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CERTAIN REAL HYPERSURFACES OF A COMPLEX SPACE FORM II

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0. Introduction

We denote by $M_n(c)$ a complete and simply connected complex n-dimensional Kählerian manifold of constant holomorphic sectional curvature $4c$, which is called a *complex space form.* Such an $M_n(c)$ is biholomorphically isometric to a complex projective space $P_n\mathbb{C}$, a complex Euclidean space $\mathbb{C}^n$ or a complex hyperbolic space $H_n\mathbb{C}$, according as $c > 0$, $c = 0$ or $c < 0$.

In this paper, we consider a real hypersurface M in $M_n(c)$. Typical examples of M in $P_n\mathbb{C}$ are the six model spaces of type A_1, A_2, B, C, D and E (cf. Theorem A in §1), and the ones of M in $H_n\mathbb{C}$ are the four model spaces of type A_0, A_1, A_2 and B (cf. Theorem B in §1), which are all given as orbits under certain Lie subgroups of the group consisting of all isometries of $P_n\mathbb{C}$ or $H_n\mathbb{C}$. Denote by (ϕ, ξ, η, g) the *almost contact metric structure* of M induced from the almost complex structure of $M_n(c)$, by A the shape operator and by S the Ricci tensor of M. Many differential geometers have studied M from various points of view. For example, Berndt [1] and Takagi [13] investigated the homogeneity of M. Kimura [6] proved that if all principal curvatures of M in $P_n\mathbb{C}$ are constant and ξ is principal vector of A, then M is congruent to one of model spaces. Moreover, Yano and Kon [15] studied M in $P_n\mathbb{C}$ satisfying the condition $A\phi + \phi A = k\phi$ for a constant $k \neq 0$ and Ki and Suh [3]

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investigated M in $P_n\mathbb{C}$ satisfying the condition $S\phi + \phi S = k\phi$ for a constant k. Recently, Takagi and the author of the present paper [5] studied M in $M_n(c)$, $c \neq 0$ satisfying the condition that $A^2\phi + \phi A^2$, $A\phi A$ or $A^2\phi + aA\phi A + \phi A^2$ is equal to $k\phi$ for constants a and k.

In the present paper, we shall classify a real hypersurface M in $M_n(c)$ satisfying the condition that $S\phi+\phi S$ or $S\phi S$ is equal to $k\phi$ for a constant k.

1. Preliminaries

We begin with recalling the basic properties of real hypersurfaces of a complex space form. Let N be a unit normal vector field on a neighborhood of a point p in M and J the almost complex structure of $M_n(c)$. For a local vector field X on a neighborhood of p, the images of X and N under the transformation J can be represented as

$$JX = \phi X + \eta(X)N \quad , \quad JN = -\xi,$$

where ϕ defines a skew-symmetric transformation on the tangent bundle TM of M, while η and ξ denote a 1-form and a vector field on the neighborhood of p , respectively. Moreover, it is seen that $g(\xi, X) = \eta(X)$, where g denotes the induced Riemannian metric on M. By the properties of the almost complex structure J, the set (ϕ, ξ, η, g) of tensors satisfies

$$\phi^2 = -I + \eta \otimes \xi \quad , \quad \phi\xi = 0 \quad , \quad \eta(\phi X) = 0 \quad \eta(\xi) = 1, \tag{1.1}$$

where I denotes the identity transformation. Accordingly, this set (ϕ, ξ, η, g) defines the ***almost contact metric structure*** on M. Furthermore, the covariant derivatives of the structure tensors are given by

$$(\nabla_X\phi)Y = \eta(Y)AX - g(AX, Y)\xi, \tag{1.2}$$

$$\nabla_X\xi = \phi AX, \tag{1.3}$$

where ∇ is the Riemannian connection of g. Since the ambient space is of constant holomorphic sectional curvature $4c$, the equations of Gauss and Codazzi are respectively given as follows :

$$\begin{aligned} R(X, Y)Z =& c\{g(Y, Z)X - g(X, Z)Y + g(\phi Y, Z)\phi X \\ & - g(\phi X, Z)\phi Y - 2g(\phi X, Y)\phi Z\} \\ & + g(AY, Z)AX - g(AX, Z)AY, \end{aligned} \tag{1.4}$$

$$(1.5) \quad (\nabla_X A)Y - (\nabla_Y A)X = c\{\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi\},$$

where R denotes the Riemannian curvature tensor of M. The Ricci tensor S' of M is the tensor of type $(0,2)$ given by $S'(X,Y) = tr\{Z \rightarrow R(Z,X)Y\}$. But it may be also regarded as a tensor of type $(1,1)$ and denoted by $S : TM \rightarrow TM$; it satisfies $S'(X,Y) = g(SX,Y)$. From the Gauss equation and (1.1), the Ricci tensor S is given by

$$(1.6) \qquad S = c\{(2n+1)I - 3\eta \otimes \xi\} + hA - A^2,$$

where h is the trace of A. A real hypersurface M of $M_n(c)$ is said to be *pseudo-Einstein* if the Ricci tensor S satisfies

$$SX = aX + b\eta(X)\xi$$

for some smooth functions a and b on M.

