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On a theorem of O. Mathieu

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1 Introduction

(1.1) This is an elementary account of O. Mathieu's proof [M2] of S. Donkin's conjecture that if $\mathfrak{G}$ is a connected affine algebraic group over an algebraically closed field and if M and M' are finite dimensional rational $\mathfrak{G}$-modules with good filtrations, then $M \otimes M'$ also admits a good filtration, where a $\mathfrak{G}$-filtration $0 = V^0 < V^1 < \dots$ of a $\mathfrak{G}$-module $V = \cup_{i \geq 0} V^i$ is called good iff for each i there is a Borel subgroup $\mathfrak{B}$ of $\mathfrak{G}$ and a 1-dimensional rational $\mathfrak{B}$-module λ such that V^i/V^{i-1} is isomorphic to the rational $\mathfrak{G}$-module $\mathrm{ind}_{\mathfrak{B}}^{\mathfrak{G}}(\lambda)$ induced by λ. For the history and the significance of the conjecture one may refer to [D1], [D2], [W].

Although [M2] proves much more, we will focus only on the above problem. Instead we will complement more details to the terse arguments of [M2], and also give an alternative construction of what Mathieu calls the canonical Frobenius splitting, based on [K] that requires less algebraic geometry.

(1.2) In order not to obscure the main ideas, let us recall Mathieu's program of the proof, assuming the standard notations from [J], (II.1).

First there is a reduction by [D1] that we may assume $\mathfrak{G}$ is simply connected and simple over an algebraically closed field of positive characteristic p with $\mathfrak{B}$ a fixed Borel subgroup of $\mathfrak{G}$.

By a technical reason it is easier to work over the prime field $\mathbf{F}_p$, which we will denote by K. Thus let G be a simply connected simple K-group scheme with a maximal torus T split over $\mathbf{Z}$, B a Borel subgroup of G containing T, and U the unipotent radical of B. Let $\mathbf{Grp}_K$ be the category of K-groups, and $X = \mathbf{Grp}_K(T, \mathbf{GL}_1)$. The 1-dimensional B-modules are provided by X under the natural projection $B = U \rtimes T \to T$.

If M is a B-module, define a quasicoherent $\mathcal{O}_{G/B}$-module $\mathcal{L}_{G/B}(M)$ by assigning to each open $\mathfrak{V}$ of G/B

(1) $\mathcal{L}_{G/B}(M)(\mathfrak{V}) = \mathbf{Sch}_K(\pi^{-1}\mathfrak{V}, M)^B := \{f \in \mathbf{Sch}_K(\pi^{-1}\mathfrak{V}, M) \mid$
$$f(A)(xb) = b^{-1}f(A)(x) \quad \forall x \in (\pi^{-1}\mathfrak{V})(A), b \in B(A), A \in \mathbf{Alg}_K\},$$

where $\mathbf{Sch}_K$ (resp. $\mathbf{Alg}_K$) denotes the category of K-schemes (resp. commutative K-algebras) and $\pi \in \mathbf{Sch}_K(G, G/B)$ is the quotient morphism. Then $\mathcal{L}_{G/B}(M)$ comes equipped with a structure of G-equivariant sheaf on G/B. In particular, we will abbreviate the G-module $\mathcal{L}_{G/B}(M)(G/B)$ as $H^0(M)$. Then Donkin's conjecture reads that for each λ and $\mu \in X$,

(2) $H^0(\lambda) \otimes_K H^0(\mu)$ admits a G-filtration whose successive quotients are isomorphic to some $H^0(\nu)$, $\nu \in X$.

Let $R \subseteq X$ be the root system of G relative to T and choose a positive system R^+ of R so that the roots of U are $-R^+$. We refine the standard PO (partial order) $\geq$ on X into an additive total order, denoted $\succeq$. Eg., if S is the simple system of R^+, choose a $\mathbf{Q}$-linearly independent set $(r_\alpha)_{\alpha \in S}$ in $\mathbf{R}$ with $r_\alpha > 0$ for each $\alpha \in S$. Then one gets a $\mathbf{Q}$-linear injection

(3) $$X \otimes_{\mathbf{Z}} \mathbf{Q} \longrightarrow \mathbf{R} \quad \text{such that} \quad \alpha \otimes 1 \longmapsto r_\alpha, \quad \alpha \in S.$$

Now for a B-module M let after [D1] $\mathcal{F}_\nu M$ (resp. $\mathcal{F}_\nu^+ M$), $\nu \in X$, be the largest B-submodule of M all of whose weights are $\succeq \nu$ (resp. $\succ \nu$), and set $\mathcal{F}_\nu^0 M = \mathcal{F}_\nu M / \mathcal{F}_\nu^+ M$. Then

(4) $\mathcal{F}_\nu$, $\mathcal{F}_\nu^+$, and $\mathcal{F}_\nu^0$ all define idempotents in the ring of endofunctors of $B\mathbf{Mod}$,

where $\mathfrak{H}\mathbf{Mod}$ denotes the category of $\mathfrak{H}$-modules if $\mathfrak{H}$ is a K-group. Define two subcategories of $B\mathbf{Mod}$

(5) $$\mathbf{M}_B^\nu = \{M \in B\mathbf{Mod} \mid \mathrm{soc}_B M \subseteq M_\nu\}$$

with $M_\nu = \{m \in M \mid t(m \otimes 1) = m \otimes \nu(A)(t) \quad \forall t \in T(A), A \in \mathbf{Alg}_K\}$ the

ν-weight space of M, and

(6) $\mathbf{grA}_B^\nu = \{\mathfrak{A} = \coprod_{n\geq 0} \mathfrak{A}_n$ graded K-algebra $\mid \mathfrak{A}_n \in \mathbf{M}_B^{n\nu} \quad \forall n \in \mathbf{N}$

and the multiplication on $\mathfrak{A}$ is B-equivariant$\}$.

Likewise define two subcategories of $G\mathbf{Mod}$

(7) $$\mathbf{M}^\nu = \mathbf{M}_B^\nu \cap G\mathbf{Mod}$$

and

(8) $\mathbf{grA}^\nu = \{\mathfrak{A} = \coprod_{n\geq 0} \mathfrak{A}_n$ graded K-algebra $\mid \mathfrak{A}_n \in \mathbf{M}^{n\nu} \quad \forall n \in \mathbf{N}$

and the multiplication on $\mathfrak{A}$ is G-equivariant$\}$.

Besides the B-filtrations defined by $\mathcal{F}_\nu$, $\mathcal{F}_\nu^+$, and $\mathcal{F}_\nu^0$, another important ingredient of Mathieu's proof is a Frobenius splitting introduced by V.B. Mehta, S. Ramanan, and A. Ramanathan [MR], [RR], [R]. If $\mathfrak{Y}$ is a K-scheme, let $F_\mathfrak{Y} \in \mathbf{Sch}_K(\mathfrak{Y}, \mathfrak{Y})$ be the Frobenius endomorphism of $\mathfrak{Y}$ defined by $F_\mathfrak{Y}(A) = \mathfrak{Y}(\phi_A)$ with $\phi_A \in \mathbf{Alg}_K(A, A)$ such that $a \mapsto a^p$. A Frobenius splitting of $\mathfrak{Y}$ is a left inverse in $\mathbf{Mod}_\mathfrak{Y}$ the category of $\mathcal{O}_\mathfrak{Y}$-modules to the structure morphism $F_\mathfrak{Y}^\sharp : \mathcal{O}_\mathfrak{Y} \to F_{\mathfrak{Y}*}\mathcal{O}_\mathfrak{Y}$. In particular, a K-linear left inverse ψ to ϕ_A with $\psi(a^p b) = a\psi(b)$ $\forall a, b \in A$ is called a Frobenius splitting of A.

We are now ready to describe Mathieu's program. Let λ, $\mu \in X$ and $V = H^0(\lambda) \otimes_K H^0(\mu)$. Filter V by $\mathcal{F}_\nu^0 V$, $\nu \in X$. One checks

(9) each $\mathcal{F}_\nu^0 V$ inherits the structure of G-module from M

such that $\mathcal{F}_\nu^0 V \in \mathbf{M}^\nu$.

If $E(?)$ denotes the injective hull in $G\mathbf{Mod}$, one has

(10) $$\mathcal{F}_\nu^0 V \leq \mathcal{F}_\nu E(\mathcal{F}_\nu^0 V).$$

On the other hand, if $\mathcal{F}_\nu^0 V \neq 0$, then

(11) $w_0\nu$ is the highest weight of $\mathcal{F}_\nu^0 V$ in the standard PO ,

where w_0 is the element of the Weyl group $W = N_G(T)/T$ such that $w_0 R^+ = -R^+$. If $(\mathcal{F}_\nu^0 V)^{w_0\nu} = (\mathcal{F}_\nu^0 V)/\sum_{\eta \neq w_0\nu}(\mathcal{F}_\nu^0 V)_\eta$, then

(12) $\mathcal{F}_\nu E(\mathcal{F}_\nu^0 V) \simeq H^0((\mathcal{F}_\nu^0 V)^{w_0\nu})$ that is a direct sum of $H^0(w_0\nu)$.

Hence if the equality holds in (10), we will be done.

To see that, Mathieu considers

(13) $\mathfrak{A}^0(\nu) = \coprod_{n\geq 0} \mathcal{F}^0_{n\nu}(H^0(n\lambda) \otimes_K H^0(n\mu))$ and

$$\mathcal{F}E(\mathfrak{A}^0(\nu)) = \coprod_{n\geq 0} \mathcal{F}_{n\nu}E(\mathfrak{A}^0(\nu)_n),$$

where $\mathfrak{A}^0(\nu)_n$ is the n-th component of $\mathfrak{A}^0(\nu)$. Under the cup product both $\mathfrak{A}^0(\nu)$ and $\mathcal{F}E(\mathfrak{A}^0(\nu))$ are made into objects of $\mathbf{grA}^\nu$ such that

(14) $$\mathfrak{A}^0(\nu) \leq \mathcal{F}E(\mathfrak{A}^0(\nu)).$$

If $\mathfrak{B}^0(\nu) = \coprod_{n\geq 0} \mathcal{F}^0_{n\nu}(\mathcal{L}_{G/B}(n\lambda)(Bw_0B/B) \otimes_K H^0(n\mu))$, then $\mathfrak{B}^0(\nu)$ is an object of $\mathbf{grA}^\nu_B$ under the cup product such that

(15) $$\mathfrak{A}^0(\nu) \leq \mathcal{F}E(\mathfrak{A}^0(\nu)) \leq \mathfrak{B}^0(\nu).$$

If the equality were to fail in (14), one could find

(16) $m \in \mathbf{N}^+$ and $a \in \mathcal{F}E(\mathfrak{A}^0(\nu))_m \setminus \mathfrak{A}^0(\nu)$ such that $a^p \in \mathfrak{A}^0(\nu)_{pm}$.

But

(17) $\mathfrak{B}^0(\nu)$ admits a Frobenius splitting $\sigma^0(\nu)$ that stabilizes $\mathfrak{A}^0(\nu)$.

Hence

(18) $$a = \sigma^0(\nu)(a^p) \in \sigma^0(\nu)(\mathfrak{A}^0(\nu)_{pm}) \subseteq \mathfrak{A}^0(\nu),$$

that is a contradiction.

The assertion (16) was a difficult (to the present author) point in [M2]. I have included full detail of the argument in (2.11/12). In (17) I will use the Frobenius splitting on $\mathfrak{G}/\mathfrak{B}$ associated to a lowest weight vector of the Steinberg module, that incidentally splits all the Schubert subschemes simultaneously [K].

(1.3) To be precise, let us intoroduce some more notations and recall some standard facts from the representation theory.

