Nihonkai Math. J.
Vol.7 (1996), 155-176

Moreau-Rockafellar Type Theorems for Generalized Subgradient Operator*)

Jia-Wen Chen and Hang-Chin Lai

**Department of Applied Mathematics, Kaohsiung Polytechnic Institute
Kaohsiung, Taiwan 84008**

Abstract

In this paper, we shall consider the integral functional defined on a real Banach space X given by

$$F(x) = \int_T f_t(x)d\mu(t) \qquad \text{for all } x \in X,$$

where $f_t: X \to \Re \cup \{+\infty\}$ is finite at $z \in X$ and (T, Σ, μ) is a positive finite measure space. The purpose of this paper is to show the following formula:

$$(*) \qquad \partial^{\uparrow}\left(\int_T f_t(z)d\mu(t)\right) \subseteq \int_T \partial^{\uparrow} f_t(z)d\mu(t),$$

where $f_t: X \to \Re \cup \{+\infty\}$ is nonconvex which is not necessary locally Lipschitzian at z for all $t \in T$. This new result (*) is strongly enough to cover the known results in the convex case as well as the locally Lipschitzian case.

(Running head: Moreaus-Rockafellar type theorem)

1991 MSC: 49K27, 49K24, 90C26, 90C30

Key word: subdifferential, generalized gradients, directional Lipschitzian, regularity, generalized directional derivative.

*) This work was partially supported by the National Science Council, Taiwan, R.O.C.

1. Introduction

Let X be a real Banach space with dual space X^*. We shall consider the integral functional defined on X as the form:

(1.1) $$F(x) = \int_T f_t(x)d\mu(t) \qquad \text{for all } x \in X,$$

where $f_t: X \to \Re \cup \{+\infty\}$ is finite at $z \in X$ not necessary locally Lipschitz and (T, Σ, μ) is a position finite measure space such that for each $x \in X$, the mapping $t \to f_t(x) \equiv f(t,x)$ is measurable. In this paper, our goal is to characterize the following inclusion:

(1.2) $$\partial^{\uparrow}[\int_T f_t(z)d\mu(t)] \subseteq \int_T \partial^{\uparrow} f_t(z)d\mu(t),$$

where $\partial^{\uparrow} F(z)$ and $\partial^{\uparrow} f_t(z)$ denote the generalized gradients of F and f_t at z in Rockafellar sense (see the definition in next section), respectively.

It is known that the formula (1.2) is an important tool to study the theories of dynamic optimization in calculus of variations and to solve the solutions of nonsmooth optimal control problem. For example, as $f_t(\cdot)$ is locally Lipschitz for each t and $\mu(T) < \infty$, Aubin and Clarke use the inclusion (1.2) in [1] to obtain some necessary conditions of solutions in the generalized optimal control problem, and when T is finite, Rockafellar use (1.2) in [9] to get the Lagrange multiplier rulers in mathematical programming.

In 1972, Ioffe and Levin [5] characterized (1.2) in the case of convex $f_t(\cdot)$ whenever T is countable or X is separable (see [5, p.8]). Moreover if each $f_t(\cdot)$ is regular, then the equality holds in (1.2). In this case, $\partial^{\uparrow} f_t(\cdot) = \partial f_t(\cdot)$ and (1.2) becomes

(1.3) $$\partial F(z) = \partial[\int_T f_t(z)d\mu(t)] = \int_T \partial f_t(z)d\mu(t),$$

where $\partial F(z) = \{x \in X^* | F(y) - F(z) \geq \langle x, y-z \rangle \text{ for all } y \in X\}$ stands for the subdifferential of F at z, here $\langle x, y \rangle$ is the dual pair for X^* and X.

Rockafellar in [6] defined the generalized gradient for arbitrary functions, and proved that (1.2) holds for $T=\{1,2\}$ (cf. [6, Theorem 2]) & [8, Theorem 5G]). By induction, we can obtain that (1.2) holds when T is finite, that is,

(1.4) $$\partial^{\uparrow}(f_1+f_2+\cdots+f_n)(z)\subseteq\partial^{\uparrow}f_1(z)+\partial^{\uparrow}f_2(z)+\cdots+\partial^{\uparrow}f_n(z),$$

under the assumptions that each f_i is directionally Lipschitzian (see the next section) at $z\in X$ and satisfies the following condition:

(1.5) $$\{v\in X\,|\,f_1^{\uparrow}(z;v)<\infty\}\cap\bigcap_{i=2}^{n}Int\{v\in X\,|\,f_i^{\uparrow}(z;v)<\infty\}\neq\emptyset.$$

In 1981, Clarke in [2, Theorem 1] extends (1.3) to the case of locally Lipschitzian $f_t(\cdot)$. In this paper, we will establish (1.2) in a more general situation, where the integrand f_t is not necessarily convex as well as not locally Lipschitzian at z. If f_t is regular at z in Rockafellar sense, then the equality holds in (1.2) so is (1.3). In section 2, we describe the generalized directional derivatives in Clarke sense. In section 3, we establish the inclusion (1.2) without the assumptions of convexity and Lipschitzian. Finally, we will prove some results concerning generalized gradients of integral functional on the X-valued subspaces of $L^{\infty}(T;X)$ whenever $\mu(T)<\infty$.

2. Generalized Directional Derivatives and Generalized Gradients

Let $N(x)$ be the family of all neighborhoods at $x\in X$. Let $f\colon X\to\Re\cup\{+\infty\}$ be an extended real-valued function on X. Let $epi\,f=\{(x,\alpha)\in X\times\Re\,|\,f(x)\leq\alpha\}$ be the ***epigraph*** of f. In Hiriart-Urruty [4, Definition 6], the ***generalized directional derivative*** of f at x in the direction v, ***in Rockafellar sense***, is give by

(2.1) $$f^{\uparrow}(x;v)=\sup_{V\in N(v)}\ \inf_{\substack{N\in(x,f(x))\\ \bar{\lambda}>0}}\ \sup_{\substack{(y,\alpha)\in epi\,f\cap N\\ \lambda\in(0,\bar{\lambda}]}}\ \inf_{d\in V}\frac{f(y+\lambda d)-\alpha}{\lambda}.$$

For convenience, we use the expression

(2.2) $$(y,\alpha)\downarrow x_f$$

to denote $(y,\alpha)\in epi\, f$ with $y\to x$ and $\alpha\to f(x)$. Then (2.1) can be written as the following equivalent form: (cf. Rockafellar [7, §4] and [6, §2])

$$(2.3)\qquad f^{\uparrow}(x,v)=\lim_{\varepsilon\downarrow 0}\limsup_{\substack{(y,\alpha)\downarrow x_f\\ \lambda\downarrow 0}}\inf_{d\in v+\varepsilon B}\frac{f(y+\lambda d)-\alpha}{\lambda},$$

where B is the open unit ball centered at 0 in X.