Now we quote the following in order to prove our results.

Theorem A ([13]). *Let M be a homogeneous real hypersurface of $P_n\mathbb{C}$. Then M is a tube of radius r over one of the following Kähler submanifolds:*

(A_1) *a hyperplane $P_{n-1}\mathbb{C}$, where $0 < r < \frac{\pi}{2}$,*

(A_2) *a totally geodesic $P_k\mathbb{C}$ $(1 \leq k \leq n-2)$, where $0 < r < \frac{\pi}{2}$,*

(B) *a complex quadratic Q_{n-1}, where $0 < r < \frac{\pi}{4}$,*

(C) *$P_1\mathbb{C} \times P_{(n-1)/2}\mathbb{C}$, where $0 < r < \frac{\pi}{4}$ and $n(\geq 5)$ is odd,*

(D) *a complex Grassmann $G_{2,5}\mathbb{C}$, where $0 < r < \frac{\pi}{4}$ and $n = 9$,*

(E) *a Hermitian symmetric space $SO(10)/U(5)$, where $0 < r < \frac{\pi}{4}$ and $n = 15$.*

Theorem B ([1]). *Let M be a real hypersurface of $H_n\mathbb{C}$. Then M has constant principal curvatures and ξ is principal if and only if M is locally congruent to one of the following:*

(A_0) *a horosphere in $H_n\mathbb{C}$,*

(A_1) *a geodesic hypersphere $H_0\mathbb{C}$ or a tube over a hyperplane $H_{n-1}\mathbb{C}$,*

(A_2) *a tube over a totally geodesic $H_k\mathbb{C}$ $(1 \leq k \leq n-2)$,*

(B) *a tube over a totally real hyperbolic space $H_n\mathbb{R}$.*

Theorem C ([10], [11]). *Let M be a real hypersurface of $M_n(c)$. Then M satisfies $A\phi = \phi A$ if and only if M is locally congruent to one of type A_1 and A_2 when $c > 0$, and of type A_0, A_1 and A_2 when $c < 0$.*

Theorem D ([2], [7], [10]). *Let M be a real hypersurface of $M_n(c)$ whose Ricci tensor is pseudo-Einstein. Then M is locally congruent to one of type A_1, A_2 and B when $c > 0$, and of type A_0 and A_1 when $c < 0$.*

Proposition A ([3], [9]). *Let M be a real hypersurface of $M_n(c)$, $c \neq 0$. If ξ is principal, then the corresponding principal curvature α is locally constant.*

Here we consider the case where the structure vector ξ is *principal*, namely, $A\xi = \alpha\xi$. It follows from (1.5) that

$$2A\phi A = 2c\phi + \alpha(A\phi + \phi A) \tag{1.7}$$

and hence, if $AX = \lambda X$ for any vector field X orthogonal to ξ, then we get

$$(2\lambda - \alpha)A\phi X = (\alpha\lambda + 2c)\phi X. \tag{1.8}$$

Accordingly, it turns out that in the case where $\alpha^2 + c \neq 0$, ϕX is also principal vector with principal curvature $\mu = (\alpha\lambda + 2c)/(2\lambda - \alpha)$, that is, we obtain

$$\begin{aligned} &A\phi X = \mu\phi X, \\ &2\lambda - \alpha \neq 0, \quad \mu = (\alpha\lambda + 2c)/(2\lambda - \alpha). \end{aligned} \tag{1.9}$$

2. Real hypersurfaces satisfying $S\phi + \phi S = k\phi$

We denote by $M_n(c)$ a complex space form with the metric of constant holomorphic sectional curvature $4c$ and M a real hypersurface in $M_n(c), c \neq 0$. In this section, we are concerned with M satisfying the following condition:

$$(2.1) \qquad S\phi + \phi S = k_1\phi \quad (k_1 = constant).$$

From (1.6) we obtain the condition (2.1) is equivalent to

$$(2.2) \qquad A^2\phi + \phi A^2 - h(A\phi + \phi A) = k\phi, \quad k = 2c(2n+1) - k_1.$$

Then we first prove the following.