If $\mathfrak{H}$ and $\mathfrak{K}$ are K-groups with $\mathfrak{K} \leq \mathfrak{H}$, one has an exact functor (cf. [J], (I.5.9))

(1) $$\mathcal{L}_{\mathfrak{H}/\mathfrak{K}} : \mathfrak{K}\mathbf{Mod} \longrightarrow \mathbf{Mod}_{\mathfrak{H}/\mathfrak{K}}$$

generalizing the construction of (1.2)(1). If $\mathfrak{H}'$ is a K-subgroup of $\mathfrak{H}$, $\mathfrak{V}$ an open of $\mathfrak{H}/\mathfrak{K}$ such that $\pi^{-1}\mathfrak{V}$ is $\mathfrak{H}'$-stable under the multiplication from the left, and if $M \in \mathfrak{K}\mathbf{Mod}$, then $\mathcal{L}_{\mathfrak{H}/\mathfrak{K}}(M)(\mathfrak{V}) = \mathbf{Sch}_K(\pi^{-1}\mathfrak{V}, M)^{\mathfrak{K}}$ affords an $\mathfrak{H}'$-module such that

(2) $(xf)(A')(y) = f(A')(x^{-1}y),$

$$x \in \mathfrak{H}'(A), A \in \mathbf{Alg}_K, A' \in \mathbf{Alg}_A, y \in (\pi^{-1}\mathfrak{V})(A'),$$

where $\pi : \mathfrak{H} \longrightarrow \mathfrak{H}/\mathfrak{K}$ is the quotient morphism. In particular, $\mathcal{L}_{\mathfrak{H}/\mathfrak{K}}(M)(\mathfrak{H}/\mathfrak{K})$ defines an $\mathfrak{H}$-module that we denote by $\mathrm{ind}_{\mathfrak{K}}^{\mathfrak{H}} M$. Then $\mathrm{ind}_{\mathfrak{K}}^{\mathfrak{H}}$ defines a functor $\mathfrak{K}\mathbf{Mod} \to \mathfrak{H}\mathbf{Mod}$, that is left adjoint to the forgetful functor $\mathfrak{H}\mathbf{Mod} \to \mathfrak{K}\mathbf{Mod}$: $\forall M \in \mathfrak{K}\mathbf{Mod}$ and $V \in \mathfrak{H}\mathbf{Mod}$, there is a K-linear isomorphism

(3) $$\mathfrak{K}\mathbf{Mod}(V, M) \longrightarrow \mathfrak{H}\mathbf{Mod}(V, \mathrm{ind}_{\mathfrak{K}}^{\mathfrak{H}} M) \quad \text{written} \quad \psi \longmapsto \hat{\psi}$$

with $\psi = \varepsilon \circ \hat{\psi}$, where $\varepsilon = \varepsilon_M \in \mathfrak{K}\mathbf{Mod}(\mathrm{ind}_{\mathfrak{K}}^{\mathfrak{H}} M, M)$ such that

(4) $$f \longmapsto f(K)(1), \quad f \in \mathrm{ind}_{\mathfrak{K}}^{\mathfrak{H}} M = \mathbf{Sch}_K(\mathfrak{H}, M)^{\mathfrak{K}}.$$

The isomorphism (3) is called a Frobenius reciprocity.

We will denote the algebra of distributions on $\mathfrak{K}$ by $\mathrm{Dist}(\mathfrak{K})$. If $\mathfrak{K}$ is noetherian and integral, and if M' is a K-linear subspace of M, then (cf. [J], (I.7.15))

(5) $$M \leq M' \text{ in } \mathfrak{K}\mathbf{Mod} \text{ iff } M \leq M' \text{ in } \mathrm{Dist}(\mathfrak{K})\mathbf{Mod}.$$

Let $\alpha \in R$ and U_α the associated root subgroup of G. Writing $K[U_\alpha] = K[t]$, define $X_\alpha^{(n)} \in \mathrm{Dist}(U_\alpha)$, $n \in \mathbf{N}$, by

(6) $$X_\alpha^{(n)}(t^m) = \delta_{nm} \quad \forall m \in \mathbf{N}.$$

Under the comultiplication of $\mathrm{Dist}(U_\alpha)$ one has

(7) $$X_\alpha^{(n)} \longmapsto \sum_{i=0}^{n} X_\alpha^{(i)} \otimes X_\alpha^{(n-i)}.$$

Let X^+ be the set of dominant weights. If $\lambda \in X^+$, then (cf. [J], (II.2.2))

(8) λ is the highest weight of $H^0(\lambda)$

$$\text{with } H^0(\lambda)_\lambda = H^0(\lambda)^{U^+} \text{ of dimension 1.}$$

Moreover (cf. [J], (II.2.3/2.4))

(9) $H^0(\lambda)$ has a simple socle of highest weight λ,

that we will denote by $L(\lambda)$.

(1.4) We list the nonstandard notations employed in this note.

$$
\begin{array}{rcl}
K & = & \mathbf{F}_p \\
H^0_B(?) & = & \mathrm{ind}_T^B(?) \\
E(?) & & \text{injective hull in } G\mathbf{Mod} \\
E_B(?) & & \text{injective hull in } B\mathbf{Mod} \\
\mathfrak{X} & = & G/B \\
\mathfrak{X}_1 & = & G/G_1B \quad \text{with } G_1 = \ker F_G \text{ the Frobenius kernel of } G \\
q & \in & \mathbf{Sch}_K(G/B, G/G_1B) \quad \text{the quotient morphism} \\
\mathfrak{V}_0 & = & Bw_0B/G_1B = w_0U^+B/G_1B \quad \text{with } U^+ = w_0^{-1}Uw_0 \\
\mathfrak{V}_0^q & = & q^{-1}\mathfrak{V}_0 \\
M^\nu & = & M/\sum_{\eta\in X\setminus\{\nu\}} M_\eta \quad \text{if } M \text{ is a } B\text{-module with } \nu \text{ a maximal weight} \\
 & & \text{of } M \text{ in the standard PO on } X.
\end{array}
$$

For $\mathbf{M}_B^\nu$, $\mathbf{grA}_B^\nu$, $\mathbf{M}^\nu$, and $\mathbf{grA}^\nu$ see (1.2)(5), (6), (7), and (8), respectively.

During the preparation of the manuscript, I learned the publication of [vdK], that covers an entire aspect of [M2]. There is also a novel proof of Donkin's conjecture by J. Paradowski [P], who uses quantum groups. I'd like to thank the referee for critical reading of the manuscript, hoping the present note may still be of some help in reading [M2].

2 Filtrations of B-modules

Lemma 2.1 (i) *The simple B-modules are 1-dimensional.*

(ii) *If* $M \in B\mathbf{Mod}$ *and* $\nu \in X$, *then* $\mathcal{F}_\nu^0 M \in \mathbf{M}_B^\nu$.

(iii) $\forall \nu \in X$, $E_B(\nu) \simeq H_B^0(\nu) \simeq K[U] \otimes_K \nu$ *in* $B\mathbf{Mod}$, *hence the formal character of* $E_B(\nu)$ *is* $\mathrm{ch}\, E_B(\nu) = e^\nu \prod_{\alpha\in R^+} \frac{1}{1-e^\alpha}$.

(iv) *If* $\lambda \in X^+$, *then* $\mathrm{soc}_B H^0(\lambda) = H^0(\lambda)_{w_0\lambda} = L(\lambda)_{w_0\lambda} = \mathrm{soc}_B L(\lambda)$ *while* $\mathrm{hd}_B L(\lambda) = L(\lambda)^\lambda$, *both of dimension* 1.

Proof. (i) follows from the local finiteness of B-modules (cf. [J], (I.2.13)). Then (ii) follows from (i). Also $\mathrm{soc}_B H_B^0(\nu) = \nu$ by the Frobenius reciprocity. As ν is injective in $T\mathbf{Mod}$, $H_B^0(\nu)$ remains injective in $B\mathbf{Mod}$ again by the Frobenius reciprocity, hence $H_B^0(\nu) \simeq E_B(\nu)$. Then

$$\text{(10)} \qquad \mathrm{ch}\ E_B(\nu) = \mathrm{ch}\ H_B^0(\nu) = \mathrm{ch}\ \mathbf{Sch}_K(B,\nu)^T = e^\nu \prod_{\alpha\in R^+} \frac{1}{1-e^\alpha}.$$

Finally, as $w_0\lambda$ is the lowest weight of $H^0(\lambda)$ (cf. (1.3)(8))

$$\text{(11)} \qquad H^0(\lambda)_{w_0\lambda} \leq \mathrm{soc}_B H^0(\lambda).$$

On the other hand,

$$\text{(12)} \qquad \begin{aligned} \mathrm{soc}_B H^0(\lambda) &\leq H^0(\lambda)^U \quad \text{by (i)} \\ &= w_0(H^0(\lambda)^{U^+}) = H^0(\lambda)_{w_0\lambda} \quad \text{(cf. (1.3)(8))}. \end{aligned}$$

Hence $H^0(\lambda)_{w_0\lambda} = \mathrm{soc}_B H^0(\lambda)$. As $\dim H^0(\lambda)_{w_0\lambda} = \dim H^0(\lambda)_\lambda = 1$ (cf. (1.3)(8)), one also obtains

$$\text{(13)} \qquad L(\lambda)_{w_0\lambda} = \mathrm{soc}_B L(\lambda) = \mathrm{soc}_B H^0(\lambda).$$

Then

$$\text{(14)} \quad \mathrm{hd}_B L(\lambda) \simeq (\mathrm{soc}_B(L(\lambda)^*))^* \simeq (\mathrm{soc}_B(L(-w_0\lambda)^*))^* \simeq (-\lambda)^* \simeq \lambda,$$

hence $\mathrm{rad}_B L(\lambda) = \sum_{\eta\neq\lambda} L(\lambda)_\eta$.

Lemma 2.2 *Let* $\nu \in X$.

(i) $\mathcal{F}_\nu$ *and* $\mathcal{F}_\nu^+$ *are both left exact. Also* $\mathcal{F}_\nu^0$ *preserves imbeddings.*

(ii) $\mathcal{F}_\nu$, $\mathcal{F}_\nu^+$, *and* $\mathcal{F}_\nu^0$ *all commute with filtered direct limits.*

(iii) *If* $M \in G\mathbf{Mod}$, *then* $\mathcal{F}_\nu M$, $\mathcal{F}_\nu^+ M$, *and* $\mathcal{F}_\nu^0 M$ *all inherit the structure of* G*-modules such that* $\mathcal{F}_\nu^0 M \in \mathbf{M}^\nu$.

Proof. (i) and (ii) follow from the definitions.

(iii) As $\mathrm{Dist}(G) = \mathrm{Dist}(U^+)\mathrm{Dist}(B)$,

$$(1) \qquad \mathrm{Dist}(G)(\mathcal{F}_\nu M) = \mathrm{Dist}(U^+)(\mathcal{F}_\nu M) \subseteq \mathcal{F}_\nu M,$$

hence $\mathcal{F}_\nu M \leq M$ in G**Mod** (cf. (1.3)(5)). Likewise

$$(2) \qquad \mathcal{F}_\nu^+ M \leq M \quad \text{in } G\mathbf{Mod}.$$

Then $\mathcal{F}_\nu^0 M$ comes equipped with a structure of G-module with $\mathcal{F}_\nu^0 M \in \mathbf{M}^\nu$ by (2.1)(ii).

Eg. 2.3 Let $\nu, \eta \in X$. As $\mathrm{soc}_B E_B(\eta) = \eta$ and as $\mathrm{ch}\, E_B(\eta) = e^\eta \prod_{\alpha \in R^+} \frac{1}{1-e^\alpha}$ by (2.1), one finds

$$(1) \qquad \mathcal{F}_\nu E_B(\eta) = \begin{cases} E_B(\eta) & \text{if } \nu \preceq \eta \\ 0 & \text{otherwise,} \end{cases}$$

$$(2) \qquad \mathcal{F}_\nu^+ E_B(\eta) = \begin{cases} E_B(\eta) & \text{if } \nu \prec \eta \\ 0 & \text{otherwise,} \end{cases}$$

$$(3) \qquad \mathcal{F}_\nu^0 E_B(\eta) = \begin{cases} E_B(\eta) & \text{if } \nu = \eta \\ 0 & \text{otherwise.} \end{cases}$$

Hence one sees from (2.2)(ii) that

(4) if $M \in B$**Mod** is injective,

then $\mathcal{F}_\nu M$, $\mathcal{F}_\nu^+ M$, and $\mathcal{F}_\nu^0 M$ all remain injective .