If f is ***lower semicontinuous*** (*l.s.c.*) at x, then (2.3) is equivalent to

$$(2.4)\qquad f^{\uparrow}(x;v)=\lim_{\varepsilon\downarrow 0}\limsup_{\substack{y\downarrow x_f\\ \lambda\downarrow 0}}\inf_{d\in v+\varepsilon B}\frac{f(y+\lambda d)-f(y)}{\lambda},$$

where $y\downarrow x_f$ means that $y\to x$ as well as $f(y)\to f(x)$.

A function $f:X\to\overline{\Re}=(-\infty,\infty]$ is said to be ***locally Lipschitz*** at $x\in X$ if there exists a neighborhood V of x and a constant $K>0$ such that

$$(2.5)\qquad |f(x_1)-f(x_2)|\le K\|x_1-x_2\| \qquad \text{for all } x_1,x_2\in V.$$

For any $v\in X$, the ***generalized directional derivative*** of f at x in the direction v, ***in Clarke sense***, is defined by (cf. Clarke [2, Definition 1])

$$(2.6)\qquad \begin{aligned} f^{\circ}(x;v)&=\limsup_{\substack{y\to x\\ \lambda\downarrow 0}}\frac{f(y+\lambda v)-f(y)}{\lambda} \qquad \text{(admits } \pm\infty)\\ &=\inf_{\substack{V\in N(x)\\ \bar\lambda>0}}\ \sup_{\substack{y\in V\\ \lambda\in(0,\bar\lambda]}}\frac{f(y+\lambda v)-f(y)}{\lambda}. \end{aligned}$$

If a function f is locally Lipschitz at x, then

$$(2.7)\qquad f^{\uparrow}(x;v)=f^{\circ}(x;v) \qquad \text{for all } v\in X.$$

If $f:X\to\overline{\Re}$ is convex on X, then the ***directional derivative*** of f at x in the direction $v\in X$ is given by

$$(2.8)\qquad f'(x;v)=\lim_{\lambda\downarrow 0}\frac{f(x+\lambda v)-f(x)}{\lambda} \qquad \text{(admits } \pm\infty),$$

where f is finite at x.

Remark 2.1. If $f: X \to \overline{\Re}$ is both convex and locally Lipschitz at $x \in X$, then

(2.9) $$f^{\uparrow}(x;v) = f^{\circ}(x;v) = f'(x;v) \qquad \text{for all } v \in X.$$

According to the definition of $f^{\uparrow}$ and f°, we define the generalized gradient of f at x as follows.

Definition 2.1. (cf. Rockafellar [6, §2]) Let $f: X \to \overline{\Re}$ be finite at x. The ***generalized gradient*** of f at x, ***in Rockafellar sense***, denoted by $\partial^{\uparrow} f(x)$, namely Rockafellar generalized gradient, is defined by the set

(2.10) $$\partial^{\uparrow} f(x) = \{z \in X^{*} |\ f^{\uparrow}(x;v) \geq < z,v > \ \text{for all } v \in X\},$$

where $< z,v >$ is dual pair for X^{*} and X. The ***generalized gradient*** of f at x, ***in Clarke sense***, denoted by $\partial^{\circ} f(x)$ namely Clarke generalized gradient, is defined by the set

(2.11) $$\partial^{\circ} f(x) = \{z \in X^{*} |\ f^{\uparrow}(x;v) \geq < z,v > \ \text{for all } v \in X\}.$$

Evidently, the generalized gradients are weak*-closed subset of X^{*}.

If f is locally Lipschitz at x, then

(2.12) $$\partial^{\uparrow} f(x) = \partial^{\circ} f(x) \neq \varnothing.$$

The ***subdifferential*** of $f: X \to \overline{\Re}$ at $x \in X$ is defined by

$$\partial f(x) = \{z \in X^{*} |\ f(y) - f(x) \geq < z, y - x > \ \text{for all } y \in X\}.$$

If $\partial^{\uparrow} f(x) \neq \varnothing$ and f is convex, then the generalized gradient of f at x in Rockafellar sense agrees with the subdifferential of f at x (cf. [7, Theorem 5]), that is,

(2.13) $$\partial^{\uparrow} f(x) = \partial f(x).$$

Proposition 2.1. *Let $f: X \to \overline{\Re}$ be finite at the point $x \in X$ and let the generalized derivative $f^{\uparrow}(x;d)$ be finite in some direction $d \in X$. Then*

(a) $f^{\uparrow}(x;0) = 0$ *and* $\partial^{\uparrow} f^{\uparrow}(x;0) = \partial f^{\uparrow}(x;0) = \partial^{\uparrow} f(x)$.

(b) *if f has a continuous derivative at x, then* $\partial^{\uparrow} f(x) = \{Df(x)\}$,

where $Df(x)$ is the Gâteaux *derivative at f at x.*

Remark 2.2. Let $f: X \to \overline{\Re}$ be finite at $x \in X$. The generalized directional derivative $f^{\uparrow}(x;\cdot)$ is sublinear and *l.s.c.* (see Rockafellar [7, Theorem 2]). Thus in Proposition 2.1 (a), $\partial f^{\uparrow}(x;v)$ is the subdifferential of the convex function: $d \to f^{\uparrow}(x;d)$ at $v \in X$.

***Proof of Proposition* 2.1.** (a). In Remark 2.2, the function $d \to f^{\uparrow}(x;d)$ is *l.s.c.* and sublinear. Obviously, $f^{\uparrow}(x;0) = 0$. Since $f^{\uparrow}(x;\cdot)$ is convex, we have

$$\begin{aligned}\partial^{\uparrow} f(x) &= \{z \in X^* |\ f^{\uparrow}(x;v) \geq \langle z,v \rangle \ \text{ for all } v \in X\} \\ &= \{z \in X^* |\ f^{\uparrow}(x;v) - f^{\uparrow}(x;0) \geq \langle z,v \rangle \ \text{ for all } v \in X\} \\ &= \partial f^{\uparrow}(x;0) = \partial^{\uparrow} f^{\uparrow}(x;0).\end{aligned}$$

(b). If f has a continuous derivative at x, it is locally Lipschitz at x, then

$$f^{\uparrow}(x;v) = f^{\circ}(x;v) = \langle Df(x), v \rangle \qquad \text{for all } v \in X.$$

This implies that $\partial f^{\uparrow}(x) = \{Df(x)\}$. Q.E.D.