Lemma 2.1. *Let M be a real hypersurface in $M_n(c)$, $c \neq 0$. If it satiesfies $S\phi + \phi S = k\phi$ for a function k and $A\xi$ is principal such that $\eta(A^3\xi) \neq trA$, then ξ is principal.*

Proof. The condition (2.2) yields $\phi A^2\xi - h\phi A\xi = 0$. From our assumption there is the function $\lambda = \eta(A^3\xi)$ on M such that $A^2\xi = \lambda A\xi$. Then we have $(\lambda - h)A\xi \in \ker\phi$, that is, $(\lambda - h)A\xi = \mu\xi$ for a function μ on M. Since $\lambda \neq h$, we see that ξ is principal. □

Remark 1. In general, "ξ is principal" implies "$A\xi$ is principal". But the converse is not true.

Remark 2. Let M be a real hypersurface in $M_n(c)$, $c \neq 0$. If M satisfies the condition $A^{2m-1}\phi + \phi A^{2m-1} = k\phi$ for $1 \leq m \leq n$, then we can easily verify the fact that ξ is principal. In fact, let $\lambda_1, \ldots, \lambda_d$ are the distinct principal curvatures. Then, since $\phi A^{2m-1}\xi = 0$, we get $\xi \in V_{\lambda_i}$ for some i $(1 \leq i \leq d)$ and hence we obtain ξ is principal.

However, if M satisfies the condition $A^{2m}\phi + \phi A^{2m} = k\phi$ for $1 \leq m \leq n$, then we have $\phi A^{2m}\xi = 0$, which means $\xi \in V_{\lambda_i} \oplus V_{-\lambda_i}$ for some i $(1 \leq i \leq d)$.

Remark 3. Yano and Kon [15] in $P_n\mathbb{C}$ and Suh [12] in $H_n\mathbb{C}$ showed that M satisfying the condition $A\phi + \phi A = k\phi$ for a constant $k \neq 0$ is locally congruent to one of type A_1 and B, and of type A_0, A_1 and B, respectively. Recently, Takagi and the author of the present paper [5] proved that M in $M_n(c)$, $c \neq 0$ satisfying the following two conditions: (i) $A\phi A$, $A^2\phi + \phi A^2$ or $A^2\phi + aA\phi A + \phi A^2$ is equal to $k\phi$ for constants a and k and (ii) ξ is principal is locally congruent to one of type A_1, A_2 with $r = \pi/4$ and B when $c > 0$, and of type A_0, A_1 and B when $c < 0$.

Now we need the following.

Lemma 2.2([3]). *Let M be a connected complete real hypersurface in $P_n\mathbb{C}$ and assume that ξ is principal. If it satisfies (2.1), then M is locally congruent to type A_1, type B or some hypersurface of type A_2.*

According to Lemmas 2.1 and 2.2 the following is immediate.

Theorem 2.3. *Let M be a real hypersurface in $P_n\mathbb{C}$. Assume that $A\xi$ is principal such that $\eta(A^3\xi) \neq trA$. Then it satisties $S\phi + \phi S = k\phi$ for a constant k if and only if M is locally congruent to type A_1, type B or some hypersurface of type A_2.*

For a real hypersurface of $H_n\mathbb{C}$ we have the following.

Theorem 2.4. *Let M be a real hypersurface in $H_n\mathbb{C}$. Assume that $A\xi$ is principal such that $\eta(A^3\xi) \neq trA$. Then it satisties $S\phi + \phi S = k\phi$ for a constant k if and only if M is locally congruent to one of the following:*

(A_0) *a horosphere in $H_n\mathbb{C}$,*

(A_1) *a geodesic hypersphere $H_0\mathbb{C}$ or a tube over a hyperplane $H_{n-1}\mathbb{C}$,*

(B) *a tube over a totally real hyperbolic space $H_n\mathbb{R}$.*

Proof. We may set $c = -1$. Owing to (1.9) and Lemma 2.1, our condition (2.2) reduces

$$(2.3) \qquad (\lambda^2 + \mu^2) - h(\lambda + \mu) = k, \quad k = -2(2n+1) - k_1,$$

where $AX = \lambda X$ and $A\phi X = \mu\phi X$ for any vector field X orthogonal to ξ. From Proposition A and Lemma 2.1 we can consider the following two cases: (I) $\alpha^2 - 4 \neq 0$ and (II) $\alpha^2 - 4 = 0$.

Case (I): Since $2\lambda-\alpha \neq 0$, we see from (1.9) that ϕX is also a principal (unit) vector orthogonal to ξ with the corresponding principal curvature $\mu = (\alpha\lambda - 2)/(2\lambda - \alpha)$. Then (2.3) gives us

$$
\begin{aligned}
(2.4) \qquad & 4\lambda^4 - 4(\alpha + h)\lambda^3 + 2(\alpha^2 + h\alpha - 2k)\lambda^2 \\
& + 4(h - \alpha + k\alpha)\lambda + 4 - 2h\alpha - k\alpha^2 = 0.
\end{aligned}
$$

This, together with our assumption and Proposition A, tells us that M has at most five distinct constant principal curvatures. Thus according to Theorem B, M is a homogeneous one. Then taking account of Berndt's classification theorem [1], we obtain that M is congruent to one of type A_0, A_1, A_2 and B. Thus we must check whether or not these four model spaces satisfy the condition (2.2) one by one. Since $\alpha^2 \neq 4$, it is enough to check (2.2) for the type A_1, A_2 and B.

