p_1(Z)$. We note that this order is at most the number c of connected components of Z.

Now Z is defined by equations of bidegree $(\delta, 1)$, $\delta \leq d'$. Indeed, f is homogeneous of degree d', and $(P, f) \in Z$ when all the derivatives of f, except those of order d', vanish at P.

Let $L = \mathcal{O}(d', 1)$, let $M = \dim H^0(\mathbb{P}^3 \times \mathbb{P}^N, L) - 1$, and let

$$j : \mathbb{P}^3 \times \mathbb{P}^N \to \mathbb{P}^M$$

be the Segre embedding. Since $j(Z)$ is the intersection of $j(\mathbb{P}^3 \times \mathbb{P}^N)$ with linear hyperplanes, the Bézout theorem [7, §8.4] tells us that

$$c \leq \deg\bigl(j(\mathbb{P}^3 \times \mathbb{P}^N)\bigr).$$

Hence c is bounded by a function of d. □

COROLLARY 1. *Given any projective line* $\Lambda \subset \mathbb{P}^n$, *there exists a finite set* Φ *of order at most* $A(d) + d$ *in* Λ *such that, if* $P \in \Lambda - \Phi$, *the linear projection of center* P *maps* C *birationally onto its image.*

Proof. Since C is not equal to Λ, the cardinality of $C \cap \Lambda$ is at most d. So the corollary follows from Theorem 1.

REMARK. The proof of Theorem 1 provides an upper bound for $A(d)$. Indeed,

$$\deg(j) = \binom{3+N}{3} d'^3,$$

$d' \leq d/2$, and

$$N = \dim H^0\bigl(\mathbb{P}^3, \mathcal{O}(d')\bigr) = \frac{d'^3}{6} + d'^2 + \frac{11}{6}d' + 1.$$

In particular, when $d \geq 12$, we get

$$\log\bigl(A(d) + d\bigr) \leq 12 \log(d) - 12.$$

□

§2. Successive minima

2.1.

Let K be a number field, let $[K : \mathbb{Q}]$ be its degree over $\mathbb{Q}$, let $\mathcal{O}_K$ be its ring of integers, let $S = \mathrm{Spec}(\mathcal{O}_K)$ be the associated scheme, and let Σ be the set of complex embeddings of K. Consider a Hermitian vector bundle (E, h) over S; that is, E is a torsion-free $\mathcal{O}_K$-module of finite rank N, and for all $\sigma \in \Sigma$, the associated complex vector space $E_\sigma = E \otimes_{\mathcal{O}_K} \mathbb{C}$ is equipped with a Hermitian scalar product h_σ. If $\bar{\sigma}$ is the conjugate of σ, we assume that the complex conjugation $E_\sigma \simeq E_{\bar{\sigma}}$ is an isometry. If $v \in E$, we define

$$\|v\| = \max_{\sigma \in \Sigma} \sqrt{h_\sigma(v, v)}.$$

If i is a positive integer, $i \leq N$, we let μ_i be the infimum of the set of real numbers r such that there exist $v_1, \ldots, v_i \in E$, linearly independent over K, such that $\log \|v_\alpha\| \leq r$ for all $\alpha \leq i$. The number μ_i is thus the logarithm of the ith successive minimum of (E, h). Let

$$(1) \qquad \mu = \frac{\mu_1 + \cdots + \mu_N}{N}.$$

2.2.

If $E^\vee = \mathrm{Hom}(E, \mathcal{O}_K)$ is the dual of E, we let $\mathbb{P}(E^\vee)$ be the associated projective space, representing lines in $E^\vee$. Let $E_K^\vee = E^\vee \otimes_{\mathcal{O}_K} K$, and let $X_K \subset \mathbb{P}(E_K^\vee)$ be a smooth geometrically irreducible curve of genus g and degree d. We assume that the embedding of X_K into $\mathbb{P}(E_K^\vee)$ is defined by a complete linear series on X_K. We also assume that $d \geq 2g + 1$. The rank of E is thus $N = d + 1 - g$.