In this paper, we need $Int\{v \in X |\ f^{\uparrow}(x;v) < \infty\} \neq \varnothing$. Since this fact is not necessary true for arbitrary functions, thus we need to consider functions satisfying this property. For this reason, we give some definitions as follows.

A function $f: X \to \overline{\Re}$ is said to be ***directionally Lipschitzian*** at $x \in X$ in the direction $v \in X$ (cf. Rockafellar [6, §2]) if

$$f^{+}(x,v) = \limsup_{\substack{(y,\alpha)\downarrow x_f \\ w \to v \\ \lambda \downarrow 0}} \frac{f(y+\lambda d)-\alpha}{\lambda}. \tag{2.14}$$

We say that f is ***directionally Lipschitzian*** at x if f is directionally Lipschitzian at x for some direction $v \in X$.

If f is *l.s.c.* at x, then (2.14) is equivalent to

$$f^+(x,v) = \limsup_{\substack{y \downarrow x_f \\ w \to v \\ \lambda \downarrow 0}} \frac{f(y+\lambda w) - f(y)}{\lambda}. \tag{2.15}$$

Remark 2.3. Let $f: X \to \overline{\Re}$ be finite at $x \in X$. If f is directionally Lipschitzian at $x \in X$, Rockafellar proved in [7, Theorem 3] (cf. Clarke [3, Theorem 2.9.5]) that

$$\begin{aligned} &\{v \in X \mid f^+(x;v) < \infty\} = Int\{v \in X \mid f^{\uparrow}(x;v) < \infty\} \quad \text{and} \\ &f^+(x;v) = f^{\uparrow}(x;v) \quad \text{for all } v \in Int\{v \in X \mid f^{\uparrow}(x;v) < \infty\}. \end{aligned} \tag{2.16}$$

A function f is ***regular*** at x ***in Rockafellar sense***, namely Rockafellar regular, if f is finite at x and

$$f^{\uparrow}(x;v) = \liminf_{\substack{w \to v \\ \lambda \downarrow 0}} \frac{f(x+\lambda w) - f(x)}{\lambda} \qquad \text{for all } v \in X. \tag{2.17}$$

In Clarke [3, Definition 2.3.4], a function f is said to be ***regular*** at x if the one-sided directional derivative

$$\begin{aligned} &f'(x;v) = \lim_{\lambda \downarrow 0} \frac{f(x+\lambda v) - f(x)}{\lambda} \text{ exists} \quad \text{and} \\ &f'(x;v) = f^{\circ}(x;v) \quad \text{for all } v \in X. \end{aligned} \tag{2.18}$$

Actually, if f is locally Lipschitz at x, then (2.17) coincides with (2.18). We state this fact as the following proposition.

Proposition 2.2. *Let $f: X \to \overline{\Re}$ be locally Lipschitz at a point $x \in X$. Then f is Rockafellar regular at x if and only if f is regular at x.*

Proof: (Necessity) Let f be Rockafellar regular at x. Then

$$f^{\uparrow}(x;v) = \liminf_{\substack{w \to v \\ \lambda \downarrow 0}} \frac{f(x+\lambda w) - f(x)}{\lambda} \qquad \text{for all } v \in X.$$

Since f is locally Lipschitz at x, $f^{\uparrow}(x;v) = f^{\circ}(x;v)$ exists for all $v \in X$. This shows that

$$f^{\circ}(x;v) = f^{\uparrow}(x;v) = \liminf_{\substack{w \to v \\ \lambda \downarrow 0}} \frac{f(x+\lambda w) - f(x)}{\lambda}.$$

The right hand side of the above equality becomes

$$f^{\circ}(x;v) \leq \liminf_{\lambda \downarrow 0} \frac{f(x+\lambda v) - f(x)}{\lambda}$$

$$\leq \limsup_{\lambda \downarrow 0} \frac{f(x+\lambda v) - f(x)}{\lambda}$$

$$\leq \liminf_{\substack{w \to v \\ \lambda \downarrow 0}} \frac{f(x+\lambda v) - f(x)}{\lambda} = f^{\circ}(x;v).$$

It follows that

$$f'(x;v) = \lim_{\lambda \downarrow 0} \frac{f(x+\lambda v) - f(x)}{\lambda} \quad \text{exists and } f'(x;v) = f^{\circ}(x;v) \quad \text{for all } v \in X.$$

Hence f is regular at x.

(Sufficiency) Let f be regular at x. Then $f'(x;v) = f^{\circ}(x;v)$ exists for all $v \in X$.

We will show that

$$f^{\uparrow}(x;v) = \liminf_{\substack{w \to v \\ \lambda \downarrow 0}} \frac{f(x+\lambda w) - f(x)}{\lambda}.$$

Since f is locally Lipschitz at x, $f'(x;v) = f^{\circ}(x;v) = f^{\uparrow}(x;v)$ for all $v \in X$ and

$$\left| \lim_{\substack{w \to v \\ \lambda \downarrow 0}} \frac{f(x+\lambda w) - f(x)}{\lambda} \right| \leq \lim_{w \to v} K\|w - v\| = 0$$

where K is a Lipschitzian constant. That is,

$$\lim_{\substack{w \to v \\ \lambda \downarrow 0}} \left(\frac{f(x+\lambda w) - f(x)}{\lambda} - \frac{f(x+\lambda v) - f(x)}{\lambda} \right) = 0;$$

$$\lim_{\substack{w \to v \\ \lambda \downarrow 0}} \frac{f(x+\lambda w) - f(x)}{\lambda} = \lim_{\lambda \downarrow 0} \frac{f(x+\lambda v) - f(x)}{\lambda};$$

$$f^{\uparrow}(x;v) = f'(x;v).$$

Hence f is regular at x in Rockafellar sense. Q.E.D.

3. Integral Functionals on a Banach Space

In order to get a rigorous formulation in the differential inclusion (1.2), we need the following assumptions and definitions.

Assumption 3.1. Let $z \in X$ and let the generalized directional derivative $f_t^{\uparrow}(z;\cdot)$ of $f_t(\cdot)$ at z satisfy the following conditions:

(i). For any $v \in X$, $f_t^{\uparrow}(z;v)$ is measurable in t.