First of all, let M be the type B. Then from the table in [1], we get $\alpha = 2\tanh(2r)$, $\lambda = \tanh(r)$ and $\mu = \coth(r)$, which implies

$$\lambda + \mu = \frac{4}{\alpha} \quad \text{and} \quad \lambda\mu = 1.$$

Combining this with (2.3), we find $k = (4/\alpha)^2 - h(4/\alpha) - 2$. If we substitute this into (2.4), then we have

$$
\begin{aligned}
& 4\alpha^2\lambda^4 - (4\alpha^3 + 4\alpha^2 h)\lambda^3 + 2(\alpha^4 + \alpha^3 h + 4\alpha^2 + 8\alpha h - 32)\lambda^2 \\
& - 4(3\alpha^3 + 3\alpha^2 h - 16\alpha)\lambda + 2\alpha^4 + 2\alpha^3 h - 12\alpha^2 = 0.
\end{aligned}
$$

Then this equation can be decomposed into

$$(\alpha\lambda^2 - 4\lambda + \alpha)(2\alpha\lambda^2 - 2(\alpha^2 + \alpha h - 4)\lambda + \alpha^3 + \alpha^2 h - 6\alpha) = 0.$$

Since the roots $\tanh(r)$ and $\coth(r)$ of the type B satisfy the quadratic equation $\alpha\lambda^2 - 4\lambda + \alpha = 0$, we see that the type B satisfies (2.2).

Next, let M be one of type A_1 and A_2. Then owing to Theorem C, our condition (2.2) is eqivalent to

$$(2.5) \qquad A^2\phi - hA\phi = \frac{k}{2}\phi, \quad k = -2(2n+1) - k_1.$$

If M is the type A_2, then M has three distinct constant principal curvatures $\alpha = 2\coth(2r)$, $\lambda = \tanh(r)$ and $\mu = \coth(r)$, where $0 < \lambda < 1$. Thus we have

$$\coth^2(r) - \tanh^2(r) - h(\coth(r) - \tanh(r)) = 0,$$

which implies $\tanh(r) + \coth(r) = h$ because of $\tanh(r) \neq \coth(r)$, that is, $\alpha = h$. Substituting this into (2.5) we get $k = -2$ and hence we have $k_1 = -4n$. Then (2.1) implies $S\phi + \phi S = -4n\phi$. Combining this with (1.6) and Theorem C, it follows $S\phi = \phi S = -2n\phi$. Then $S = -2nI + b\eta \otimes \xi$ for some function b on M. Thus we obtain the type A_2 satisfying (2.1) is pseudo-Einstein. But it is contrary to Theorem D. Therefore the type A_2 can not occur. If M is the type A_1, then M has two distinct constant principal curvatures $\alpha = 2\coth(2r)$ and $\lambda = \tanh(r)$ if $0 < \lambda < 1$ or $\lambda = \coth(r)$ if $\lambda > 1$. Thus (2.5) yields $k = -2(1+2(n-1)\tanh^2(r))$ or $k = -2(1+2(n-1)\coth^2(r))$. Therefore for such constant k the type A_1 satisfies (2.5).

Case (II): First, we consider the subcase where $\alpha = 2$. Then (1.8) gives forth to

$$(\lambda - 1)A\phi X = (\lambda - 1)\phi X.$$

Let us take an open set $M_0 = \{x \in M | \lambda(x) \neq 1\}$. Then $A\phi X = \phi X$ on M_0, which implies $\mu = 1$ on M_0. Combining this with (2.3), we get $\lambda^2 - h\lambda + (1 - h - k) = 0$ on M_0, which means λ is constant on M_0. On the other hand, we have $\lambda = 1$ on $M - M_0$. Thus, the continuity of principal curvatures yields the fact that if the set $M - M_0$ is not empty, then $\lambda = 1$ on M. Hence M is the type A_0. For the case where M_0 coincides with the whole M, we find $2\lambda - \alpha \neq 0$ and this case was discussed in the Case (I).

Conversely, let M be the type A_0. Then M has two distinct constant principal curvatures $\alpha = 2$ and $\lambda = 1$. Substituting these into (2.5), we get $k = 2(1 - h) = 2(1 - 2n)$. Thus for such constant k, the type A_0 satisfies (2.5), namely, (2.2).

Next, let $\alpha = -2$. Then, by the same method as the above we have M is the type A_0. □

According to lemma 2.1 and Theorem 2.4 the following is immediate.