If X is the Zariski closure of X_K in $\mathbb{P}(E^\vee)$, and if $\overline{\mathcal{O}(1)}$ is the canonical Hermitian line bundle on $\mathbb{P}(E^\vee)$, the Faltings height of X_K is the real number

$$h(X_K) = \widehat{\deg}\big(\hat{c}_1(\overline{\mathcal{O}(1)})^2 \mid X\big)$$

(see [2, (3.1.1) and (3.1.5)]).

2.3.

For any positive integer $i \leq N$, we define the integer f_i by the formulas

$$f_i = i - 1 \qquad \text{if } i - 1 \leq d - 2g$$

and

$$f_i = i - 1 + \alpha \quad \text{if } i - 1 = d - 2g + \alpha, \qquad 0 \leq \alpha \leq g.$$

Assume that k and i are two positive integers, $k \leq N$, $i \leq N$. We let

$$h_{i,k} = \begin{cases} f_i & \text{if } i \leq k, i = N - 1, \text{ or } i = N, \\ f_k & \text{if } k \leq i \leq N - 2. \end{cases}$$

Finally, if $2 \leq k \leq N$, we let

$$B_k = \max_{i=2,\ldots,N} \frac{h_{i,k}^2}{(i-1)h_{i,k} - \sum_{j=1}^{i-1} h_{j,k}}.$$

PROPOSITION 1. *For every k such that $2 \leq k \leq N - 3$, the following inequality holds:*

$$B_k(\mu_{N+1-k} - \mu_1) + \frac{h(X_K)}{[K : \mathbb{Q}]} + 2d\mu \geq (2d - NB_k)(\mu - \mu_1) - 2d\log\big(A(d) + d\big).$$

2.4.

From Proposition 1, we shall deduce the following result.

THEOREM 2. *If $g > 0$, for every integer k such that $4 \le k \le d - 2g - 6$, the following inequality holds:*

$$\mu_{N+1-k} - \mu \ge -k\Big(\frac{h(X_K)}{2d[K:\mathbb{Q}]} + \mu\Big) - 12d\log(d) + 12d.$$

REMARKS. (i) It is proved in [13] that, when $d \ge 2g + 1$,

$$\frac{h(X_K)}{2d[K:\mathbb{Q}]} + \mu \ge \frac{2g(d-2g)}{d^2 + d - 2g^2}(\mu - \mu_1).$$

In particular,

$$\frac{h(X_K)}{2d[K:\mathbb{Q}]} + \mu \ge 0.$$

(ii) From Bombieri and Vaaler's version of Minkowski's theorem on successive minima (see [4]), we know that

$$-\mu \ge \frac{\widehat{\deg}(E,h)}{N[K:\mathbb{Q}]} - \frac{\log|D_K|}{2[K:\mathbb{Q}]} - K(N),$$

where D_K is the absolute discriminant of K and the constant $K(N)$ depends only on N. If h_{L^2} is the L^2-metric on $H^0(X, \mathcal{O}(1))$ (see [3, (1.2.3)]), the quantity

$$\frac{1}{[K:\mathbb{Q}]}\Big(\frac{h(X_K)}{2d} - \frac{\widehat{\deg}(H^0(X, \mathcal{O}(1)), h_{L^2})}{N}\Big)$$

is the normalized height of X_K introduced by Bost [3, (1.2.4)].

2.5.

To prove Proposition 1, fix a positive integer $k \le N - 3$ and choose elements $x_1, \ldots, x_N$ in E, linearly independent over K and such that

$$\log\|x_i\| = \mu_{N-i+1}, \quad 1 \le i \le N.$$

Fix integers n_α, $\alpha = k+1, \ldots, N-2$, to be specified later (in §2.7). If $1 \le i \le N$, we define

(2)
$$v_i = \begin{cases} x_i + n_i x_{i-1} & \text{if } k+1 \le i \le N-2, \\ x_i & \text{otherwise.} \end{cases}$$

We get a complete flag $E_K = V_1 \supset V_2 \supset \cdots \supset V_N$ by defining V_i to be the linear span of $v_i, v_{i+1}, \ldots, v_N$.