(ii). $f_t(\cdot)$ is continuous at z for all $t \in T$.

(iii). $f_t^{\uparrow}(z;0) = 0$ for all $t \in T$ and $\bigcap_{t \in T} Int(Dom\, f_t^{\uparrow}(z;\cdot)) \neq \varnothing$,

where $Dom\, f_t^{\uparrow}(z;\cdot) = \{v \in X |\ f_t^{\uparrow}(z;v) < \infty\}$.

Definition 3.1. An integrand $f_t : X \to \Re \cup \{+\infty\}$ is said to be ***locally pseudo Lipschitzian*** at $z \in X$ in the direction $v \in X$ if there exist $W \in N(v)$, $K_1 \in L^1(T;\Re^+)$, $K_2 \in L^1(T;\Re^+)$ and a real number $\bar{\lambda} > 0$ such that

(3.1) $$\frac{f_t(x+\lambda w) - f_t(x)}{\lambda} \leq K_1(t)\|w\| + K_2(t),$$

for all $t \in T, w \in W, \lambda \in (0,\bar{\lambda}]$ and $x \in B_\varepsilon(z)$, where $B_\varepsilon(z) = \{x \in X |\ \|x - z\| < \varepsilon\}$ for some $\varepsilon > 0$.

Remark 3.1. If $f_t : X \to \overline{\Re}$ is *l.s.c.* as well as locally pseudo Lipschitzian at $z \in X$ in the direction $v \in X$, then by the definition 3.1, the inequality (3.1) implies that

$$f_t^{+}(z,v)=\limsup_{\substack{x\downarrow z_f\\ w\to v\\ \lambda\downarrow 0}}\frac{f_t(x+\lambda w)-f_t(x)}{\lambda}\le K_1(t)\|v\|+K_2(t),$$

so that f_t is directionally Lipschitzian at $z\in X$ in the direction $v\in X$ (see (2.15)). The converse of this result may not be true. Indeed the locally pseudo Lipschitzian implies that $f_t^{\uparrow}(z;v)$ is bounded above by a L^1-function $K_1(t)\|v\|+K_2(t)$, but the directionally Lipschitzian does not satisfy this condition.

If f_t is ***locally Lipschitz*** at z in the sense that for a function $K\in L^1(T;\Re^{+})$, then

(3.2) $$|f_t(s_1)-f_t(s_2)|\le K(t)\|s_1-s_2\| \qquad \text{for all } s_1,s_2\in B_\eta(z), \text{ and } t\in T,$$

where $\eta>0$. It follows that f_t is also locally pseudo Lipschitzian at z in any direction $v\in X$ if we take $W=v+\frac{\eta}{2}B$, $K_1(t)=K(t)$ and $K_2(t)=0$ for all $t\in T$ and choose $\lambda=\frac{\eta}{2(\|v\|+\eta)}$, $\varepsilon=\frac{\eta}{2}$ in definition 3.1. That is (3.1) holds.

The converse of this result is not true, for example, if we define $f_t:\Re\to\Re$ by

$$f_t=\begin{cases} t\sqrt{x} & \text{if } x\ge 0\\ 0 & \text{if } x<0\end{cases}, \quad \text{for all } t\in[1,2].$$

Then by the definition of the generalized directional derivative, it is easily to get

$$f_t^{\uparrow}(0;v)=\begin{cases}\infty & \text{if } v>0\\ 0 & \text{if } v\le 0\end{cases}.$$

This shows that $Dom f_t^{\uparrow}(0;\cdot)=\{v\in\Re|\ f_t^{\uparrow}(0;v)<\infty\}=(-\infty,0)$ and f_t is locally pseudo Lipschitzian at $z=0$ in any direction $v\in(-\infty,0)$ for all $t\in[1,2]$. Indeed, taking $K_1(t)=K_2(t)=0$ for all $t\in[1,2]$ in definition 3.1, we have

$$\frac{f_t(x+\lambda w)-f_t(x)}{\lambda}\le 0 \qquad \text{for all } w\in Int(Dom f_t^{\uparrow}(0;\cdot))=(-\infty,0).$$

This shows that f_t is locally pseudo Lipschitzian at 0 in any direction $v\in(-\infty,0)$ for all $t\in[1,2]$. But f_t is not locally Lipschitz at 0.

Remark 3.2. A function $f: X \to \overline{\Re}$ is locally Lipschitz at x_0 if and only if f is directional Lipschitzian at x_0 with $v = 0$ (see (2.15)). That is $0 \in Int(Dom(f^{\uparrow}(x_0;\cdot)))$ (see Remark 2.3).

In this section, we will extend Moreau-Rockafellar theorem from finite sums of functions like (1.4) to the integral functional form like (1.2) over a positive finite measure space (T, Σ, μ). To this end, for each $x \in X$ we let the mapping $t \in T \to f_t(x) = f(t,x)$ be measurable for each $t \in T$.

Now we let $f_t(\cdot)$ be finite at $z \in X$ and define

$$g(v) = \int_T f_t^{\uparrow}(z;v)d\mu(t) \qquad \text{for all } v \in X.$$

By Assumption 3.1 (i), we obtain that for any $v \in X$, $f_t^{\uparrow}(z;v)$ is measurable in *t*. This implies that *g* is well defined. Moreover, we need the following assumptions.

Assumption 3.2. Either the normal cone $N_{Dom\,g}(0)$ contains only one element $\{0\}$ or $Dom\,g = Dom\,f_t^{\uparrow}(z;v)$ on a set $S \subseteq T$ of positive measure, where $Dom\,g = \{v \in | g(v) < \infty\}$ and $N_{Dom\,g}(0) = \{\zeta \in X^* | < \zeta, v > \leq 0 \text{ for all } v \in Dom\,g\}$.