Theorem 2.5. *Let M be a real hypersurface in $H_n\mathbb{C}$. Assume that ξ is principal. Then it satisties $S\phi + \phi S = k\phi$ for a constant k if and only if M is locally congruent to one of the following:*

(A_0) *a horosphere in $H_n\mathbb{C}$,*

(A_1) *a geodesic hypersphere $H_0\mathbb{C}$ or a tube over a hyperplane $H_{n-1}\mathbb{C}$,*

(B) *a tube over a totally real hyperbolic space $H_n\mathbb{R}$.*

3. Real hypersurfaces satisfying $S\phi S = k\phi$

Let M be a real hypersurface in a complex space form $M_n(c), c \neq 0$. In this section, we will consider M satisfying the following condition:

$$S\phi S = k_1\phi \quad (k_1 = constant). \tag{3.1}$$

From (1.6) it follows that the condition (3.1) is equivalent to

$$\begin{gathered} c(2n+1)h(A\phi + \phi A) - c(2n+1)(A^2\phi + \phi A^2) \\ +h^2 A\phi A - h(A^2\phi A + A\phi A^2) + A^2\phi A^2 = k\phi, \\ k = k_1 - c^2(2n+1)^2. \end{gathered} \tag{3.2}$$

Then we first have the following.

Theorem 3.1. *Let M be a real hypersurface in $P_n\mathbb{C}$, $n \geq 3$. Then it satisties $S\phi S = k\phi$ for a constant k and ξ is principal if and only if M is locally congruent to one of the following:*

(A_1) *a tube of radius r over a hyperplane $P_{n-1}\mathbb{C}$, where $0 < r < \frac{\pi}{2}$,*

(B) *a tube of radius r over a complex quadratic Q_{n-1}, where $0 < r < \frac{\pi}{4}$.*

Proof. Assume that ξ is principal. Let X be a principal (unit) vector orthogonal to ξ with the corresponding principal curvature λ. Then we see from (1.9) that ϕX is also a principal curvature (unit) vector orthogonal to ξ with the corresponding principal curvature $\mu = (\alpha\lambda + 2)/(2\lambda - \alpha)$, where we have set $c = 1$. Thus our condition (3.2) means

(3.3)
$$\lambda^2\mu^2 - (2n+1)(\lambda^2+\mu^2) - h\lambda\mu(\lambda+\mu) + h(2n+1)(\lambda+\mu) + h^2\lambda\mu = k, \quad k = k_1 - (2n+1)^2.$$

Then we get

(3.4)
$$\begin{aligned}&(\alpha^2 - 2\alpha h - 8n - 4)\lambda^4 + 2(4\alpha + \alpha h^2 + 4\alpha n + 4hn)\lambda^3\\ &- (\alpha^2(2 + h^2 + 4n) + 4\alpha h(1+n) + 4(k - h^2 - 1))\lambda^2\\ &+ 2(\alpha(2k - h^2 - 4n - 2) + 4hn)\lambda\\ &- \alpha^2 k - 2\alpha h(2n+1) - 8n - 4 = 0.\end{aligned}$$

Owing to Proposition A , (3.4) tells us that M has at most five distinct constant principal curvatures. Thus, accorrding to a theorem due to Kimura [6] M is homogeneous one. By virtue of the classification theorem in [13], M is one of type A_1, A_2, B, C, D and E. Hence, in order to prove our theorem we must check the condition (3.2) one by one for the above six model spaces.

First, let M be one of type C, D and E. Then from the table in [13], it follows that
$$\lambda + \mu = -\frac{4}{\alpha} \quad \text{and} \quad \lambda\mu = -1,$$
where $\lambda = \cot(r - \pi/4)$, $\mu = -\tan(r - \pi/4)$ (resp. $\lambda = \cot r$, $\mu = -\tan r$) and $\alpha = 2\cot 2r$. Thus taking account of this and (3.3) we find $k = -(2n+1)h(4/\alpha) - (2n+1)(2\alpha^2+16)/\alpha^2 - h^2 - h(4/\alpha) + 1$. The substitution of this into (3.4) gives rise to

(3.5)
$$\begin{aligned}&(\alpha^4 - 2\alpha^3 h - 8\alpha^2 n - 4\alpha^2)\lambda^4 + 2(\alpha^3(h^2 + 4n + 4)\lambda^3\\ &+ 4\alpha^2 hn)\lambda^3 - (\alpha^4(h^2 + 4n + 2) + 4\alpha^3 h(n+1)\\ &- 8\alpha^2(h^2 + 2n + 1) - 32\alpha h(n+1) - 128n - 64)\lambda^2\\ &- (2\alpha^3(3h^2 + 12n + 4) + 8\alpha^2 h(3n+4) + 64\alpha(2n+1))\lambda\\ &+ \alpha^4(h^2 + 4n + 1) + 2\alpha^3 h(2n+3) + 12\alpha^2(2n+1) = 0.\end{aligned}$$