Lemma 2.4 *Let $\nu \in X$ and $\lambda \in X^+$.*

(i) *One has*

$$\mathcal{F}_\nu L(\lambda) = \begin{cases} L(\lambda) & \text{if } \nu \preceq w_0\lambda \\ 0 & \text{otherwise,} \end{cases}$$

$$\mathcal{F}_\nu^+ L(\lambda) = \begin{cases} L(\lambda) & \text{if } \nu \prec w_0\lambda \\ 0 & \text{otherwise,} \end{cases}$$

$$\mathcal{F}^0_\nu L(\lambda) = \begin{cases} L(\lambda) & \textit{if } \nu = w_0\lambda \\ 0 & \textit{otherwise,} \end{cases}$$

$$\mathcal{F}_\nu H^0(\lambda) = \begin{cases} H^0(\lambda) & \textit{if } \nu \preceq w_0\lambda \\ 0 & \textit{otherwise,} \end{cases}$$

$$\mathcal{F}^+_\nu H^0(\lambda) = \begin{cases} H^0(\lambda) & \textit{if } \nu \prec w_0\lambda \\ 0 & \textit{otherwise,} \end{cases}$$

$$\mathcal{F}^0_\nu H^0(\lambda) = \begin{cases} H^0(\lambda) & \textit{if } \nu = w_0\lambda \\ 0 & \textit{otherwise.} \end{cases}$$

(ii) *If* $Q(\lambda) = E(L(\lambda))$, *then in* G**Mod**

$$\mathcal{F}_{w_0\lambda} Q(\lambda) = \mathcal{F}^0_{w_0\lambda} Q(\lambda) \simeq H^0(\lambda).$$

Proof. (i) follows from (2.1)(iv).

(ii) Put $\nu = w_0\lambda$. If $\mathcal{F}^+_\nu Q(\lambda) \neq 0$, then

$$L(\lambda) = \mathrm{soc}_G Q(\lambda) \leq \mathcal{F}^+_\nu Q(\lambda) \quad \text{by (2.2)(iii),}$$

hence $w_0\lambda \succ \nu$, absurd. Consequently, it is enough to show

(1) $$\mathcal{F}_\nu Q(\lambda) \simeq H^0(\lambda) \quad \text{in } G\mathbf{Mod}.$$

As $\mathrm{soc}_G H^0(\lambda) = L(\lambda)$ (cf. (1.3)(9)), one may regard $H^0(\lambda) \leq Q(\lambda)$. Then

(2) $$\mathcal{F}_\nu Q(\lambda) \geq \mathcal{F}_\nu H^0(\lambda) = H^0(\lambda) \quad \text{by (i).}$$

On the other hand, $Q(\lambda)$ admits a good filtration (cf. [J], (II.4.18)) with

(3) $$(Q(\lambda) : H^0(\mu)) = [H^0(\mu) : L(\lambda)] \quad \forall \mu \in X^+,$$

where the LHS (resp. RHS) is the multiplicity of $H^0(\mu)$ (resp. $L(\lambda)$) in the good filtration (resp. the composition series) of $Q(\lambda)$ (resp. $H^0(\mu)$). Moreover, we may assume $H^0(\lambda)$ appears at the bottom of the filtration [J], (II.6.20).

Put $V = Q(\lambda)/H^0(\lambda)$ and just suppose $(\mathcal{F}_\nu Q(\lambda))/H^0(\lambda) \neq 0$. Then

$$\exists \mu \in X^+ : L(\mu) \leq (\mathcal{F}_\nu Q(\lambda))/H^0(\lambda) \leq V.$$

Let $v \in L(\mu) \setminus 0$, and $0 = V^0 < V^1 < \dots$ a good filtration of V with $V^i/V^{i-1} \simeq H^0(\lambda_i)$. If $v \in V^j \setminus V^{j-1}$, then

$$L(\mu) \leq V^j/V^{j-1} \simeq H^0(\lambda_j),$$

hence $\mu = \lambda_j$ (cf. (1.3)(9)). But $\lambda_j > \lambda$ by (3), hence

$$\nu = w_0\lambda > w_0\lambda_j = w_0\mu \quad \text{a weight of } L(\mu),$$

absurd.

Proposition 2.5 *Let $\nu \in X$, $M \in \mathbf{M}_B^\nu$, and fix an imbedding of M into $E_B(M)$ in $B\mathbf{Mod}$.*

(i) *If $M \neq 0$, then ν is the lowest weight of M in the standard PO. In particular, $\mathcal{F}_\nu M = M$. Also $\mathcal{F}_\nu^+ M = 0$, hence $\mathcal{F}_\nu^0 M = M$.*

(ii) $\mathrm{soc}_B E_B(M) = \mathrm{soc}_B M = M_\nu$, *hence $E_B(M) \in \mathbf{M}_B^\nu$. Also $E_B(M) \simeq H_B^0(M_\nu)$ in $B\mathbf{Mod}$ with $\hat{\pi}_\nu \in B\mathbf{Mod}(M, H_B^0(M_\nu))$ injective if $\pi_\nu \in T\mathbf{Mod}(M, M_\nu)$ is the natural projection.*

(iii) *If $\eta \in X$, $M' \in \mathbf{M}_B^\eta$, and if $i_B : M \otimes_K M' \to E_B(M) \otimes_K E_B(M')$ is the natural imbedding, then $\mathcal{F}_{\nu+\eta}^0(i_B)$ restricts to an isomorphism from*

$$\begin{aligned} \mathrm{soc}_B \mathcal{F}_{\nu+\eta}^0(M \otimes_K M') &= \{\mathcal{F}_{\nu+\eta}^0(M \otimes_K M')\}_{\nu+\eta} \\ &\simeq M_\nu \otimes_K M'_\eta \quad \textit{under the natural projection} \end{aligned}$$

onto

$$\mathrm{soc}_B \mathcal{F}_{\nu+\eta}^0(E_B(M) \otimes_K E_B(M')) = \mathcal{F}_{\nu+\eta}^0(E_B(M) \otimes_K E_B(M'))_{\nu+\eta}.$$

Hence one has also isomorphisms in $B\mathbf{Mod}$

$$\mathcal{F}_{\nu+\eta}^0(E_B(M) \otimes_K E_B(M')) \simeq E_B(M_\nu \otimes_K M'_\eta) \simeq E_B(\mathcal{F}_{\nu+\eta}^0(M \otimes_K M'))$$

Proof. (i) If $m \in M_\lambda \setminus 0$, $\lambda \in X$, then $\mathrm{Dist}(B)m$ is a finite dimensional submodule of M, hence $\nu \leq \lambda$.

(ii) By (i) $M_\nu \leq \mathrm{soc}_B M$, hence $M_\nu = \mathrm{soc}_B M = \mathrm{soc}_B E_B(M)$. Then $E_B(M) \in \mathbf{M}_B^\nu$, and also $E_B(M) \simeq H_B^0(M_\nu)$ (cf. (2.1)(iii)). As $\varepsilon \circ \hat{\pi}_\nu |_{\mathrm{soc}_B M} = \mathrm{id}_{M_\nu} = \mathrm{id}_{\mathrm{soc}_B M}$, $\hat{\pi}_\nu$ is injective.

(iii) By (ii) we may assume $i_B = \hat{\pi}_\nu \otimes_K \hat{\pi}_\eta$. Then

$$\begin{aligned}
&\mathrm{soc}_B \mathcal{F}^0_{\nu+\eta}(H^0_B(M) \otimes_K H^0_B(M')) \\
&= \mathcal{F}^0_{\nu+\eta}(H^0_B(M) \otimes_K H^0_B(M'))_{\nu+\eta} \quad \text{by (ii) and (2.1)(ii)} \\
&\simeq H^0_B(M)_\nu \otimes_K H^0_B(M')_\eta \quad \text{as } \nu \text{ (resp. } \eta\text{) is the lowest weight of } H^0_B(M) \\
&\qquad \text{(resp. } H^0_B(M')\text{) by (i) and the character formula (2.1)(iii)} \\
&= M_\nu \otimes_K M'_\eta \\
&\simeq \mathcal{F}^0_{\nu+\eta}(M \otimes_K M')_{\nu+\eta} = \mathrm{soc}_B \mathcal{F}^0_{\nu+\eta}(M \otimes_K M'),
\end{aligned}$$

where the two isomorphisms are both induced by the natural projections.

As any T-module is injective, $H^0_B(M) \otimes_K H^0_B(M')$ is injective in $B\mathbf{Mod}$ by the Frobenius reciprocity (cf. [J], (I.3.10) or by the tensor identity). As $\mathcal{F}^0_{\nu+\eta}(H^0_B(M) \otimes_K H^0_B(M'))$ remains injective by (2.3)(4), the last assertion also follows.

Corollary 2.6 *Let $\nu \in X$, $A \in \mathbf{grA}^\nu_B$, and $E_B(A) = \coprod_{n\geq 0} E_B(A_n)$. Then one can make $E_B(A)$ into an object of $\mathbf{grA}^\nu_B$ uniquely such that $A \leq E_B(A)$ in $\mathbf{grA}^\nu_B$. If A is commutative (resp. commutative and reduced), then $E_B(A)$ is also commutative (resp. commutative and reduced).*

Proof. Take after (2.5)

(1) $$E_B(A_n) = H^0_B(A_{n,n\nu}) \quad \forall n \in \mathbf{N},$$

and fix an imbedding

(2) $$j^B_n = \hat{\pi}_{n\nu} \in B\mathbf{Mod}(A_n, H^0_B(A_{n,n\nu})).$$

Thus if $i^B_n : A_{n,n\nu} \hookrightarrow A_n$, then

(3) $$\varepsilon \circ j^B_n \circ i^B_n = \pi_{n\nu} \circ i^B_n = \mathrm{id}_{A_{n,n\nu}},$$

where $\pi_{n\nu} \in T\mathbf{Mod}(A_n, A_{n\nu})$ is the natural projection.

If $\gamma^B_{nm} : A_n \otimes_K A_m \to A_{n+m}$ is the multiplication on A, define

$$\tilde{\gamma}^B_{nm} \in B\mathbf{Mod}(E_B(A_n) \otimes_K E_B(A_m), E_B(A_{n+m}))$$

to be $\{\gamma^B_{nm}|_{A_{n,n\nu}\otimes_K A_{m,m\nu}} \circ (\varepsilon \otimes_K \varepsilon)\}\hat{}$:

$$
(4)\qquad \begin{array}{ccc}
E_B(A_n)\otimes_K E_B(A_m) & \dashrightarrow & E_B(A_{n+m}) \\
{\scriptstyle \varepsilon\otimes_K\varepsilon}\downarrow & & \downarrow{\scriptstyle \varepsilon} \\
A_{n,n\nu}\otimes_K A_{m,m\nu} & \xrightarrow[\gamma^B_{nm}]{} & A_{n+m,(n+m)\nu}.
\end{array}
$$

We must show

$$
(5)\qquad \tilde{\gamma}^B_{nm}\circ(j^B_n\otimes_K j^B_m)=j^B_{n+m}\circ\gamma^B_{nm},
$$

and that the multiplication on $E_B(A)$ defined by $\tilde{\gamma}^B_{nm}$, $n,m\in\mathbf{N}$, is associative: $\forall n,m,\ell\in\mathbf{N}$,

$$
(6)\qquad \tilde{\gamma}^B_{n,m+\ell}\circ(E_B(A_n)\otimes_K\tilde{\gamma}^B_{m\ell})=\tilde{\gamma}^B_{n+m,\ell}\circ(\tilde{\gamma}^B_{nm}\otimes_K E_B(A_\ell)).
$$

First by (2.5)(ii) and (i)

$$
(7)\qquad \mathcal{F}^0_{(n+m)\nu}E_B(A_{n+m})=E_B(A_{n+m}),
$$

hence both sides of (5) factor through $\mathcal{F}^0_{(n+m)\nu}(A_n\otimes_K A_m)$. Then to see (5), it is enough to show that both sides of (5) induced on $\mathcal{F}^0_{(n+m)\nu}(A_n\otimes_K A_m)$ agree on $\mathrm{soc}_B\mathcal{F}^0_{(n+m)\nu}(A_n\otimes_K A_m)$. On the other hand, the natural projection induces by (2.5)(iii) an isomorphism

$$
(8)\qquad A_{n,n\nu}\otimes_K A_{m,m\nu}\longrightarrow \mathrm{soc}_B\mathcal{F}^0_{(n+m)\nu}(A_n\otimes_K A_m).
$$

Hence we have only to check with $i^B_{nm}=i^B_n\otimes_K i^B_m$ and $j^B_{nm}=j^B_n\otimes_K j^B_m$

$$
(9)\qquad \tilde{\gamma}^B_{nm}\circ j^B_{nm}\circ i^B_{nm}=j^B_{n+m}\circ\gamma^B_{nm}\circ i^B_{nm},
$$

that will also imply by (2.5)(iii) the unicity of the multiplication on $E_B(A)$ extending that of A. But the image of the LHS of (9) is contained in $E_B(A_{n+m})_{(n+m)\nu}$, that is $A_{n+m,(n+m)\nu}$ by (2.5)(ii). Hence

$$
\begin{aligned}
(10)\qquad \text{LHS} &= j^B_{n+m}\circ i^B_{n+m}\circ\varepsilon\circ\tilde{\gamma}^B_{nm}\circ j^B_{nm}\circ i^B_{nm}\\
&= j^B_{n+m}\circ\gamma^B_{nm}\circ i^B_{nm}\circ(\varepsilon\otimes_K\varepsilon)\circ j^B_{nm}\circ i^B_{nm} \quad \text{by (4)}\\
&= \text{RHS} \quad \text{by (3)},
\end{aligned}
$$

as desired.