When m is large enough, the cup-product map

$$\varphi : E_K^{\otimes m} \to H^0\big(X_K, \mathcal{O}(m)\big)$$

is surjective; hence $H^0(X_K, \mathcal{O}(m))$ is generated by the monomials

$$v_1^{\alpha_1} \cdots v_N^{\alpha_N} = \varphi(v_1^{\otimes \alpha_1} \cdots v_N^{\otimes \alpha_N}),$$

$\alpha_1 + \cdots + \alpha_N = m$. A *special basis* of $H^0(X_K, \mathcal{O}(m))$ is a basis made of such monomials.

Let $r_1, r_2, \ldots, r_N$ be N real numbers, and let $\boldsymbol{r} = (r_1, \ldots, r_N)$. We define the weight of v_i to be r_i, we define the weight of a monomial in $E_K^{\otimes m}$ to be the sum of the weights of the v_i occurring in it, and we define the weight of a monomial $u \in H^0(X_K, \mathcal{O}(m))$ to be the minimum $wt_{\boldsymbol{r}}(u)$ of the weights of the monomials in the v_i mapping to u by φ. The weight $wt_{\boldsymbol{r}}(\mathcal{B})$ of a special basis $\mathcal{B}$ is the sum of the weights of its elements, and $w_{\boldsymbol{r}}(m)$ is the minimum of the weights of special bases of $H^0(X_K, \mathcal{O}(m))$.

When $r_1 \geq r_2 \geq \cdots \geq r_N = 0$ are natural integers, there exists $e_{\boldsymbol{r}} \in \mathbb{N}$ such that, as m goes to infinity,

$$w_{\boldsymbol{r}}(m) = e_{\boldsymbol{r}} \frac{m^2}{2} + O(m)$$

(see [12], [11, Corollary 3.3]).

Our next goal is to find an upper bound for $e_{\boldsymbol{r}}$.

2.6.

For every positive integer $i \leq N$, we let e_i be the drop in degree of X_K when projected from $\mathbb{P}(E_K^\vee)$ to $\mathbb{P}(V_i^\vee)$. A criterion of Gieseker (see [8], [11, Corollary 3.8]) tells us that $e_{\boldsymbol{r}} \leq S$ with

$$S = \min_{1 = i_0 < \cdots < i_\ell = N} \sum_{j=0}^{\ell-1} (r_{i_j} - r_{i_{j+1}})(e_{i_j} + e_{i_{j+1}}).$$

Note that S is an increasing function in each variable e_i. Furthermore, it follows from Clifford's theorem and Riemann-Roch that

$$(3) \qquad\qquad\qquad\qquad e_i \leq f_i$$

for every positive $i \leq N$ (see [11, proof of Theorem 4.4]). (Note that in [11, Theorem 4.4], the filtration of V_0 has length $n + 1$, while $n = \dim V_0$. In our case, we start the filtration with V_1, hence the discrepancy between our definition of f_i and that of [11].)

2.7.

We now specify our choice of the integers n_i in (2). This is where our argument will differ from that of [13, Theorem 4], which corresponds to the choice $n_i = 0$ for every i.

Let $w_1, \dots, w_N \in E_K^\vee$ be the dual basis of $v_1, \dots, v_N$. The linear projection from $\mathbb{P}(V_i^\vee)$ to $\mathbb{P}(V_{i+1}^\vee)$ has center the image $\dot{w}_i$ of w_i.

If $y_1, \dots, y_N \in E_K^\vee$ is the dual basis of $x_1, \dots, x_N$, then we get

$$w_i = \begin{cases} y_i + n_i z_i & \text{if } k \leq i \leq N - 3, \\ y_i & \text{otherwise,} \end{cases}$$