Then (3.5) can be decomposed into

(3.6)
$$
\begin{aligned}
&(\alpha\lambda^2 + 4\lambda - \alpha)((\alpha^3 - 2\alpha^2 h - 4\alpha - 8\alpha n)\lambda^2 \\
&+ (2\alpha^2(h^2 + 4n + 2) + 8\alpha h(n+1) + 32n + 16)\lambda \\
&- \alpha^3(4n + h^2 + 1) - 2\alpha^2 h(2n+3) - 12\alpha(2n+1)) = 0.
\end{aligned}
$$

Since $\cot(r - \pi/4)$ and $-\tan(r - \pi/4)$ satisfy the quadratic equation $\alpha\lambda^2 + 4\lambda - \alpha = 0$, another roots $\cot r$ and $-\tan r$ of the types C, D and E must satisfy

$$
\begin{aligned}
&((\alpha^3 - 2\alpha^2 h - 4\alpha - 8\alpha n)\lambda^2 + (2\alpha^2(h^2 + 4n + 2) + 8\alpha h(n+1) + 32n \\
&+ 16)\lambda - \alpha^3(4n + h^2 + 1) - 2\alpha^2 h(2n+3) - 12\alpha(2n+1) = 0.
\end{aligned}
$$

However, since $\cot r$ and $-\tan r$ are the roots of the quadratic equation $\lambda^2 - \alpha\lambda - 1 = 0$, comparing these two quadratic equations, we have

$$
\begin{aligned}
&\alpha^3 - 2h\alpha^2 - 4(2n+1)\alpha - 1 = 0, \\
&2(h^2 + 4n + 2)\alpha^2 + (8h(n+1) + 1)\alpha + 16(2n+1) = 0, \\
&(4n + h^2 + 1)\alpha^3 + 2h(2n+3)\alpha^2 + 12(2n+1)\alpha - 1 = 0.
\end{aligned}
$$

Taking account of α and h of these types C, D and E, we have a contradiction. Hence the type C, D and E can not occur.

Next, let M be the type B. From the table in [13], we see that $\lambda + \mu = -4/\alpha$ and $\lambda\mu = -1$, where $\lambda = \cot(r - \pi/4)$, $\mu = -\tan(r - \pi/4)$ and $\alpha = 2\cot 2r$. Then taking account of (3.6) we see that the type B satisfies the condition (3.2).

Last, let M be one of type A_1 and A_2. Then owing to Theorem C , (3.2) equals to

(3.7)
$$
\begin{aligned}
&\lambda^4 - 2h\lambda^3 + (h^2 - 2(2n+1))\lambda^2 + 2(2n+1)h\lambda = k, \\
&k = k_1 - (2n+1)^2.
\end{aligned}
$$

If M is the type A_2, then M has three distinct principal curvatures $\alpha = 2\cot 2r$, $\lambda = -\tan r$ and $\mu = \cot r$. Thus we have

$$
\begin{aligned}
&2h(2n+1)(\cot r + \tan r) + (h^2 - 2(2n+1))(\cot^2 r \\
&- \tan^2 r) - 2h(\cot^3 r + \tan^3 r) + \cot^4 r - \tan^4 r = 0,
\end{aligned}
$$

which yields

$$(h - \cot r + \tan r)(\cot r + \tan r)(\cot^2 r + \tan^2 r - h(\cot r - \tan r) - 4n - 2) = 0.$$

Then we get $\alpha = h$ or $\alpha^2 - \alpha h - 4n = 0$ because of $\cot r + \tan r \neq 0$. First, let $\alpha = h$. Substituting this into (3.7) we get $k = -2(2n+1)$ and hence we have $k_1 = 4n^2 - 1$. Then (3.1) implies $S\phi S = (4n^2 - 1)\phi$. Combining this with (1.6) and Theorem C, it follows $S\phi = \phi S = \pm\sqrt{4n^2 - 1}\phi$. Then $S = \pm\sqrt{4n^2 - 1}I + b\eta \otimes \xi$ for some function b on M, that is, M is pseudo-Einstein. But, owing to well-known theorem (cf. [2], [7], [15]) of pseudo-Einstein real hypersurfaces in $P_n\mathbb{C}$, we see that this is not the case. Next, let $\alpha^2 - h\alpha - 4n = 0$. Then we get $\alpha(\alpha - h) = 4n$. Since M is type A_2, we have $h = \alpha + 2(p-1)\cot r - 2(q-1)\tan r$. Substituting this into the above equation, we obtain $(p-1)\cot^2 r + (q-1)\tan^2 r = -2(n+1) + p + q$. This implies $p + q \geq 2(n+1)$ and hence it is contrary to the fact that $4 \leq p + q \leq n + 1$. Therefore this is not the case, too. Therefore, the type A_2 does not occur.