Now we come to our main theorem which we state as follows:

Theorem 3.1. *Let $f_t(\cdot)$ be locally pseudo Lipschitzian at $z \in X$ in any direction $v \in Int(Dom(f_t^{\uparrow}(z;\cdot)))$ for all $t \in T$. Suppose that $f_t^{\uparrow}(z;\cdot)$ satisfies Assumptions* 3.1 *and* 3.2. *If* (a) *T is countable or* (b) *X is separable, then we have*

$$\partial^{\uparrow}\left(\int_T f_t(z)d\mu(t)\right) \subseteq \int_T \partial^{\uparrow} f_t(z)d\mu(t). \tag{3.3}$$

If each $f_t(\cdot)$ is Rockafellar regular at z and there exist $\overline{\lambda} > 0$ and functions $K_3, K_4 \in L^1(T;\Re)$ satisfying the inequality

$$\frac{f_t(z+\lambda v) - f_t(z)}{\lambda} \geq K_3(t)\|v\| + K_4(t) \text{ for all } \lambda \in (0,\overline{\lambda}], t \in T, v \in X, \tag{3.4}$$

then F is Rockafellar regular at z and the equality in (3.3) *holds.*

Remark 3.3. The inclusion of (3.3) is explained as follows: To every ζ in $\partial^{\uparrow}F(z)\subseteq X^*$ there corresponds a mapping $t\to\zeta_t$ from T into X^* with $\zeta_t\in\partial^{\uparrow}f_t(z)$ for a.a. $t\in T$ such that the mapping $t\to<\zeta_t,v>$ is summable and

$$<\zeta,v>=\int_T<\zeta_t,v>d\mu(t) \qquad \text{for all } v\in X.$$

***Proof of Theorem* 3.1.** Define $F:X\to\Re\cup\{\pm\infty\}$ by

$$F(x)=\int_T f_t(x)d\mu(t) \qquad \text{for all } x\in X.$$

First, we will show that

(3.5) $$F^{\uparrow}(z;v)\le\int_T f_t^{\uparrow}(z;v)d\mu(t) \qquad \text{for all } v\in X.$$

Without loss of generality, we can assume that $v\in Dom\, f_t^{\uparrow}(z;\cdot)$ for almost all $t\in T$, otherwise, $\int_T f_t^{\uparrow}(z;v)d\mu(t)=\infty$ and (3.5) holds. Since integrals over sets of measure zero are zero, we can assume that $v\in Dom\, f_t^{\uparrow}(z;\cdot)$ for all $t\in T$.

Let $v\in Dom\, f_t^{\uparrow}(z;\cdot)$. We prove (3.5) holds by the following two cases.

Case 1: Let $v\in Int(Dom\, f_t^{\uparrow}(z;\cdot))$ for all $t\in T$. Since $f_t(\cdot)$ is continuous on X and $\mu(T)<\infty$, $F(\cdot)$ is also continuous on X. Like in (2.4), we recall the notations:

$x\downarrow z_F$ means that $x\to z$ and $F(x)\to F(z)$;

$x\downarrow z_f$ means that $x\to z$ and $f_t(x)\to f_t(z)$;

and $B_\varepsilon(v)$ denotes the neighborhood $\{w\in X\,\|\,\|w-v\|<\varepsilon\}=v+\varepsilon B$,

where B is the unit open ball of X centered at origin 0.

Note that by assumption 3.1(i) $f_t^{\uparrow}(z;\cdot)$ is measurable in t. If T is countable, then $f_t^{\uparrow}(z;\cdot)$ is clearly measurable in t.

By the hypotheses that $f_t(\cdot)$ is locally pseudo Lipschitzian at $z\in X$ in any direction $v\in Int(Dom(f_t^{\uparrow}(z;\cdot)))$ for all $t\in T$, it is directional Lipschitzian at $z\in X$ in any direction $v\in Int(Dom(f_t^{\uparrow}(z;\cdot)))$ for all $t\in T$ (see Remark 3.1). Then from the assumption 3.1(ii) and the expresion (2.16), we would have that

$$\int_T f_t^{\uparrow}(z;v)d\mu(t)=\int_T f_t^{+}(z;v)d\mu(t)$$

$$= \int_T \limsup_{\substack{x \downarrow z_f \\ w \to v \\ \lambda \downarrow 0}} \frac{f_t(x+\lambda w) - f(x)}{\lambda} d\mu(t)$$

(3.6) $$\geq \limsup_{\substack{x \downarrow z_f \\ w \to v \\ \lambda \downarrow 0}} \int_T \frac{f_t(x+\lambda w) - f(x)}{\lambda} d\mu(t).$$

The last inequality follows from Fatou's lemma. Indeed, since f_t is locally pseudo Lipschitzian at $z \in X$ in the direction $v \in X$ for all $t \in T$, there exist $B_\eta(v) \in N(v)$, $K_1 \in L^1(T;\Re^+)$, $K_2 \in L^1(T;\Re^+)$ and a real number $\overline{\lambda} > 0$ such that

(3.7) $$\frac{f_t(x+\lambda w) - f_t(x)}{\lambda} \leq K_1(t)\cdot\|w\| + K_2(t) \leq K_1(t)\cdot(\|v\|+\eta) + K_2(t),$$

for all $w \in B_\eta(v), \lambda \in (0,\overline{\lambda}], x \in B_\varepsilon(z)$ with some $\varepsilon > 0$.

This shows that the right-hand side of (3.7) is bounded above by an integrable function which are independent of w, x and λ. So Fatou's lemma is applicable and so (3.6) holds.

From (3.6), we obtain

$$\int_T f_t^\uparrow(z;v)d\mu(t) \geq \limsup_{\substack{x \downarrow z_f \\ w \to v \\ \lambda \downarrow 0}} \int_T \frac{f_t(x+\lambda w) - f(x)}{\lambda} d\mu(t)$$

$$= \limsup_{\substack{x \downarrow z_F \\ w \to v \\ \lambda \downarrow 0}} \frac{F(x+\lambda w) - F(x)}{\lambda}$$

$$\geq \lim_{\varepsilon \downarrow 0} \limsup_{\substack{x \downarrow z_F \\ \lambda \downarrow 0}} \inf_{w \in v + \varepsilon B} \frac{F(x+\lambda w) - F(x)}{\lambda} = F^\uparrow(z;v).$$

Hence (3.5) holds for all $v \in Int(Dom\, f_t^\uparrow(z;\cdot))$ for all $t \in T$.