Likewise, to see (6), as both sides of (6) factor through $\mathcal{F}^0_{(n+m+\ell)\nu}(E_B(A_n)\otimes_K E_B(A_m)\otimes_K E_B(A_\ell))$ and as $j_n^B\otimes_K j_m^B\otimes_K j_\ell^B$ induces a bijection

(11) $A_{n,n\nu}\otimes_K A_{m,m\nu}\otimes_K A_{\ell,\ell\nu}\longrightarrow$

$$\mathrm{soc}_B\mathcal{F}^0_{(n+m+\ell)\nu}(E_B(A_n)\otimes_K E_B(A_m)\otimes_K E_B(A_\ell)),$$

one has only to show with $j^B_{nm\ell}=j_n\otimes_K j_m\otimes_K j_\ell$ and $i^B_{nm\ell}=i_n\otimes_K i_m\otimes_K i_\ell$

(12) $\tilde{\gamma}^B_{n,m+\ell}\circ(E_B(A_n)\otimes_K\tilde{\gamma}^B_{m,\ell})\circ j^B_{nm\ell}\circ i^B_{nm\ell}=\tilde{\gamma}^B_{n+m,\ell}\circ j^B_{nm\ell}\circ i^B_{nm\ell}.$

But the image of the LHS is contained in $A_{n+m+\ell,(n+m+\ell)\nu}$, hence (12) follows as in (10).

If A is commutative, then one can argue likewise to show

(13) $$\tilde{\gamma}^B_{mn}\circ\tau_{nm}=\tilde{\gamma}^B_{nm}\quad\forall n,m\in\mathbf{N},$$

where $\tau_{nm}: E_B(A_n)\otimes_K E_B(A_m)\to E_B(A_m)\otimes_K E_B(A_n)$ is the transposition. Hence $E_B(M)$ will be commutative.

Assume finally that A is commutative and reduced. Just suppose $x\in E_B(A_n)\setminus 0$ is nilpotent. Then the B-submodule $\mathrm{Dist}(B)x=KBx$ (cf. (1.3)(5) and [J], (I.2.13)) of $E_B(A_n)$ would consist of nilpotents. But

$$0\neq\mathrm{soc}_B(KBx)\leq\mathrm{soc}_B E_B(A_n)=\mathrm{soc}_B(A_n).$$

Hence A_n would contain a nonzero nilpotent, absurd.

Proposition 2.7 *Let $\nu,\eta\in X$, and $M\in\mathbf{M}^\nu$, $N\in\mathbf{M}^\eta$.*

(i) *If $M\neq 0$, then $w_0\nu$ is the highest weight of M in the standard PO.*

(ii) *If we regard $M\leq E(M)$, then*

$$M_\nu=\mathrm{soc}_B M=(\mathrm{soc}_G M)_\nu=\mathrm{soc}_B\mathcal{F}_\nu E(M)=\{\mathcal{F}_\nu E(M)\}_\nu,$$

hence $\mathcal{F}_\nu E(M)\in\mathbf{M}^\nu$ that is an injective hull of M in $\mathbf{M}^\nu$.

(iii) $\mathrm{hd}_B\mathrm{soc}_G M=(\mathrm{soc}_G M)^{w_0\nu}\simeq M^{w_0\nu}$ *in $B\mathbf{Mod}$ with the isomorphism induced by the inclusion.*

(iv) $\mathcal{F}_\nu E(M)\simeq H^0(M^{w_0\nu})$ *in $G\mathbf{Mod}$ with $\hat{\pi}^{w_0\nu}\in G\mathbf{Mod}(M,H^0(M^{w_0\nu}))$ injective if $\pi^{w_0\nu}\in B\mathbf{Mod}(M,M^{w_0\nu})$ is the natural projection. In particular, $\mathcal{F}_\nu E(M)$ is a direct sum of $H^0(w_0\nu)$, and* $\mathrm{soc}_G\mathcal{F}_\nu E(M)\simeq\mathrm{soc}_G M$ *is a direct sum of $L(w_0\nu)$.*

(v) *In* B**Mod**

$$\begin{aligned}\mathrm{hd}_B\mathrm{soc}_G\mathcal{F}^0_{\nu+\eta}(M\otimes_K N) &= \{\mathrm{soc}_G\mathcal{F}^0_{\nu+\eta}(M\otimes_K N)\}^{w_0(\nu+\eta)}\\ &\simeq \{\mathcal{F}^0_{\nu+\eta}(M\otimes_K N)\}^{w_0(\nu+\eta)} \quad \textit{induced by the inclusion}\\ &\simeq (M\otimes_K N)^{w_0(\nu+\eta)} \quad \textit{induced by the natural projection}\\ &\simeq M^{w_0\nu}\otimes_K N^{w_0\eta}.\end{aligned}$$

(vi) *The cup product induces an isomorphism in* G**Mod**

$$\mathcal{F}^0_{\nu+\eta}(H^0(M^{w_0\nu})\otimes_K H^0(N^{w_0\eta})) \longrightarrow H^0(M^{w_0\nu}\otimes_K N^{w_0\eta}).$$

(vii) *If* $i : M\otimes_K N \to \mathcal{F}_\nu E(M)\otimes_K \mathcal{F}_\eta E(N)$ *is the natural injection (cf.* (2.5)(i)), *then* $\mathcal{F}^0_{\nu+\eta}(i)$ *restricts to an isomorphism in* G**Mod** (*resp.* B**Mod**) *from*

$$\mathrm{soc}_G\mathcal{F}^0_{\nu+\eta}(M\otimes_K N) \quad (\textit{resp.}\quad \mathrm{soc}_B\mathcal{F}^0_{\nu+\eta}(M\otimes_K N) = \{\mathcal{F}^0_{\nu+\eta}(M\otimes_K N)\}_{\nu+\eta}\simeq M_\nu\otimes_K N_\eta)$$

onto

$$\mathrm{soc}_G\mathcal{F}^0_{\nu+\eta}(\mathcal{F}_\nu E(M)\otimes_K \mathcal{F}_\eta E(N)) \quad (\textit{resp.}\quad \mathrm{soc}_B\mathcal{F}^0_{\nu+\eta}(\mathcal{F}_\nu E(M)\otimes_K\mathcal{F}_\eta E(N)) = \{\mathcal{F}^0_{\nu+\eta}(\mathcal{F}_\nu E(M)\otimes_K\mathcal{F}_\eta E(N))\}_{\nu+\eta}).$$

Proof. (i) follows from (2.5)(i).

(ii)-(iv) If $L(\lambda)\le \mathrm{soc}_G M$ with $\lambda\in X^+$, then

$$\begin{aligned}L(\lambda)_{w_0\lambda} &= \mathrm{soc}_B L(\lambda) \quad \text{by (2.1)(iv)}\\ &\le \mathrm{soc}_B M = M_\nu \quad \text{by (2.5)(ii)},\end{aligned}\tag{1}$$

hence $w_0\lambda=\nu$. Consequently,

$$\mathrm{soc}_G M \text{ is a direct sum of } L(w_0\nu).\tag{2}$$

If we write $\mathrm{soc}_G M = \coprod_\Lambda L(w_0\nu)$, then in G**Mod**

$$\begin{aligned}\mathcal{F}_\nu E(M) &\simeq \mathcal{F}_\nu E(\mathrm{soc}_G M) = \mathcal{F}_\nu E(\coprod_\Lambda L(w_0\nu))\\ &= \coprod_\Lambda H^0(w_0\nu) \quad \text{by (2.2)(ii) and (2.4)(ii)}\\ &\simeq H^0(\mathrm{hd}_B\mathrm{soc}_G M) \quad \text{by (2.1)(iv)}.\end{aligned}\tag{3}$$

If we regard $M \le E(\mathrm{soc}_G M)$, then

$$\begin{aligned}(4)\qquad \mathrm{soc}_B M &= \mathrm{soc}_B \mathcal{F}_\nu M \quad \text{by (2.5)(i)}\\ &\le \mathrm{soc}_B \mathcal{F}_\nu E(\mathrm{soc}_G M) = \mathrm{soc}_B \mathcal{F}_\nu E(\coprod_\Lambda L(w_0\nu))\\ &= \mathrm{soc}_B \coprod_\Lambda H^0(w_0\nu) \quad \text{by (3)}\\ &= \{\coprod_\Lambda H^0(w_0\nu)\}_\nu = \{\coprod_\Lambda L(w_0\nu)\}_\nu \quad \text{by (2.1)(iv)}\\ &= \{\mathcal{F}_\nu E(\mathrm{soc}_G M)\}_\nu = (\mathrm{soc}_G M)_\nu\\ &= \mathrm{soc}_B \mathrm{soc}_G M \quad \text{by (2.5)(ii)}\\ &\le \mathrm{soc}_B M.\end{aligned}$$

Hence if we regard $M \le E(M)$, then

$$\begin{aligned}(5)\quad M_\nu &= \mathrm{soc}_B M \quad \text{by (2.5)(ii)}\\ &= \mathrm{soc}_B \mathcal{F}_\nu E(M) = \{\mathcal{F}_\nu E(M)\}_\nu = (\mathrm{soc}_G M)_\nu = \mathrm{soc}_B \mathrm{soc}_G M.\end{aligned}$$

In particular,

(6) $$\mathcal{F}_\nu E(M) \in \mathbf{M}^\nu.$$

Also

(7) $$M_{w_0\nu} = \{\mathcal{F}_\nu E(M)\}_{w_0\nu} = (\mathrm{soc}_G M)_{w_0\nu},$$

hence one gets in $B\mathbf{Mod}$

$$\begin{aligned}(8)\qquad \mathrm{hd}_B \mathrm{soc}_G M &= (\mathrm{soc}_G M)^{w_0\nu} \quad \text{by (2) and (2.1)(iv)}\\ &\simeq M^{w_0\nu} \quad \text{by (i).}\end{aligned}$$

Then by (3)

(9) $$\mathcal{F}_\nu E(M) \simeq H^0(M^{w_0\nu}).$$

As $\hat{\pi}^{w_0\nu}\,|_{\mathrm{soc}_G M}$ preserves $(\mathrm{soc}_G M)_{w_0\nu}$ and as $\mathrm{soc}_G M$ is a direct sum of $L(w_0\nu)$, $\hat{\pi}^{w_0\nu}\,|_{\mathrm{soc}_G M}$ is injective, hence

(10) $$\hat{\pi}^{w_0\nu} \in G\mathbf{Mod}(M, H^0(M^{w_0\nu})) \text{ is injective.}$$

(v) By (2.2)(iii) $\mathcal{F}^0_{\nu+\eta}(M \otimes_K N) \in \mathbf{M}^{\nu+\eta}$, hence (iii) implies in B**Mod**

$$
\begin{aligned}
(11)\quad \mathrm{hd}_B \mathrm{soc}_G \mathcal{F}^0_{\nu+\eta}(M \otimes_K N) &= \{\mathrm{soc}_G \mathcal{F}^0_{\nu+\eta}(M \otimes_K N)\}^{w_0(\nu+\eta)} \\
&\simeq \{\mathcal{F}^0_{\nu+\eta}(M \otimes_K N)\}^{w_0(\nu+\eta)} \quad \text{induced by the inclusion by (8)} \\
&\simeq (M \otimes_K N)^{w_0(\nu+\eta)} \quad \text{induced by the natural projection as} \\
&\qquad \{\mathcal{F}^0_{\nu+\eta}(M \otimes_K N)\}_{\nu+\eta} \simeq (M \otimes_K N)_{\nu+\eta} \text{ by (2.5)(iii)} \\
&\simeq M^{w_0\nu} \otimes_K N^{w_0\eta}.
\end{aligned}
$$

(vi) The cup product $H^0(M^{w_0\nu}) \otimes_K H^0(N^{w_0\eta}) \to H^0(M^{w_0\nu} \otimes_K N^{w_0\eta})$ is surjective (cf. [J], (II.14.20)). Put $V = H^0(M^{w_0\nu}) \otimes_K H^0(N^{w_0\eta})$.