If M is the type A_1, then M has two distinct principal curvatures $\alpha = 2\cot 2r$ and $\lambda = \cot r$. Thus from (3.7) it follows that for constant k such that $k = \cot^4 r - 2h\cot^3 r + (h^2 - 2(2n+1))\cot^2 r + 2(2n+1)h\cot r$, the type A_1 satisfies (3.2). □

For a real hypersurface of $H_n\mathbb{C}$ we have the following.

Theorem 3.2. *Let M be a real hypersurface in $H_n\mathbb{C}$, $n \geq 2$. Then it satisties $S\phi S = k\phi$ for a constant k and ξ is principal if and only if M is locally congruent to one of the following:*

(A_0) *a horosphere in $H_n\mathbb{C}$,*

(A_1) *a geodesic hypersphere $H_0\mathbb{C}$ or a tube over a hyperplane $H_{n-1}\mathbb{C}$,*

(B) *a tube over a totally real hyperbolic space $H_n\mathbb{R}$.*

Proof. Assume that ξ is principal. Let X be a principal (unit) vector orthogonal to ξ with the corresponding principal curvature λ. From Proposition A and (1.9) we can consider the following two cases: (I) $\alpha^2 - 4 \neq 0$ and (II) $\alpha^2 - 4 = 0$.

Case (I): Since $2\lambda-\alpha \neq 0$, we see from (1.9) that ϕX is also a principal (unit) vector orthogonal to ξ with the corresponding principal curvature $\lambda = (\alpha\lambda - 2)/(2\lambda - \alpha)$, where we have set $c = -1$. Thus our condition (3.2) means

(3.8)
$$\lambda^2\mu^2 + (2n+1)(\lambda^2+\mu^2) - h\lambda\mu(\lambda+\mu) - h(2n+1)(\lambda+\mu) \\ + h^2\lambda\mu = k, \quad k = k_1 - (2n+1)^2.$$

Then we get

(3.9)
$$\begin{aligned} &(\alpha^2 - 2\alpha h + 8n + 4)\lambda^4 + 2(\alpha h^2 - 4\alpha - 4\alpha n - 4hn)\lambda^3 \\ &+ (\alpha^2(2 - h^2 + 4n) + 4\alpha h(1+n) - 4(k + h^2 - 1))\lambda^2 \\ &+ 2(\alpha(2k + h^2 - 4n - 2) + 4hn)\lambda \\ &- \alpha^2 k - 2\alpha h(2n+1) + 8n + 4 = 0. \end{aligned}$$

Owing to Proposition A , (3.9) tells us that M has at most five distinct constant principal curvatures. Thus, accorrding to a theorem due to Berndt [1] M is homogeneous one, that is, M is congruent to one of type A_0, A_1, A_2 and B. Thus by the same argument as the above theorem we must check the condition (3.2) one by one for these four model spaces. Since $\alpha^2 \neq 4$, it is enough to check (3.2) for the type A_1, A_2 and B.

First of all, let M be the type B. Then from the table in [1], we get $\alpha = 2\tanh(2r)$, $\lambda = \tanh(r)$ and $\mu = \coth(r)$, which implies

$$\lambda + \mu = \frac{4}{\alpha} \quad \text{and} \quad \lambda\mu = 1.$$

Combining this with (3.8), we obtain $k = -(2n+1)h(4/\alpha) + (2n+1)(16 - 2\alpha^2)/\alpha^2 + h^2 - h(4/\alpha) + 1$. The substitution of this into (3.9) gives rise to

(3.10)
$$\begin{aligned} &(\alpha^4 - 2\alpha^3 h + 8\alpha^2 n + 4\alpha^2)\lambda^4 + 2(\alpha^3(h^2 - 4n - 4) \\ &- 4\alpha^2 hn)\lambda^3 + (\alpha^4(4n - h^2 + 2) + 4\alpha^3 h(n+1) \\ &+ 8\alpha^2(2n + 1 - h^2) + 32\alpha h(n+1) - 64(2n+1))\lambda^2 \\ &+ (2\alpha^3(3h^2 - 12n - 4) - 8\alpha^2 h(3n+4) + 64\alpha(2n+1))\lambda \\ &+ \alpha^4(4n + 1 - h^2) + 2\alpha^3 h(2n+3) - 12\alpha^2(2n+1) = 0. \end{aligned}$$

Then (3.10) can be decomposed into

$$(\alpha\lambda^2 - 4\lambda + \alpha)((\alpha^3 - 2\alpha^2 h + 4\alpha(2n+1))\lambda^2$$
$$+ (2\alpha^2(h^2 - 4n - 2) - 8\alpha h(n+1) + 32n + 16)\lambda$$
$$+ \alpha^3(n + 1 - h^2) + 2\alpha^2 h(2n+3) - 12\alpha(2n+1)) = 0.$$

Since the roots $\tanh(r)$ and $\coth(r)$ of the type B satisfy the quadratic equation $\alpha\lambda^2 - 4\lambda + \alpha = 0$, we see that the type B satisfies (3.2).