Case 2: Let $v \in Dom\, f_t^\uparrow(z;\cdot)$ for all $t \in T$. By the assumption 3.1 (iii) there exists a vector $v_0 \in Int(Dom\, f_t^\uparrow(z;\cdot))$ for all $t \in T$. Then by the convexity of $f_t^\uparrow(z;\cdot)$, we have that for all $\lambda \in [0,1)$, $v_\lambda = (1-\lambda)v_0 + \lambda v_0 \in Int(Dom f_t^{\uparrow}(z;\cdot))$ for all $t \in T$. From the Case 1, we obtain,

$$(3.8)\qquad \begin{aligned} F^{\uparrow}(z;v_\lambda) &\le \int_T f_t^{\uparrow}(z;v_\lambda)d\mu(t) \\ &= \int_T f_t^{\uparrow}(z;(1-\lambda)v_0 + \lambda v)d\mu(t) \\ &\le (1-\lambda)\int_T f_t^{\uparrow}(z;v_0)d\mu(t) + \lambda \int_T f_t^{\uparrow}(z;v)d\mu(t). \end{aligned}$$

Since $F^{\uparrow}(z;\cdot)$ is *l.s.c.*, letting $\liminf\limits_{\lambda\to 1^-}$ in the two sides of (3.8), we obtain

$$\int_T f_t^{\uparrow}(z;v)d\mu(t) \ge \liminf_{\lambda\to 1^-} F^{\uparrow}(z;v_\lambda) \ge F^{\uparrow}(z;v).$$

Hence (3.5) holds for $v \in Dom f_t^{\uparrow}(z;\cdot)$ for all $t \in T$. This shows that (3.5) holds for all $v \in X$.

Define function $g: X \to \Re \cup \{+\infty\}$ by

$$(3.9)\qquad g(v) = \int_T f_t^{\uparrow}(z;v)d\mu(t) \quad \text{for all } v \in X.$$

Then g is a convex function on X that is not identically equal $+\infty$ since $g(0) = 0$. By (3.5), we have

$$(3.10)\qquad F^{\uparrow}(z;v) \le g(v) \quad \text{for all } v \in X.$$

Since $F^{\uparrow}(z;\cdot)$ is *l.s.c.* and sublinear, thus if $F^{\uparrow}(z;0) \ne 0$, then $F^{\uparrow}(z;0) = -\infty$. This implies that $\partial^{\uparrow}F(z) = \varnothing$, and so Theorem 3.1 holds in this case.

So we can set $F^{\uparrow}(z;0) = 0$, and by Proposition 2.1, we have

$$\partial^{\uparrow}F(z) = \partial^{\uparrow}F^{\uparrow}(z;0) = \partial F^{\uparrow}(z;0).$$

It follows from (3.10) that

$$(3.11)\qquad \partial F^{\uparrow}(z;0) \subseteq \partial g(0).$$

From (3.7), it is immediately that

$$\sup_{w\in B_\eta(v)} f_t^{\uparrow}(z;w) \le K_1(t)\cdot(\|v\| + \eta) + K_2(t).$$

Since $f_t^{\uparrow}(z;\cdot)$ is convex and *l.s.c.* on *X*, it follows from Ioffe-Levin theorem (cf. [5, Theorem 3, p.38]) that

$$\partial g(0) = \int_T \partial f_t^{\uparrow}(z;0)d\mu(t) + \Lambda(0),$$

where $\Lambda(0) = N_{Dom g}(0) = \{\zeta \in X^* | <\zeta, v> \leq 0 \text{ for all } v \in Dom g\}$.

By Assumption 3.2 and Ioffe-Levin theorem (cf. [5, Theorem 3 & Remark, p.28]), we obtain

$$\partial g(0) = \int_T \partial f_t^\uparrow(z;0) d\mu(t).$$

From (3.11) and Proposition 2.1, it yields

$$\partial^\uparrow F(z) \subseteq \partial g(0) \subseteq \int_T \partial f_t^\uparrow(z;0) d\mu(t) = \int_T \partial^\uparrow f_t(z) d\mu(t).$$

Therefore (3.3) is proved.

Finally, we show that the equality in (3.3) holds if f_t is Rockafellar regular at z. Indeed, from (3.5) and the regularity of f_t at z in Rockafellar sense, we obtain

$$\begin{aligned} F^\uparrow(z;v) &\leq \int_T f_t^\uparrow(z;v) d\mu(t) \\ &= \int_T \liminf_{\substack{w \to v \\ \lambda \downarrow 0}} \frac{f_t(z+\lambda w) - f_t(z)}{\lambda} d\mu(t). \end{aligned}$$

By (3.4), the difference quotient function is bounded below by an integrable function, thus Fatou's lemma is applicable and the last expression reduces to

$$\leq \liminf_{\substack{w \to v \\ \lambda \downarrow 0}} \int_T \frac{f_t(z+\lambda w) - f_t(z)}{\lambda} d\mu(t)$$

$$= \liminf_{\substack{w \to v \\ \lambda \downarrow 0}} \frac{F(z+\lambda w) - F(z)}{\lambda}$$

$$\leq \lim_{\varepsilon \downarrow 0} \limsup_{\substack{x \downarrow_F z \\ \lambda \downarrow 0}} \inf_{w \in v + \varepsilon B} \frac{F(z+\lambda w) - F(z)}{\lambda} = F^\uparrow(z;v) \qquad \text{for all } v \in X.$$

From the above result, this shows that F is regular at z in Rockafellar sense and

(3.12) $$F^\uparrow(z;v) = \int_T f_t^\uparrow(z;v) d\mu(t) \qquad \text{for all } v \in X.$$

Next for any $\zeta_t \in \partial^\uparrow f_t(z)$ for almost all $t \in T$, it corresponds an element $\zeta = \int_T <\zeta_t, v> d\mu(t) \in X^*$ such that for any $v \in X$,

$$F^{\uparrow}(z;v) = \int_T f_t^{\uparrow}(z;v)d\mu(t)$$
$$\geq \int_T <\zeta_t, v> d\mu(t) = <\zeta, v>.$$

This shows that $\zeta \in \partial^{\uparrow} F(z)$ and hence the proof is complete.

Q.E.D.

Corollary 3.2. *(Clarke* [2, *Theorem* 1]) *Suppose that $f_t: X \to \overline{\Re}$ satisfies the following conditions:*

(i). *For each $x \in X$, the function $t \to f_t(x)$ is measurable.*

(ii). *For some $K \in L^1(T;\Re^+)$ and $V \in N(z)$, one has*

$$|f_t(x) - f_t(y)| \leq K(t) \cdot \|x - y\| \text{ for all } x, y \in V \text{ and } t \in T.$$

Then

$$\partial^{\circ} F(z) = \partial^{\circ}\left(\int_T f_t(z)d\mu(t)\right) \subseteq \int_T \partial^{\circ} f_t(z)d\mu(t). \tag{3.13}$$

If $f_t(\cdot)$ is regular at z for all $t \in T$, then F is regular at z and the equality in (3.13) *holds.*

Proof: **Since $f_t: X \to \overline{\Re}$ is locally Lipschitz at *z*, by (3.2) and (2.12), f_t is also locally pseudo Lipschitzian at *z* in any direction $v \in X$ and $\partial^{\circ} f_t(z) = \partial^{\uparrow} f_t(z)$. Note that Assumption 3.1, Assumption 3.2 and the hypotheses in Theorem 3.1 are satisfied, so that (3.13) holds.**

Let f_t be regular at *z*. Then *F* is regular at *z* in which *F* is locally Lipschitz at *z*. By Proposition 2.2 and Theorem 3.1, the equality holds in (3.13).