As $H^0(M^{w_0\nu} \otimes_K N^{w_0\eta}) = \mathcal{F}^0_{\nu+\eta} H^0(M^{w_0\nu} \otimes_K N^{w_0\eta})$ by (2.4)(i), the cup product factors through $\mathcal{F}^0_{\nu+\eta}V$ to induce a morphism in G**Mod** $\gamma : \mathcal{F}^0_{\nu+\eta}V \to H^0(M^{w_0\nu} \otimes_K N^{w_0\eta})$. On the other hand, $\mathrm{soc}_G \mathcal{F}^0_{\nu+\eta}V$ is a direct sum of $L(w_0(\nu+\eta))$ by (iv). As γ preserves $(\mathcal{F}^0_{\nu+\eta}V)_{w_0(\nu+\eta)}$ by (11), $\gamma\,|_{\mathrm{soc}_G \mathcal{F}^0_{\nu+\eta}V}$ is injective. Hence γ is invertible.

(vii) By (iv) we may assume $i = \hat{\pi}^{w_0\nu} \otimes_K \hat{\pi}^{w_0\eta}$, so $\mathcal{F}_\nu E(M) \otimes_K \mathcal{F}_\eta E(N) = V$ of (vi). Then one has in B**Mod**

$$
\begin{aligned}
(12)\quad (\mathrm{soc}_G \mathcal{F}^0_{\nu+\eta}V)_{\nu+\eta} &= \mathrm{soc}_B \mathcal{F}^0_{\nu+\eta}V = (\mathcal{F}^0_{\nu+\eta}V)_{\nu+\eta} \quad \text{by (ii)} \\
&\simeq V_{\nu+\eta} \quad \text{under the natural projection as } \nu+\eta \text{ is the lowest weight of } V \\
&= H^0(M^{w_0\nu})_\nu \otimes_K H^0(N^{w_0\eta})_\eta \\
&\simeq M_\nu \otimes_K N_\eta \quad \text{induced by } i \text{ as } M_{w_0\nu} \otimes_K N_{w_0\eta} \simeq \\
&\qquad H^0(M^{w_0\nu})_{w_0\nu} \otimes_K H^0(N^{w_0\eta})_{w_0\eta} \\
&= (M \otimes_K N)_{\nu+\eta} \\
&\simeq \{\mathcal{F}^0_{\nu+\eta}(M \otimes_K N)\}_{\nu+\eta} \\
&= \mathrm{soc}_B \mathcal{F}^0_{\nu+\eta}(M \otimes_K N) = \{\mathrm{soc}_G \mathcal{F}^0_{\nu+\eta}(M \otimes_K N)\}_{\nu+\eta}.
\end{aligned}
$$

Hence also

$$
(13)\qquad \mathrm{soc}_G \mathcal{F}^0_{\nu+\eta}V \simeq \mathrm{soc}_G \mathcal{F}^0_{\nu+\eta}(M \otimes_K N)
$$

as both are direct sums of copies of $L(w_0(\nu+\eta))$ by (iv).

Remark 2.8 It follows that $\mathbf{M}^\nu \neq \{0\}$ iff $w_0\nu \in X^+$, in which case $\mathbf{M}^\nu$ has a unique simple $L(w_0\nu)$ with the injective hull $H^0(w_0\nu)$.

Corollary 2.9 *Let $\nu \in X$, $A \in \mathbf{grA}^\nu$, and $\mathcal{F}E(A) = \coprod_{n \geq 0} \mathcal{F}_{n\nu} E(A_n)$. One can make $\mathcal{F}E(A)$ uniquely into an object of $\mathbf{grA}^\nu$ such that $A \leq \mathcal{F}E(A)$ in $\mathbf{grA}^\nu$ and that $\mathcal{F}E(A) \leq E_B(A)$ in $\mathbf{grA}^\nu_B$. If A is commutative (resp. commutative and reduced), then $\mathcal{F}E(A)$ remains commutative (resp. commutative and reduced).*

Proof. Take after (2.7)(iv) (resp. (2.5)(ii))

(1) $\mathcal{F}_{n\nu}E(A_n) = H^0(A_n^{nw_0\nu})$ (resp. $E_B(A)_n = E_B(A_n) = H^0_B(A_{n,n\nu})$),

and fix an imbedding

(2) $$j_n = \hat{\pi}^{nw_0\nu} \in G\mathbf{Mod}(A_n, H^0(A_n^{nw_0\nu})).$$

By (2.1)(iv)

(3) $$\mathrm{soc}_B H^0(A_n^{nw_0\nu}) = H^0(A_n^{nw_0\nu})_{n\nu} \simeq w_0 A_{n,nw_0\nu} \simeq A_{n,n\nu},$$

hence the projection $H^0(A_n^{nw_0\nu}) \longrightarrow H^0(A_n^{nw_0\nu})_{n\nu}$ in $T\mathbf{Mod}$ induces an injection $h_n \in B\mathbf{Mod}(H^0(A_n^{nw_0\nu}), H^0_B(A_{n,n\nu}))$ such that

(4) $$\varepsilon \circ h_n \circ j_n \mid_{A_{n,n\nu}} = \mathrm{id}_{A_{n,n\nu}}.$$

Recall from (2.5)(ii) the imbedding $j_n^B = \hat{\pi}_{n\nu} : A_n \to H^0_B(A_{n,n\nu})$. One has

(5) $$h_n \circ j_n = j_n^B \quad \forall n \in \mathbf{N}.$$

If $\gamma_{nm} \in G\mathbf{Mod}(A_n \otimes_K A_m, A_{n+m})$ is the multiplication on A, define the multiplication on $\mathcal{F}E(A)$ to be $\tilde{\gamma}_{nm} = \{\gamma_{nm}^{(n+m)w_0\nu} \circ (\varepsilon \otimes_K \varepsilon)\}\hat{} \in G\mathbf{Mod}(H^0(A_n^{nw_0\nu}) \otimes_K H^0(A_m^{mw_0\nu}), H^0(A_{n+m}^{(n+m)w_0\nu}))$:

(6) $$\begin{array}{ccc} H^0(A_n^{nw_0\nu}) \otimes_K H^0(A_m^{mw_0\nu}) & \xrightarrow{\tilde{\gamma}_{nm}} & H^0(A_{n+m}^{(n+m)w_0\nu}) \\ {\scriptstyle \varepsilon \otimes_K \varepsilon}\downarrow & \circlearrowright & \downarrow{\scriptstyle \varepsilon} \\ A_n^{nw_0\nu} \otimes_K A_m^{mw_0\nu} & \xrightarrow[\gamma_{nm}^{(n+m)w_0\nu}]{} & A_{n+m}^{(n+m)w_0\nu} \\ {\scriptstyle \pi^{nw_0\nu} \otimes_K \pi^{mw_0\nu}}\uparrow & \circlearrowright & \uparrow{\scriptstyle \pi^{(n+m)w_0\nu}} \\ A_n \otimes_K A_m & \xrightarrow[\gamma_{nm}]{} & A_{n+m} \end{array}$$

We must show

(7) $$\tilde{\gamma}_{nm} \circ (j_n \otimes_K j_m) = j_{n+m} \circ \gamma_{nm}$$

and

$$(8)\qquad \tilde{\gamma}^B_{nm}\circ(h_n\otimes_K h_m)=h_{n+m}\circ\tilde{\gamma}_{nm}.$$

As $H^0(A_{n+m}^{(n+m)w_0\nu})=\mathcal{F}^0_{(n+m)\nu}H^0(A_{n+m}^{(n+m)w_0\nu})$ by (2.4)(i), both sides of (7) factor through $\mathcal{F}^0_{(n+m)\nu}(A_n\otimes_K A_m)$. Hence to see (7), it is enough to show that the both sides of (7) induce the same morphisms on $\mathrm{soc}_G\mathcal{F}_{(n+m)\nu}(A_n\otimes_K A_m)$, that will also imply by (2.7)(vii) the unicity of the multiplication on $\mathcal{F}E(M)$ extending that of A. But the natural projection induces by (2.7)(v) an isomorphism

$$(9)\qquad A_n^{nw_0\nu}\otimes_K A_m^{mw_0\nu}\longrightarrow \mathrm{hd}_B\mathrm{soc}_G\mathcal{F}^0_{(n+m)\nu}(A_n\otimes_K A_m).$$

If $i_r: A_{r,rw_0\nu}\hookrightarrow A_r$, $r\in\mathbf{N}$, then with $i_{nm}=i_n\otimes_K i_m$ and $j_{nm}=j_n\otimes_K j_m$

$$(10)\qquad j_{n+m}\circ\gamma_{nm}\circ i_{nm}=\tilde{\gamma}_{nm}\circ j_{nm}\circ i_{nm}.$$

As $\mathrm{soc}_G\mathcal{F}^0_{(n+m)\nu}(A_n\otimes_K A_m)$ is a direct sum of copies of $L((n+m)w_0\nu)$ by (2.7)(iv), (7) follows.

Likewise by (2.5)(ii)/(i) both sides of (8) factor through $\mathcal{F}^0_{(n+m)\nu}(H^0(A_n^{nw_0\nu})\otimes_K H^0(A_m^{mw_0\nu}))$, hence one has only to check that both sides induce the same morphisms on $\mathrm{soc}_B\mathcal{F}^0_{(n+m)\nu}(H^0(A_n^{nw_0\nu})\otimes_K H^0(A_m^{mw_0\nu}))$. But $\hat{\pi}^{nw_0\nu}\otimes_K\hat{\pi}^{mw_0\nu}$ induces by (2.7)(vii) a bijection

$$(11)\qquad A_{n,n\nu}\otimes_K A_{m,m\nu}\longrightarrow \mathrm{soc}_B\mathcal{F}^0_{(n+m)\nu}(H^0(A_n^{nw_0\nu})\otimes_K H^0(A_m^{mw_0\nu})).$$

In the notation of (2.6) one has

$$(12)\qquad \begin{aligned}\tilde{\gamma}^B_{nm}\circ(h_n\otimes_K h_m)\circ j_{nm}\circ i^B_{nm} &= j^B_{n+m}\circ\gamma_{nm}\circ i^B_{nm}\\ &= h_{n+m}\circ\tilde{\gamma}_{nm}\circ j_{nm}\circ i^B_{nm},\end{aligned}$$

hence (8).

As $\mathcal{F}E(A)\leq E_B(A)$ in $\mathbf{grA}^\nu_B$, the last assertions follow from (2.6).