Next, let M be one of type A_1 and A_2. Then owing to Theorem C (3.8) is equivalent to

(3.11)
$$\lambda^4 - 2h\lambda^3 + (h^2 + 2(2n+1))\lambda^2 - 2(2n+1)h\lambda = k,$$
$$k = k_1 - (2n+1)^2.$$

If M is the type A_2, then M has three distinct constant principal curvatures $\alpha = 2\coth(2r)$, $\lambda = \tanh(r)$ and $\mu = \coth(r)$, where $0 < \lambda < 1$. Thus by means of (3.11) we have

$$\tanh^4(r) - \coth^4(r) - 2h(\tanh^3(r) - \coth^3(r)) + (h^2 + 2(2n+1))$$
$$(\tanh^2(r) - \coth^2(r)) - 2(2n+1)h(\tanh(r) - \coth(r)) = 0,$$

which yields

$$(h - \coth(r) - \tanh(r))(\tanh(r) - \coth(r))(\coth^2(r)$$
$$+ \tanh^2(r) - h(\coth(r) + \tanh(r)) + 4n + 2) = 0.$$

Then we get $\alpha = h$ or $\alpha^2 - h\alpha + 4n = 0$ because of $\coth(r) - \tanh(r) \neq 0$. First, let $\alpha = h$. Substituting this into (3.11) we get $k = 2(2n+1)$ and hence we have $k_1 = (2n+1)(2n+3)$. Then (3.1) implies $S\phi S = (2n+1)(2n+3)\phi$. Combining this with (1.6) and Theorem C, it follows $S\phi = \phi S = \pm\sqrt{(2n+1)(2n+3)}\phi$. Then $S = \pm\sqrt{(2n+1)(2n+3)}I + b\eta \otimes \xi$ for some function b on M, that is, M is pseudo-Einstein. However, owing to Theorem D, we see that this is not the case. Next, let $\alpha^2 - h\alpha + 4n = 0$. Then we get $\alpha(\alpha - h) = -4n$. Since we may say $\alpha \neq h$, we have $\alpha = 4n/(h - \alpha)$. On the other hand, type A_2 satisfies the quadratic equation $\alpha\lambda^2 - 4\lambda + \alpha = 0$. Combining these two equations we get $\tanh^2(r) = \{p-(n+1)\}/\{(n+1)-q\}$ or $\coth^2(r) = \{q-(n+1)\}/\{(n+1)-p\}$. This is contrary to the fact that $4 \leq p+q \leq n+1$. Therefore this is not the case, too. Consequently, the type A_2 can not occur. If M is the type A_1, then M has two distinct constant principal curvatures $\alpha = 2\coth(2r)$ and $\lambda = \tanh(r)$ if $0 < \lambda < 1$ or $\lambda = \coth(r)$ if $\lambda > 1$. Then from (3.11), it follows that for constant k such that $k = \tanh^4(r) - 2h\tanh^3(r) + (h^2 + 2(2n+1))\tanh^2(r) - 2(2n+1)h\tanh(r)$ or $k = \coth^4(r) - 2h\coth^3(r) + (h^2+2(2n+1))\coth^2(r) - 2(2n+1)h\coth(r)$, the type A_1 satisfies (3.8).

Case (II): First, we consider the subcase where $\alpha = 2$. Then (1.8) gives forth to

$$(\lambda - 1)A\phi X = (\lambda - 1)\phi X.$$

Let us take an open set $M_0 = \{x \in M | \lambda(x) \neq 1\}$. Then $A\phi X = \phi X$ on M_0, which implies $\mu = 1$. Combining this with (3.8), we get $(2(n+1)-h)\lambda^2 + (h^2 - 2h(n+1))\lambda + (2n+1)(1-h) - k = 0$ on M_0, which means λ is constant on M_0. On the other hand, we have $\lambda = 1$ on $M - M_0$. Thus, the continuity of principal curvatures yields the fact that if the set $M - M_0$ is not empty, then $\lambda = 1$ on M. Hence M is the type A_0. For the case where M_0 coincides with the whole M, we find $2\lambda - \alpha \neq 0$ and this case was discussed in the Case (I).

Conversely, let M be the type A_0. Then M has two distinct constant principal curvatures $\alpha = 2$ and $\lambda = 1$. Substituting these into (3.11), we obtain $k = h^2 - 4(n+1)h + 4n + 3 = 3 - 4n - 4n^2$. Thus for such constant k the type A_0 satisfies (3.11), namely, (3.2).

Next, let $\alpha = -2$. Then, by the same method as the above we have M is the type A_0. □

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