where $z_i + y_{i+1}$ is a linear combination of $y_{i+2}, y_{i+3}, \dots$ with coefficients depending only on $n_{i+1}, n_{i+2}, \dots$. When $n \neq m$ are two integers, the vectors $y_i + n z_i$ and $y_i + m z_i$ are linearly independent over K; therefore, their images in $\mathbb{P}(V_i^\vee)$ are distinct. Since $e_{N-3} \leq f_{N-3}$ and $g > 0$, we get $e_{N-3} \leq d - 3$; therefore, the image of X_K in $\mathbb{P}(V_i^\vee)$, $i \leq N - 3$, has degree at least 3. Furthermore, $\dim \mathbb{P}(V_i^\vee) \geq 3$. By Theorem 1 and Corollary 1, it follows that we can choose n_i such that $0 \leq n_i < A(d) + d$ and the projection from $\mathbb{P}(V_i^\vee)$ to $\mathbb{P}(V_{i+1}^\vee)$ does not change the degree of the image of X_K. We fix the integers n_i, $k \leq i \leq N - 3$, with this property. Hence we have

$$(4) \qquad\qquad e_i = e_k \quad \text{whenever } k \leq i \leq N - 2.$$

2.8.

From (3) and (4), we conclude that

$$e_i \leq h_{i,k} \quad \text{if } 1 \leq i \leq N$$

(see §2.3). Hence, by Morrison's main combinatorial theorem [11, Corollary 4.3], for any decreasing sequence of real numbers $r_1 \geq r_2 \geq \cdots \geq r_N$ we have, if $k \geq 2$,

$$S \leq \psi(\boldsymbol{r})$$

with

$$\psi(\boldsymbol{r}) = B_k \cdot \sum_{i=1}^{N} (r_i - r_N).$$

So, when $r_1 \geq r_2 \geq \cdots \geq r_N = 0$ is a decreasing sequence of integers,

$$(5) \qquad\qquad e_{\boldsymbol{r}} \leq \psi(\boldsymbol{r}).$$

Let

$$(6) \qquad s_i = \begin{cases} \log\|x_{i-1}\| + \log(A+d) & \text{if } k+1 \le i \le N-2, \\ \log\|x_i\| + \log(A+d) & \text{otherwise,} \end{cases}$$

and let $r_i = s_i - s_N$. The sequence $r_1, \ldots, r_N = 0$ is decreasing, and (2) implies that

$$\log\|v_i\| \le \log\|x_{i-1}\| + \log(1+n_i) \le s_i \quad \text{if } k+1 \le i \le N-2,$$

and

$$\log\|v_i\| = \log\|x_i\| \le s_i \quad \text{otherwise.}$$

We endow $\mathcal{O}(1)$ with the metric induced by the metric h on E. We choose a Hermitian metric, invariant by complex conjugation, on the complex points of X, and we endow $M = H^0(X, \mathcal{O}(m))$ with the associated L^2-metric. After multiplying the metric on X by a fixed constant, we know that, for every m, the morphism φ is norm decreasing. Therefore, if $u = \varphi(v_1^{\otimes \alpha_1} \cdots v_N^{\otimes \alpha_N})$ is a monomial in M, we have

$$\log\|u\| \le \sum_{i=1}^N \alpha_i \log\|v_i\| \le \sum_{i=1}^N \alpha_i s_i = \sum_{i=1}^N \alpha_i r_i + m s_N.$$

By definition of $wt_{\boldsymbol{r}}(u)$, we get

$$\log\|u\| \le wt_{\boldsymbol{r}}(u) + m s_N.$$

Let $\widehat{\deg}(\bar{M})$ be the arithmetic degree of the Hermitian vector bundle $\bar{M} = (M, h_{L^2})$ over S. From the inequality above and the Hadamard inequality, we deduce that, for any special basis $\mathcal{B}$ of M,

$$\widehat{\deg}(\bar{M}) \ge -[K:\mathbb{Q}] \sum_{u \in \mathcal{B}} \big(wt_{\boldsymbol{r}}(u) + m s_N\big).$$

This implies that

$$\widehat{\deg}(\bar{M}) \ge -[K:\mathbb{Q}]\big(w_{\boldsymbol{r}}(m) + m h^0(X_K, \mathcal{O}(m)) s_N\big).$$