Lemma 2.10 ([M1], Lemma 13) *Let A be a commutative graded K-algebra and A' a proper graded subalgebra of A. Assume A (resp. A') is an A'-module (resp. K-algebra) of finite type. If the inclusion $i: A'\to A$ induces a bijection $\mathrm{Max}(A\otimes_K\overline{K})\to\mathrm{Max}(A'\otimes_K\overline{K})$ of the maximal spectra, then there is a homogeneous element $s\in A\setminus A'$ such that $s^p\in A'$, where $\overline{K}$ is an algebraic closure of K.*

Proof. If one can find such $s \in (A \otimes_K \overline{K}) \setminus (A' \otimes_K \overline{K})$, write $s = \sum_j s_j \otimes \xi_j$ with $(\xi_j)_j$ linearly independent over K. Then $\sum_j s_j^p \otimes \xi_j^p = s^p \in A' \otimes_K \overline{K}$, hence $\sum_j s_j^p \otimes \xi_j^p = 0$ in $(A/A') \otimes_K \overline{K}$. But $(\xi_j^p)_j$ remain linearly independent over K: if $0 = \sum_j c_j \xi_j^p$ in $\overline{K}$ with $c_j \in K$, then $0 = \sum_j c_j^p \xi_j^p = (\sum_j c_j \xi_j)^p$, hence $0 = \sum_j c_j \xi_j$, so $c_j = 0\ \forall j$. Consequently $s_j^p \in A'\ \forall j$. As there is some j with $s_j \notin A'$, by considering $A \otimes_K \overline{K}$ and $A' \otimes_K \overline{K}$ instead of A and A', one may replace K by $\overline{K}$.

If a is a homogeneous element of A, define a homogeneous ideal of A'

(1) $$I(a) = \{x \in A' \mid xa \in A'\}.$$

As A' is noetherian, there is homogeneous $c \in A$ such that

(2) $I(c)$ is maximal among the $I(a)$ properly contained in A'.

In particular,

(3) $$c \notin A'.$$

Put $C = A'[c]$ in A and $\mathfrak{p} = I(c)$. By the maximality of $\mathfrak{p}$

(4) $\mathfrak{p}$ is a prime ideal of A'.

Moreover,

(5) $\mathfrak{p}$ forms an ideal of C.

For as C is integral over A' (cf. [AM], (5.1)), there is $\mathfrak{q} \in \mathrm{Spec}(C)$ such that $\mathfrak{q} \cap A' = \mathfrak{p}$ (cf. [AM], (5.10)). Then $\mathfrak{p}c \subseteq \mathfrak{p}C \cap A' \subseteq \mathfrak{q} \cap A' = \mathfrak{p}$.

We have two cases to consider, either $\sqrt{\mathfrak{p}} = \mathfrak{p}$ in C or otherwise. Suppose first $\sqrt{\mathfrak{p}} \neq \mathfrak{p}$. Then

(6) $$\exists \text{ homogeneous } s \in \sqrt{\mathfrak{p}} \setminus \mathfrak{p} \text{ and } n \in \mathbf{N}^+ : s^{p^n} \in \mathfrak{p}\ .$$

As $\mathfrak{p}$ is prime in A', $s \notin A'$, hence the assertion.

Suppose next $\sqrt{\mathfrak{p}} = \mathfrak{p}$. From a commutative diagram of the natural maps

(7) $$\begin{array}{ccccc} C/\mathfrak{p} & \longleftarrow & C & \xrightarrow{i'} & A \\ \uparrow{\scriptstyle \bar{i}_c} & & \uparrow{\scriptstyle i_c} & \nearrow{\scriptstyle i} & \\ A'/\mathfrak{p} & \longleftarrow & A' & & \end{array}$$

one gets a commutative diagram

$$\begin{array}{ccccc} \mathrm{Max}(C/\mathfrak{p}) & \hookrightarrow & \mathrm{Max}(C) & \xleftarrow{\mathrm{Max}(i')} & \mathrm{Max}(A) \\ {\scriptstyle \mathrm{Max}(\overline{i_c})}\downarrow & & {\scriptstyle \mathrm{Max}(i_c)}\downarrow & \swarrow {\scriptstyle \mathrm{Max}(i)} & \\ \mathrm{Max}(A'/\mathfrak{p}) & \hookrightarrow & \mathrm{Max}(A') & & \end{array} \tag{8}$$

As A (resp. C) is a C-module (resp. an A'-module) of finite type, $\mathrm{Max}(i')$, $\mathrm{Max}(i_c)$, and $\mathrm{Max}(\overline{i_c})$, are all surjective (cf. [AM], (5.10)/(5.8)) and also closed (cf. [AM], Ex. 5.1). But $\mathrm{Max}(i)$ is bijective by the hypothesis, hence

(9) $\quad \mathrm{Max}(i')$, $\mathrm{Max}(i_c)$, and $\mathrm{Max}(\overline{i_c})$, are all homeomorphisms .

In particular, $\mathrm{Max}(C/\mathfrak{p})$ is irreducible, hence

(10) $\quad \mathfrak{p} = \sqrt{\mathfrak{p}}$ is a prime ideal of C .

Also (cf. [H], (4.6))

(11) the fractional field $\mathrm{Frac}(C/\mathfrak{p})$ of $C/\mathfrak{p}$ is finite
and purely inseparable over $\mathrm{Frac}(A'/\mathfrak{p})$.

Hence there is $r \in \mathbf{N}$, $a \in A'$, and $b \in A' \setminus \mathfrak{p}$ such that $bc^{p^r} = a \mod \mathfrak{p}$. Then $(bc)^{p^r} \in A'$. As $bc \notin A'$, the assertion follows.

Theorem 2.11 *Let $\lambda \in X^+$, $A = \amalg_{n\geq 0} H^0(n\lambda)$ graded K-algebra with the multiplication given by the cup product, and $A' = K[L(\lambda)]$ graded subalgebra of A generated by $L(\lambda) \leq H^0(\lambda) = A_1$.*

(i) *A is an A'-module of finite type.*

(ii) *If $\overline{K}$ is an algebraic closure of K, the inclusion $A' \hookrightarrow A$ induces homeomorphisms*

$$\mathrm{Proj}(A)(\overline{K}) \to \mathrm{Proj}(A')(\overline{K}) \quad \textit{and} \quad \mathrm{Max}(A \otimes_K \overline{K}) \to \mathrm{Max}(A' \otimes_K \overline{K}).$$

(iii) *If A'' is an intermediate graded algebra between A and A' with $A'' < A$, then there is a homogeneous element $a \in A \setminus A''$ such that $a^p \in A''$.*

Proof. By considering $G \times_K \overline{K}$, $H^0(n\lambda) \otimes_K \overline{K} \simeq \mathrm{ind}_{B\times_K\overline{K}}^{G\times_K\overline{K}}(n\lambda \otimes_K \overline{K})$, and $L(\lambda) \otimes_K \overline{K}$ the simple socle of $H^0(n\lambda) \otimes_K \overline{K}$ instead of G, $H^0(n\lambda)$, and

$L(\lambda)$, respectively, we may replace K by $\overline{K}$, so in the rest of the proof we assume $K = \overline{K}$.

Let $f_- \in (H^0(\lambda)^*)_{-\lambda} \setminus 0$ and $[f_-]$ the line through f_- in $\mathbf{P}(H^0(\lambda)^*)(K)$. If $i : L(\lambda) \to H^0(\lambda)$ is the inclusion, then

(1) $$\ker(i^*) \cap G(K)f_- = \emptyset \quad \text{as } L(\lambda)_\lambda = H^0(\lambda)_\lambda.$$

Hence (cf. [Mi], p.154) $G(K)[f_-] \subseteq \mathrm{D}_+(S_K^+(L(\lambda)))(K)$ with $\mathrm{Proj}(S_K(i)) : \mathrm{D}_+(S_K^+(L(\lambda))) \to \mathrm{Proj}(S_K(L(\lambda)))$ inducing a morphism $\theta : G(K)[f_-] \to G(K)[f_- \circ i]$ via

(2) $$x[f_-] \longmapsto x[f_- \circ i], \quad x \in G(K),$$

where $S_K(L(\lambda))$ is the symmetric algebra of $L(\lambda)$ over K, $S_K^+(L(\lambda)) = \amalg_{n>0} S_K(L(\lambda))_n$ the irrelevant ideal of $S_K(L(\lambda))$, and

$$\mathrm{D}_+(S_K^+(L(\lambda))) = \{\mathfrak{p} \in \mathrm{Proj}(S_K(H^0(\lambda))) \mid \mathfrak{p} \not\supseteq S_K^+(L(\lambda))\}.$$

Let P be the parabolic subgroup of G with $P(K) = C_{G(K)}([f_-])$. One has an isomorphism (cf. [J], (II.14.19))

(3) $$(G/P)(K) \longrightarrow G(K)[f_-] \quad \text{via} \quad x \longmapsto x[f_-], \quad x \in G(K)$$

that gives a closed imbedding of G/P into $\mathbf{P}(H^0(\lambda)^*)$ such that

(4) $\mathcal{L}_{G/P}(\lambda)$ is the associated very ample sheaf on G/P.

In particular,

(5) $G(K)[f_-]$ is an irreducible closed subvariety of $\mathbf{P}(H^0(\lambda)^*)(K)$,

hence also (cf. [H], (21.1)(b))

(6) $G(K)[f_- \circ i] = \operatorname{im}\theta$ is an irreducible closed subvariety of $\mathbf{P}(L(\lambda)^*)(K)$.

Moreover,

(7) θ is a homeomorphism.

To see that, as θ is closed (cf. [H], (21.1)(a)/(b)), it is enough to show that θ is injective. For that we claim a stronger statement that

(8) $$i^* \mid_{KG(K)f_-} : KG(K)f_- \longrightarrow L(\lambda)^* \text{ is injective,}$$

inducing a bijection $\theta' : KG(K)f_- \to KG(K)(f_- \circ i)$.

Suppose $f_- \circ i = \theta'(f_-) = \theta'(\xi x f_-) = \xi x(f_- \circ i)$, $\xi \in K$ and $x \in G(K)$. If W_P is the Weyl group of P, one can write (cf. [S], (10.3.3)(5))

$$(9) \qquad x = uwy, \quad u \in U(K), w \in W/W_P, y \in P(K).$$

Let $v_+ \in L(\lambda)_\lambda \setminus 0$ with $f_-(v_+) = (f_- \circ i)(v_+) = 1$. Then

$$\begin{aligned}(10) \qquad 1 &= f_-(uv_+) \qquad \text{(cf. [J], (II.1.19)(6))} \\ &= (f_- \circ i)(uv_+) \qquad \text{as } uv_+ \in L(\lambda) \\ &= (\xi x(f_- \circ i))(uv_+) = (\xi uwyf_-)(uv_+) = \xi(wyf_-)(v_+) \\ &= \xi\xi'(wf_-)(v_+) \quad \text{for some } \xi' \in K \text{ as } P(K) \text{ fixes } [f_-] \\ &= \xi\xi' f_-(w^{-1}v_+),\end{aligned}$$

hence $w \in W_P$, so we may assume $w = 1$. Then $\xi\xi' = 1$, hence

$$\begin{aligned}(11) \qquad \xi x f_- &= \xi u y f_- = \xi\xi' u f_- = u f_- \\ &= f_- \quad \text{as } f_- \text{ has the lowest weight of } H^0(\lambda)^*,\end{aligned}$$

as desired.