Q.E.D.

Now, we give an example to explain that Theorem 3.1 holds for a non locally Lipschitzian function . It will show that Theorem 3.1 extends the Clarke's result (like Corollary 3.2).

Example: Define $f_t: \Re \to \Re$ by

$$f_t(x) = \begin{cases} t\sqrt{x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}, \quad \text{for all } t \in [1,3].$$

Then by definition, it is easy to get

$$f_t^{\uparrow}(0;v) = \begin{cases} \infty & \text{if } v > 0 \\ 0 & \text{if } v \leq 0. \end{cases}$$

This shows that $Dom\, f_t^{\uparrow}(0;\cdot) = (-\infty, 0]$ and f_t is also locally pseudo Lipschitzian at $z = 0$ in any direction $v \in (-\infty, 0)$ for all $t \in [1,3]$. Indeed, taking $K_1(t) = K_2(t) = 0$ for all $t \in [1,3]$ in definition 3.1, we have

$$\frac{f_t(x + \lambda w) - f_t(x)}{\lambda} \leq 0 \qquad \text{for all } w \in (-\infty, 0).$$

This shows that f_t is locally pseudo Lipschitzian at 0 in any direction $w \in (-\infty, 0)$ for all $t \in [1,3]$. But f_t is not locally Lipschitz at 0.

Clearly, we see that Assumption 3.1, Assumption 3.2 and the hypotheses in Theorem 3.1 are satisfied at $z = 0$. Now to compute the formula (3.3), by the definition of generalized gradient, we obtain

$$F(x) = \int_1^3 f_t(x)dt = \begin{cases} 4\sqrt{x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0, \end{cases}$$

$$\partial^{\uparrow} F(0;v) = \begin{cases} \infty & \text{if } x \geq 0 \\ 0 & \text{if } x < 0. \end{cases}$$

It follows that $\partial^{\uparrow} F(0) = [0, \infty) \subseteq \int_1^3 \partial^{\uparrow} f_t(0)dt = \int_1^3 [0, \infty)dt = [0, \infty)$. That is,

$$\partial^{\uparrow} F(0) = [0, \infty) = \int_1^3 \partial^{\uparrow} f_t(0)dt = \int_1^3 [0, \infty)dt.$$

In fact, $f_t(\cdot)$ is regular at $z = 0$ in Rockafellar sense. This shows that the equality holds in (3.3) whenever $f_t(\cdot)$ is regular at $z = 0$ in Rockafellar sense.

Q.E.D.

4. Integral functionals on $L^\infty(T;X)$

In this section, we let (T,Σ,μ) be a positive finite measure space and let X be a separable Banach space. $L^\infty(T;X)$ denotes the space of essentially bounded measurable functions from T into X. Now let Y be a closed subspace of $L^\infty(T;X)$ and consider the integral functional F defined on Y by the following form:

(4.1) $$F(x)=\int_T f_t(x(t))d\mu(t) \qquad \text{for all } x\in Y.$$

Here like in section 3, for each $x\in X$, the mapping $t\to f_t(x)$ satisfies the following assumptions:

Assumption 4.1. Let $z\in Y$ and let the generalized directional derivative $f_t^\uparrow(z(t);\cdot)$ of $f_t(\cdot)$ at z satisfy the following conditions:

(i). For any $v\in X$, $f_t^\uparrow(z(t);v)$ is measurable in t.

(ii). $f_t(\cdot)$ is continuous at z for all $t\in T$.

(iii). $f_t^\uparrow(z(t);0)=0$ for all $t\in T$ and $\bigcap_{t\in T} Int(Dom\, f_t^\uparrow(z(t);\cdot))\neq\varnothing$,

where $Dom\, f_t^\uparrow(z(t);\cdot)=\{v\in X|\ f_t^\uparrow(z;v)<\infty\}$.

Now we let $z\in Y$ and $f_t: X\to\overline{\Re}$ be finite at $z(t)\in X$ for all $t\in T$ and define

$$g(v)=\int_T f_t^\uparrow(z(t);v)d\mu(t) \qquad \text{for all } v\in X.$$

By the Assumption 4.1 (i), we obtain that for any $v\in X$, $f_t^\uparrow(z(t);v)$ is measurable in t. This implies that g is well defined. To obtain our another main theorem, we need the following assumption.

Assumption 4.2. Either the normal cone $N_{Dom\,g}(0)$ contains only one element $\{0\}$ or $Dom\,g=Dom\,f_t^\uparrow(z(t);v)$ on a set $S\subseteq T$ of positive measure, where $Dom\,g=\{v\in|g(v)<\infty\}$ and $N_{Dom\,g}(0)=\{\zeta\in X^*|<\zeta,v>\leq 0 \text{ for all } v\in Dom\,g\}$.

Theorem 4.1. *Let $f_t(\cdot)$ be locally pseudo Lipschitzian at $z(t) \in X$ in any direction $v \in Int(Dom(f_t^{\uparrow}(z(t);\cdot)))$ for all $t \in T$ and let $f_t^{\uparrow}(z(t);\cdot)$ satisfy Assumptions* 4.1 *and* 4.2. *If X is separable, then*

$$\partial^{\uparrow}\left(\int_T f_t(z(t))d\mu(t)\right) \subseteq \int_T \partial^{\uparrow} f_t(z(t))d\mu(t). \tag{4.2}$$

Furthermore, if each $f_t(\cdot)$ is Rockafellar regular at $z(t) \in X$ and there exist $\overline{\lambda} > 0$ and functions $K_3, K_4 \in L^1(T;\Re)$ satisfying the inequality

$$\frac{f_t(z+\lambda v) - f_t(z)}{\lambda} \geq K_3(t)\|v\| + K_4(t) \text{ for all } \lambda \in (0,\overline{\lambda}], t \in T, v \in X, \tag{4.3}$$

then F is Rockafellar regular at z and the equality holds in (4.2).