By (5) and the definition of $e_{\boldsymbol{r}}$, since both $w_{\boldsymbol{r}}(m)$ and $\psi(\boldsymbol{r})$ are linear functions of $\boldsymbol{r}$, by approximating $\boldsymbol{r}$ by a collection of rational numbers we get that, for every positive real number η,

$$w_{\boldsymbol{r}}(m) \le \big(\psi(\boldsymbol{r}) + \eta\big)\frac{m^2}{2} + O(m)$$

(see [13, §2.2]), so we get

$$\widehat{\deg}(\bar{M}) \geq -[K:\mathbb{Q}]\big(\psi(\boldsymbol{r}) + 2ds_N + \eta\big)\frac{m^2}{2} + O(m).$$

On the other hand, by [6, Corollary to Theorem 3] and [9, Theorem 8], we have

$$\widehat{\deg}(\bar{M}) = h(X_K)\frac{m^2}{2} + O\big(m\log(m)\big).$$

Therefore, for every $\eta > 0$,

(7) $$h(X_K) \geq -[K:\mathbb{Q}]\big(\psi(\boldsymbol{r}) + 2ds_N + \eta\big).$$

By the definition of ψ, we deduce from (7) that

$$\frac{h(X_K)}{[K:\mathbb{Q}]} + 2ds_N \geq -B_k\Big(\sum_{i=1}^{N} r_i\Big),$$

and, using (6), that

(8) $$\frac{h(X_K)}{[K:\mathbb{Q}]} + 2d\mu_1 \geq -B_k\Big(\sum_{i=1}^{N}(\mu_i - \mu_1) + \mu_{N+1-k} - \mu_3\Big) - 2d\log\big(A(d)+d\big).$$

Since $\mu_3 \geq \mu_1$, (8) implies the inequality in Proposition 1.

2.9.

To make Proposition 1 more explicit, we need to evaluate B_k. For any $i = 2, \ldots, N$, we let

$$B_{i,k} = \frac{h_{i,k}^2}{(i-1)h_{i,k} - \sum_{j=1}^{i-1} h_{j,k}}$$

so that

$$B_k = \sup_i B_{i,k}.$$

LEMMA 1. *Assume that $k - 1 \leq d - 2g$. Then*

• *if $i \leq k$,*

$$B_{i,k} = 2 - \frac{2}{i};$$

• *if $k \leq i \leq N - 2$,*

$$B_{i,k} = 2 - \frac{2}{k}.$$

2.10.

To prove Lemma 1, we first assume that $i \leq k$. Then, if $j \leq i$, we have

$$h_{j,k} = f_j = j - 1.$$

Therefore,

$$\sum_{j=1}^{i-1} h_{j,k} = \sum_{j=1}^{i-1} (j-1) = \frac{(i-2)(i-1)}{2},$$

and

$$B_{i,k} = \frac{(i-1)^2}{(i-1)^2 - \frac{(i-2)(i-1)}{2}} = 2 - \frac{2}{i}.$$

Assume now that $k \leq i \leq N - 2$. Then, if $1 \leq j \leq k - 1$, we have

$$h_{j,k} = f_j = j - 1.$$

Furthermore, if $k \leq j \leq i - 1$, we get

$$h_{j,k} = f_k = k - 1.$$

Therefore,

$$B_{i,k} = \frac{(k-1)^2}{(i-1)(k-1) - \sum_{j=1}^{k-1}(j-1) - \sum_{j=k}^{i-1}(k-1)} = 2 - \frac{2}{k}.$$

$\square$

2.11.

LEMMA 2. *Assume that $g > 0$ and that $4 \leq k \leq d - 2g - 6$. Then $B_k < 2$ and*

$$2d - NB_k \geq \frac{2d}{k}.$$

To prove Lemma 2, we first compute $B_{N-1,k}$ and $B_{N,k}$ under the assumption $k - 1 \leq d - 2g$. When $d - 2g \leq i - 1 < N$, we have