Let $\mathfrak{I} \trianglelefteq S_K(H^0(\lambda))$ be the homogeneous radical ideal defining $G(K)[f_-]$, so (cf. [Mi], p. 152)

$$\begin{aligned}(12) \quad G(K)[f_-] &= \mathrm{V}_+(\mathfrak{I})(K) := \{\mathfrak{p} \in \mathrm{Proj}(S_K(H^0(\lambda)))(K) \mid \mathfrak{p} \supseteq \mathfrak{I}\} \\ &\simeq \mathrm{Proj}(S_K(H^0(\lambda))/\mathfrak{I})(K).\end{aligned}$$

As G/P is integral, $\mathfrak{I}$ is prime, hence (*cf.* [Mi], Theorem II.3.4)

$$(13) \qquad S_K(H^0(\lambda))/\mathfrak{I} \text{ is a domain.}$$

If $\mathfrak{I}' = S_K(L(\lambda)) \cap \mathfrak{I}$, then

$$(14) \qquad G(K)[f_- \circ i] = \mathrm{V}_+(\mathfrak{I}')(K) \simeq \mathrm{Proj}(S_K(L(\lambda))/\mathfrak{I}')(K).$$

Indeed, $G(K)[f_- \circ i] \subseteq \mathrm{V}_+(\mathfrak{I}')(K)$ as $\theta : \mathfrak{p} \mapsto \mathfrak{p} \cap S_K(L(\lambda))$. On the other hand, as $G(K)[f_- \circ i]$ is closed, there is a homogeneous radical ideal $\mathfrak{I}''$ of $S_K(L(\lambda))$ such that $G(K)[f_- \circ i] = \mathrm{V}_+(\mathfrak{I}'')(K)$. By the projective Nullstellensatz (cf. [F], p. 91)

$$(15) \qquad \mathfrak{I}'' = \mathrm{I}_+(\mathrm{V}_+(\mathfrak{I}'')(K)) \supseteq \mathrm{I}_+(\mathrm{V}_+(\mathfrak{I}')(K)) = \mathfrak{I}'.$$

But in $\mathbf{P}(H^0(\lambda)^*)(K)$

(16) $$\mathrm{V}_+(\mathfrak{J}'')(K) \supseteq G(K)[f_-] = \mathrm{V}_+(\mathfrak{J})(K),$$

hence by the projective Nullstellensatz

(17) $$\sqrt{\mathfrak{J}''} \subseteq \mathfrak{J} \quad \text{in} \quad S_K(H^0(\lambda)).$$

Consequently, $\mathfrak{J}'' \subseteq \sqrt{\mathfrak{J}''} \cap S_K(L(\lambda)) \subseteq \mathfrak{J} \cap S_K(L(\lambda)) = \mathfrak{J}'$, hence $\mathfrak{J}'' = \mathfrak{J}'$, as desired.

On the other hand, by (4)

(18) $$A \simeq \coprod_{n \geq 0} \Gamma(G(K)[f_-], \mathcal{O}(n)),$$

hence one has by (12) and (13) a natural injection (cf. [Mi], Theorem II.3.9)

(19) $$S_K(H^0(\lambda))/\mathfrak{J} \longrightarrow A \quad \text{in} \quad \mathbf{Alg}_K.$$

Moreover (cf. [J], (II.14.20)),

(20) $$A = K[A_1],$$

hence (19) is bijective. Then as both A' and $S_K(L(\lambda))/\mathfrak{J}'$ are generated by $L(\lambda)$ in $A \simeq S_K(H^0(\lambda))/\mathfrak{J}$, one gets an isomorphism

(21) $$S_K(L(\lambda))/\mathfrak{J}' \longrightarrow A'.$$

$$\begin{array}{ccccc}
A & & \hookleftarrow & & A' \\
\uparrow\wr & \nwarrow & & \nearrow & \uparrow\wr \\
& S_K(H^0(\lambda)) & \longleftarrow & S_K(L(\lambda)) & \\
& \swarrow & & \searrow & \\
S_K(H^0(\lambda))/\mathfrak{J} & & \hookleftarrow & & S_K(L(\lambda))/\mathfrak{J}'
\end{array}$$

Hence the inclusion $j : A' \hookrightarrow A$ induces a homeomorphism

(22) $$\mathrm{Proj}(j)(K) : \mathrm{Proj}(A)(K) \longrightarrow \mathrm{Proj}(A')(K).$$

$$\begin{array}{ccc}
\mathrm{Proj}(A)(K) & \xrightarrow{\mathrm{Proj}(j)(K)} & \mathrm{Proj}(A')(K) \\
\wr\| & & \|\wr \\
G(K)[f_-] & \xrightarrow[\theta]{} & G(K)[f_- \circ i]
\end{array}$$

Next if $r = \dim H^0(\lambda)^*$, the cone (cf. [F], p. 90) of $G(K)[f_-]$ is

$$\text{(23)}\qquad \begin{aligned} \mathrm{C}(G(K)[f_-]) &:= \{z \in \mathbf{A}^r \mid [z] \in G(K)[f_-] \quad \text{or} \quad z = 0\} \\ &= KG(K)f_- \end{aligned}$$

By the projective Nullstellensatz

$$\text{(24)}\qquad KG(K)f_- = \mathrm{V}_a(\mathfrak{I})(K) := \{\mathfrak{p} \in \mathbf{A}^r(K) \mid \mathfrak{p} \supseteq \mathfrak{I}\}$$

and

$$\text{(25)}\quad \mathfrak{I} = \mathrm{I}_a(KG(K)f_-) \quad \text{the annihilator of } KG(K)f_- \quad \text{in} \quad K[\mathbf{A}^r] \simeq S_K(H^0(\lambda)).$$

Likewise if $r' = \dim L(\lambda)^*$,

$$\text{(26)}\quad \mathrm{C}(G(K)[f_- \circ i]) = KG(K)(f_- \circ i) = \mathrm{V}_a(\mathfrak{I}')(K) \quad \text{in} \quad \mathbf{A}^{r'}(K)$$

and

$$\text{(27)}\qquad \mathfrak{I}' = \mathrm{I}_a(KG(K)(f_- \circ i)) \quad \text{in} \quad K[\mathbf{A}^{r'}] \simeq S_K(L(\lambda)).$$

Hence the inclusion $j : A' \hookrightarrow A$ induces

$$\text{(28)}\qquad \theta' = \mathrm{C}(\theta) : KG(K)f_- \longrightarrow KG(K)(f_- \circ i).$$

$$\begin{array}{ccc} \mathrm{Max}(A) & \xrightarrow{\mathrm{Max}(j)} & \mathrm{Max}(A') \\ \wr\| & & \|\wr \\ \mathrm{Max}(S_K(H^0(\lambda))/\mathfrak{I}) & \circlearrowright & \mathrm{Max}(S_K(L(\lambda))/\mathfrak{I}') \\ \| & & \| \\ KG(K)f_- & \xrightarrow[\theta']{} & KG(K)(f_- \circ i) \end{array}$$

But θ' is bijective by (8), hence (cf. [S], (4.2.4) and [DG], (I.5.1.2))

(29) A is an A'-module of finite type.

Then (cf. [AM], Ex. 5.1)

(30) $\mathrm{Max}(j)$ is closed, hence a homeomorphism.

This concludes the proof of (i) and (ii).

Finally, if $j'' : A'' \hookrightarrow A$, then $\mathrm{Max}(j'') : \mathrm{Max}(A) \to \mathrm{Max}(A'')$ is a closed surjection as A is an A''-module of finite type by (i) (cf. [AM], Ex. 5.1 and (5.10)/(5.8)), hence is a homeomorphism as the injective $\mathrm{Max}(j)$ factors through $\mathrm{Max}(j'')$. Then (iii) follows from (2.10).

Corollary 2.12 *Let* $\nu \in X$ *and* $A \in \mathbf{grA}^\nu$ *with the* 0*-th homogeneous part* $A_0 = K$. *Assume* A *is commutative and reduced. If* $A < \mathcal{F}E(A)$ *in* (2.9), *then there is* $m \in \mathbf{N}^+$ *and* $a \in \mathcal{F}E(A)_m \setminus A_m$ *such that* $a^p \in A_{pm}$.

Proof. By (2.4)(ii)

(1) $$\mathcal{F}E(A)_0 = \mathcal{F}_0E(A_0) = \mathcal{F}_0E(K) = K = A_0.$$

If $A_r < \mathcal{F}E(A)_r$, then considering the subalgebra $K[A_r]$, we may assume $r = 1$.

As $A_1 \in \mathbf{M}^\nu$, by (2.7)(iv)

(2) $$\mathcal{F}E(A)_1 = \mathcal{F}_\nu E(A_1) \text{ is a direct sum of } H^0(w_0\nu),$$

hence there is a direct summand M of $\mathcal{F}E(A)_1$ with $M \simeq H^0(w_0\nu)$ in $G\mathbf{Mod}$ such that

(3) $$M \not\subseteq A_1.$$

Put $L = \mathrm{soc}_G M$, and let $\mathfrak{A} = K[M]$ (resp. $\mathfrak{A}' = K[L]$) be the subalgebra of $\mathcal{F}E(A)$ generated by M (resp. L).

If $\mathfrak{A}'' = \mathfrak{A} \cap A$, one has in $\mathbf{grA}^\nu$

(4) $$\mathfrak{A}' \leq \mathfrak{A}'' < \mathfrak{A} \leq \mathcal{F}E(A).$$

Recall from (2.9) the multiplication $\tilde{\gamma}$ on $\mathcal{F}E(A) = \coprod_{n\geq 0} H^0(A_n^{nw_0\nu})$. If γ is the multiplication on A, one has a commutative diagram

$$
(5) \qquad
\begin{array}{ccc}
\mathcal{F}E(A)_1 \otimes_K \mathcal{F}E(A)_1 & \xrightarrow{\tilde{\gamma}} & \mathcal{F}E(A)_2 \\
\| & & \| \\
H^0(A_1^{w_0\nu}) \otimes_K H^0(A_1^{w_0\nu}) & & H^0(A_2^{2w_0\nu}) \\
{\scriptstyle \varepsilon\otimes_K\varepsilon}\downarrow & & \downarrow{\scriptstyle \varepsilon} \\
A_1^{w_0\nu} \otimes_K A_1^{w_0\nu} & \dashrightarrow & A_2^{2w_0\nu} \\
\uparrow & & \uparrow \\
A_1 \otimes_K A_1 & \xrightarrow[\gamma]{} & A_2 .
\end{array}
$$

As A is reduced, $\tilde{\gamma}$ does not annihilate $\mathfrak{A}_{1,w_0\nu} \otimes_K \mathfrak{A}_{1,w_0\nu}$. Also as $\mathcal{F}E(A)_2 \in \mathbf{M}^{2\nu}$, $\tilde{\gamma}\,|_{\mathfrak{A}_1 \otimes_K \mathfrak{A}_1}$ induces an isomorphism in $G\mathbf{Mod}$

$$
(6) \qquad \operatorname{im} \tilde{\gamma}\,|_{\mathfrak{A}_1 \otimes_K \mathfrak{A}_1} \simeq \mathcal{F}^0_{2\nu}(\mathfrak{A}_1 \otimes_K \mathfrak{A}_1) \simeq H^0(2w_0\nu).
$$

Repeating the argument yields

$$
(7) \qquad \mathfrak{A} \simeq \coprod_{n \geq 0} H^0(nw_0\nu) \quad \text{in} \quad \mathbf{grA}^\nu
$$

with the multiplication on the RHS given by the cup product.

The assertion now follows from (2.11).

3 Frobenius splittings

Lemma 3.1 ([R], Remark. 1.3(i)) *Let $A \in \mathbf{Alg}_K$. If the map $A \longrightarrow A$ via $a \mapsto a^p$ admits a left inverse, then A is reduced. In particular, if A admits a Frobenius splitting, then A is reduced.*

Proof. Let ψ be a left inverse to the p-th power map on A. If a is a nilpotent of A, there is $r \in \mathbf{N}^+$ such that $a^{p^r} = 0$. Then

$$
0 = \psi(0^p) = \psi(0) = \psi(a^{p^r}) = a^{p^{r-1}}.
$$

Repeat to get $a = 0$.

(3.2) Let $G_1 = \ker F_G$ the Frobenius kernel of G, and $\pi \in \mathbf{Sch}_K(G, G/B)$, $q \in \mathbf{Sch}_K(G/B, G/G_1B)$ the quotient morphisms. The Frobenius morphism $F_{G/B} : G/B \to G/B$ factors through q to induce an isomorphism

$$F : G/G_1B \longrightarrow G/B. \tag{8}$$

We will write $\mathfrak{X}$ (resp. $\mathfrak{X}_1$) for G/B (resp. G/G_1B). Let $\mathcal{S} = \mathcal{L}_{\mathfrak{X}}((p-1)\rho)$ and $St = \mathcal{S}(\mathfrak{X}) = H^0((p-1)\rho) = L((p-1)\rho)$ the Steinberg module.