Remark 4.1. The inclusion of (4.2) is explained as follows: To every ζ in $\partial^{\uparrow}F(z) \subseteq X^*$ there corresponds a mapping $t \to \zeta_t$ from T into X^* with $\zeta_t \in \partial^{\uparrow} f_t(z(t))$ for almost all $t \in T$ such that the mapping $t \to < \zeta_t, v >$ is summable and

$$< \zeta, v > = \int_T < \zeta_t, v(t) > d\mu(t) \qquad \text{for all } v \in Y. \tag{4.4}$$

***Proof of Theorem* 4.1.** Let $\zeta \in \partial^{\uparrow} F(z)$ and let v be any element of X. Then by Assumption 4.1, we see that the mapping $t \to f_t^{\uparrow}(z(t);v)$ is measurable in t.

In fact, the proof of Theorem 4.1 is similar to the proof of Theorem 3.1, and we deduce the following inequality

$$< \zeta, v > \leq F^{\uparrow}(z;v) = \int_T f_t^{\uparrow}(z(t);v(t))d\mu(t) \qquad \text{for all } v \in Y. \tag{4.5}$$

Now we define

$$h(v) = \int_T g_t(v(t))d\mu(t), \quad \text{where } g_t(\cdot) = f_t^{\uparrow}(z(t);\cdot).$$

Since $g_t(\cdot)$ is convex on X, $h(\cdot)$ is also convex on Y. From the expression (4.5), it implies that ζ belongs to the subdifferential of integral functional $v \to \int_T g_t(v(t))d\mu(t)$ at $v = 0$.

Note that the requirement of [5, Theorem 3, p.28] are satisfied, so that we can obtain the existence of a function $t \to \zeta_t$ such that the mapping $t \to <\zeta_t, v>$ is summable, and

$$<\zeta, v> = \int_T <\zeta_t, v(t)> d\mu(t) \qquad \text{for all } v \in Y,$$

such that $\zeta_t \in \partial g_t(0)$ for almost all $t \in T$.

Since $\partial g_t(0) \neq \varnothing$ for all $t \in T$, this implies $f_t^{\uparrow}(z(t);0) = 0$ for all $t \in T$. Then by Proposition 2.1, we obtain $\partial g_t(0) = \partial f_t^{\uparrow}(z(t);0) = \partial^{\uparrow} f_t(z(t))$ for all $t \in T$. Hence $\zeta_t \in \partial^{\uparrow} f_t(z(t))$. This shows that (4.2) holds.

Finally, we shows that the equality in (4.2) holds if f_t is regular at z in Rockafellar sense as well as the inequality (4.3) hold.

If f_t is regular at $z(t) \in X$ in Rockafellar sense, from (4.5), we obtain

$$\begin{aligned} F^{\uparrow}(z;v) &\le \int_T f_t^{\uparrow}(z(t);v(t))d\mu(t) \\ &= \int_T \liminf_{\substack{w \to v \\ \lambda \downarrow 0}} \frac{f_t(z(t)+\lambda w(t)) - f_t(z(t))}{\lambda} d\mu(t). \end{aligned} \tag{4.6}$$

By (4.3), it shows that the difference quotient function is bounded below by an integrable function, thus Fatou's lemma is applicable and the last expression deduces to

$$\le \liminf_{\substack{w \to v \\ \lambda \downarrow 0}} \int_T \frac{f_t(z(t)+\lambda w(t)) - f_t(z(t))}{\lambda} d\mu(t)$$

$$= \liminf_{\substack{w \to v \\ \lambda \downarrow 0}} \frac{F(z+\lambda w) - F(z)}{\lambda}$$

$$\le \lim_{\varepsilon \downarrow 0} \limsup_{\substack{x \downarrow_F z \\ \lambda \downarrow 0}} \inf_{w \in v+\varepsilon B} \frac{F(z+\lambda w) - F(z)}{\lambda} = F^{\uparrow}(z;v) \qquad \text{for all } v \in Y.$$

This shows that F is regular at z in Rockafellar sense and

$$F^{\uparrow}(z;v) = \int_T f_t^{\uparrow}(z(t);v(t))d\mu(t) \qquad \text{for all } v \in Y. \tag{4.7}$$

Now for any $\zeta_t \in \partial^{\uparrow} f_t(z(t)) \subseteq X^*$ for almost all $t \in T$, we define $\zeta = \int_T < \zeta_t, \cdot > d\mu(t)$. Then by (4.7), we have

$$\begin{aligned} F^{\uparrow}(z;v) &= \int_T f_t^{\uparrow}(z(t);v(t))d\mu(t) \\ &\geq \int_T < \zeta_t, v(t) > d\mu(t) \\ &= < \zeta, v > \quad \text{for all } v \in Y \subseteq L^{\infty}(T;X). \end{aligned}$$

This shows that $\zeta \in \partial^{\uparrow} F(z(t))$ and the proof is complete.

Q.E.D.

References

[1] J.P. Aubin and F.H. Clarke, Shadow prices and duality for a class of optimal control problems, ***SIAM J. Control Optim. 17***, (1979), pp.567-587.

[2] F.H. Clarke, Generalized gradients of Lipschitz functionals, ***Adv. in Math. 40,*** (1981), pp.52-67.

[3] F.H. Clarke, ***Optimization and Nonsmooth Analysis,*** John Wiley and Sons Inc., 1983.

[4] J.B. Hiriart-Urruty, Mean-value theorems in nonsmooth analysis, ***Numer. Funt. Anal. Optim. 2,*** (1980), pp.1-30.

[5] A.D. Ioffe and V.L. Levin, Subdifferentials of convex functions, ***Trans. Monscow Math. Soc. 26,*** (1972), pp.1-72.

[6] R.T. Rockafellar, Directionally Lipschitzian functions and subdifferential calculus, ***Proc. London Math. Soc. 39,*** (1979), pp.331-355.

[7] R.T. Rockafellar, Generalized directional derivatives and subgradients of nonconvex functions, ***Canad. J. Math. 32,*** (1980), pp.257-280.

[8] R.T. Rockafellar, ***The Theory of Subgradients and Its Applications to Problems of Optimization: Convex and Nonconvex Functions,*** Helderman Verlag Berlin, 1981.

[9] R.T. Rockafellar, Lagrange multipliers and subderivatives of optimal value functions in nonlinear programming, ***Math. Programming Study 17,*** (1982), pp.28-66.

[10] J.W. Chen and H.C. Lai, Moreau-Rockafellar type theorems for nonconvex, non-locally Lipschitz integral functional on $L^p(T,X)$, to appear in ***Optimization.***

Received February 19, 1996