$$f_i = 2i - 2 - d + 2g.$$

Therefore,

$$h_{N-1,k} = f_{N-1} = 2(N-2) - d + 2g = d - 2,$$

and

$$h_{N,k} = f_N = 2(N-1) - d + 2g = d.$$

On the other hand,

$$\sum_{j=1}^{N-2} h_{j,k} = \sum_{j=1}^{k} f_j + \sum_{j=k+1}^{N-2} f_k$$

$$= \sum_{j=1}^{k} (j-1) + \sum_{j=k+1}^{N-2} (k-1),$$

and

$$\sum_{j=1}^{N-1} h_{j,k} = \sum_{j=1}^{N-2} h_{j,k} + f_{N-1}.$$

Therefore,

(9)
$$B_{N-1,k} = \frac{(d-2)^2}{(N-2)(d-2) - \sum_{j=1}^{k}(j-1) - \sum_{j=k+1}^{N-2}(k-1)}$$

$$= \frac{(d-2)^2}{(N-2)(d-1-k) + \frac{k(k-1)}{2}},$$

and

(10)
$$B_{N,k} = \frac{d^2}{(N-2)d + 2 - \sum_{j=1}^{k}(j-1) - \sum_{j=k+1}^{N-2}(k-1)}$$

$$= \frac{d^2}{2 + (N-2)(d+1-k) + \frac{k(k-1)}{2}}.$$

When $i \leq N-2$, we know from Lemma 1 that

$$B_{i,k} \leq 2 - \frac{2}{k} < 2;$$

hence,

$$2d - NB_{i,k} \geq 2d - (d+1-g)\left(2 - \frac{2}{k}\right) \geq \frac{2d}{k}.$$

When $i = N-1$, we put $k = 4+p$, $p \geq 0$, and $d = 10 + 2g + p + t$, $t \geq 0$. Then (9) implies that $2 - B_{N-1,k}$ has the same sign as $2gt + 14g + p + t^2 + 12t + 38$, which is positive. On the other hand,

$$2d - NB_{N-1,k} - \frac{2d}{k}$$

has the sign of the numerator of $k - 1 - NkB_{N-1,k}/(2d)$, namely,

$$
\begin{aligned}
(2+p)t^3 &+ \big(p^2 + (5g+21)p + 10g + 36\big)t^2 \\
&+ \big((8+4g)p^2 + (8g^2 + 121 + 72g)p + 186 + 118g + 16g^2\big)t + (g-1)p^3 \\
&+ (5+26g+4g^2)p^2 + (172 + 4g^3 + 60g^2 + 240g)p \\
&+ 244 + 328g + 92g^2 + 8g^3 > 0.
\end{aligned}
$$

Similarly, (10) implies that $2 - B_{N,k}$ has the same sign as $t^2 + (2g+12)t + 42 + 10g + p$, so it is positive, and

$$
2d - NB_{N,k} - \frac{2d}{k}
$$

has the sign of

$$
\begin{aligned}
(2+p)t^2 &+ \big((9+3g)p + 12 + 6g\big)t + (g-1)p^2 + (11 + 18g + 2g^2)p \\
&- 14 + 22g + 4g^2 > 0.
\end{aligned}
$$

This ends the proof of Lemma 2.

2.12.

To prove Theorem 2, we first note that $B_k < 2$ and $2d - NB_k \geq 2d/k$ by Lemma 2. Since $\mu_1 \leq \mu_{N+1-k}$ and $\mu_1 \leq \mu$, Proposition 1 implies that

$$
2(\mu_{N+1-k} - \mu_1) + \frac{h(X_K)}{[K:\mathbb{Q}]} + 2d\mu \geq \frac{2d}{k}(\mu - \mu_1) - 2d\log\big(A(d) + d\big).
$$

By dividing this inequality by $2d$, we obtain

$$
(11) \qquad \frac{1}{d}(\mu_{N+1-k} - \mu_1) + \frac{h(X_K)}{2d[K:\mathbb{Q}]} + \mu \geq \frac{1}{k}(\mu - \mu_1) - \log\big(A(d) + d\big).
$$

Since $k < d$, we get

$$
\mu_{N+1-k} - \mu_1 + k\Big(\frac{h(X_K)}{2d[K:\mathbb{Q}]} + \mu\Big) \geq \mu - \mu_1 - d\log\big(A(d) + d\big),
$$

and, since $d \geq 12$, the remark in §1 implies Theorem 2.

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