Let $v_- \in St_{-(p-1)\rho} = \{\mathbf{Sch}_K(G, (p-1)\rho)^B\}_{-(p-1)\rho}$ such that

$$v_-|_{w_0U^+} = 1 \qquad \text{(cf. [J], (II.2.6))}, \tag{9}$$

and recall from [K] the Frobenius splitting of G/B associated with v_-. Thus let $\overline{v}_- \in \mathbf{Mod}_{\mathfrak{X}}(\mathcal{O}_{\mathfrak{X}}, \mathcal{S})$ induced by v_- :

$$\overline{v}_-(\mathfrak{U}) : 1 \longmapsto v_-|_{\mathfrak{U}} \qquad \forall \mathfrak{U} \quad \text{open of } \mathfrak{X}. \tag{10}$$

One has an isomorphism (cf. [K], (1.3)(9) and (1.7))

$$\mathcal{O}_{\mathfrak{X}_1} \otimes_K St \longrightarrow q_*\mathcal{S} \quad \text{via} \quad a \otimes v \longmapsto \widetilde{a \otimes v}, \tag{11}$$

where $a \in \mathcal{O}_{\mathfrak{X}_1}(\mathfrak{V}) = \mathbf{Sch}_K(\pi_1^{-1}(\mathfrak{V}), K)^{G_1B}$ with $\mathfrak{V}$ an open of $\mathfrak{X}_1$, $\pi_1 = q \circ \pi$, $v \in St = \mathbf{Sch}_K(G, (p-1)\rho)^B$, and

(12) $\widetilde{a \otimes v} \in (q_*\mathcal{S})(\mathfrak{V}) = \mathbf{Sch}_K(\pi_1^{-1}(\mathfrak{V}), (p-1)\rho)^B$ such that

$$\widetilde{a \otimes v}(x) = a(A)(x)v(A)(x) \quad \forall x \in \pi_1^{-1}(\mathfrak{V})(A) \quad \text{with} \quad A \in \mathbf{Alg}_K.$$

If $f_+ \in (St^*)_{(p-1)\rho}$ is the dual element of v_-, then $\tilde{\sigma}$ defined by the following commutative diagram is the Frobenius splitting of G/B associated with v_-

$$\begin{array}{ccccc} q_*\mathcal{O}_{\mathfrak{X}} & \xrightarrow{\tilde{\sigma}} & \mathcal{O}_{\mathfrak{X}_1} & & af_+(v) \\ {\scriptstyle q_*\overline{v}_-}\downarrow & & \uparrow{\scriptstyle \mathcal{O}_{\mathfrak{X}_1} \otimes_K f_+} & & \uparrow \\ q_*\mathcal{S} & \xleftarrow{\sim} & \mathcal{O}_{\mathfrak{X}_1} \otimes_K St & & \\ \widetilde{a \otimes v} & & \longleftarrow\!\!\!\!\longleftarrow\!\!\!\!\dashv & & a \otimes v. \end{array} \tag{13}$$

The structure morphism $q^\natural : \mathcal{O}_{\mathfrak{X}_1} \to q_*\mathcal{O}_{\mathfrak{X}}$ induces on each open $\mathfrak{V}$ of $\mathfrak{X}_1$ an isomorphism

$$\mathcal{O}_{\mathfrak{X}_1}(\mathfrak{V}) \longrightarrow \{a^p \mid a \in (q_*\mathcal{O}_{\mathfrak{X}})(\mathfrak{V}) = \mathbf{Sch}_K(\pi_1^{-1}\mathfrak{V}, K)^B\}. \tag{14}$$

(3.3) Take $\mathfrak{B} = Bw_0B/G_1B = w_0U^+B/G_1B$ in (3.2), which we will denote by $\mathfrak{B}_0$. Then

$$(1) \qquad \begin{aligned} (q_*\mathcal{O}_{\mathfrak{X}})(\mathfrak{B}_0) &= \mathbf{Sch}_K(w_0U^+, K) \\ &\simeq K[y] := K[y_\alpha]_{\alpha\in R^+} \\ &\simeq \mathbf{Sch}_K(w_0U^+, (p-1)\rho) = (q_*\mathcal{S})(\mathfrak{B}_0), \end{aligned}$$

where $y_\alpha(A)(w_0 \prod_{\beta\in R^+} x_\beta(A)(a_\beta)) = a_\alpha \; \forall A \in \mathbf{Alg}_K$ and $a_\beta \in A$ with $x_\beta : \mathbf{G_a} \to U^+$ the root morphism associated with $\beta \in R^+$. Under this identification one has a commutative diagram

$$(2) \qquad \begin{array}{ccccc} (q_*\mathcal{O}_{\mathfrak{X}})(\mathfrak{B}_0) & \simeq & K[y] & & \\ \big\downarrow {\scriptstyle (q_*\overline{v}_-)(\mathfrak{B}_0)} & & \big\downarrow {\scriptstyle \mathrm{id}} & & \\ (q_*\mathcal{S})(\mathfrak{B}_0) & \simeq & K[y] & & a(v|_{w_0U^+}) \\ \big\uparrow \wr & & \big\uparrow & & \big\uparrow \\ (\mathcal{O}_{\mathfrak{X}_1} \otimes_K St)(\mathfrak{B}_0) & \simeq & K[y^p] \otimes_K St & & a \otimes v. \end{array}$$

Hence choosing a K-basis $(v_i)_i$ of St consisting of weight vectors including v_-, one can write

$$(3) \qquad K[y] = \coprod_{i=1}^{p^{|R^+|}} K[y^p]v_i \quad \text{with } v_- = 1 \text{ in } K[y].$$

Then for each i

$$(4) \qquad \tilde{\sigma}(\mathfrak{B}_0)|_{K[y^p]v_i} = \begin{cases} \mathrm{id}_{K[y^p]} & \text{if } v_i = v_-, \\ 0 & \text{otherwise.} \end{cases}$$

In particular,

$$(5) \qquad \tilde{\sigma}(\mathfrak{B}_0) \text{ is } T\text{-equivariant.}$$

(3.4) Let $\alpha \in S$ and arrange

$$(1) \qquad U^+ = U_\alpha \times \prod_{\beta\in R^+\setminus\{\alpha\}} U_\beta.$$

One can choose a representative of w_0 in $N_G(T)$ (cf. [J], (II.1.4)) such that

(2) $\exists c \in \{\pm 1\} : \forall a \in A$ with $A \in \mathbf{Alg}_K, x_{w_0\alpha}(A)(-a)w_0 = w_0 x_\alpha(A)(ca)$.

Hence if we write $K[U_{w_0\alpha}] = K[t]$, the $U_{w_0\alpha}$-module structure on

$$(q_*\mathcal{O}_{\mathfrak{X}})(\mathfrak{V}_0) = \mathbf{Sch}_K(w_0U^+, K) = K[y]$$

is given by the map

(3) $K[y] \to K[y,t] = K[y] \otimes_K K[U_{w_0\alpha}]$ such that

$$\prod_{\beta \in R^+} y_\beta^{n_\beta} \longmapsto (y_\alpha + ct)^{n_\alpha} \prod_{\beta \in R^+ \setminus \{\alpha\}} y_\beta^{n_\beta}, \quad n_\beta \in \mathbf{N}.$$

If we identify $\mathcal{O}_{\mathfrak{X}_1}(\mathfrak{V}_0)$ with $K[y^p] = K[y_\beta^p]_{\beta \in R^+}$ as in (3.2)(14), then

(4) the $U_{w_0\alpha}$-module structure on $\mathcal{O}_{\mathfrak{X}_1}(\mathfrak{V}_0)$ is given by (3) upon restriction.

Put $\delta_r = X_{w_0\alpha}^{(r)} \in \mathrm{Dist}(U_{w_0\alpha})$, $r \in \mathbf{N}$. Then

$$\delta_r(\prod_{\beta \in R^+} y_\beta^{n_\beta}) = c^r \binom{n_\alpha}{r} y_\alpha^{n_\alpha - r} \prod_{\beta \in R^+ \setminus \{\alpha\}} y_\beta^{n_\beta}, \tag{5}$$

hence one can write symbolically,

$$\delta_r = \frac{c^r}{r!}\frac{\partial^r}{\partial y_\alpha^r} \quad \text{on} \quad K[y] = \coprod_{i=1}^{p^{|R^+|}} K[y^p]v_i. \tag{6}$$

By the Leibniz rule

$$\delta_r(av_i) = \sum_{j=0}^{r} \delta_j(a)\delta_{r-j}(v_i) \quad \forall a \in K[y^p]. \tag{7}$$

Assume from now on that p divides r. As the weights of St are $\{(p-1)\rho - \Sigma_{\beta \in R^+} n_\beta \beta \mid 0 \leq n_\beta \leq p-1\}$,

$$\delta_{r-j}(v_i) \in (St\,|_{w_0U^+})_{\omega_i + (r-j)w_0\alpha} = 0 \quad \text{if } p \mid j \text{ and } j \neq r, \tag{8}$$

where ω_i is the weight of v_i. Hence in (7)

$$\delta_r(av_i) = \delta_r(a)\delta_0(v_i) = \delta_r(a)v_i \quad \text{with} \quad \delta_r(a) \in K[y^p]. \tag{9}$$

Then by (3.3)(4)

$$\tilde{\sigma}(\mathfrak{V}_0) \circ \delta_r(av_i) = \begin{cases} \delta_r(a) & \text{if } v_i = v_- \\ 0 & \text{otherwise} \end{cases} \tag{10}$$
$$= \delta_r \circ \tilde{\sigma}(\mathfrak{V}_0)(av_i).$$

Hence for any $r \in \mathbf{N}$ with $p \mid r$,

$$\tilde{\sigma}(\mathfrak{V}_0) \circ \delta_r = \delta_r \circ \tilde{\sigma}(\mathfrak{V}_0) \quad \text{on} \quad (q_*\mathcal{O}_{\mathfrak{X}})(\mathfrak{V}_0). \tag{11}$$

As $w_0 U^+ B$ is open dense in G, one has a commutative diagram

$$\begin{array}{ccc} (q_*\mathcal{O}_{\mathfrak{X}})(\mathfrak{X}_1) & \xrightarrow{\tilde{\sigma}(\mathfrak{X}_1)} & \mathcal{O}_{\mathfrak{X}_1}(\mathfrak{X}_1) \\ {\scriptstyle \text{res}}\downarrow & & \downarrow{\scriptstyle \text{res}} \\ (q_*\mathcal{O}_{\mathfrak{X}})(\mathfrak{V}_0) & \xrightarrow[\tilde{\sigma}(\mathfrak{V}_0)]{} & \mathcal{O}_{\mathfrak{X}_1}(\mathfrak{V}_0) \end{array} \tag{12}$$

with the $\mathrm{Dist}(U_{w_0\alpha})$-module structure on both $q_*\mathcal{O}_{\mathfrak{X}}$ and $\mathcal{O}_{\mathfrak{X}_1}$ compatible with the restrictions. Hence together with (3.3)(5) and (3.2)(14) one obtains

Lemma 3.5 *Let* $\mathfrak{V} \in \{Bw_0B/G_1B, G/B\}$.

(i) $\tilde{\sigma}(\mathfrak{V})$ *is* T*-equivariant.*

(ii) *If* $\alpha \in S$, *then for any* $r \in \mathbf{N}$ *with* $p \mid r$,

$$\tilde{\sigma}(\mathfrak{V}) \circ X_{-\alpha}^{(r)} = X_{-\alpha}^{(r)} \circ \tilde{\sigma}(\mathfrak{V}) \quad \textit{on} \quad (q_*\mathcal{O}_{G/B})(\mathfrak{V}).$$

(iii) $\forall a \in \mathcal{O}_{\mathfrak{X}_1}(\mathfrak{V}), \quad \tilde{\sigma}(\mathfrak{V})(a^p) = a^p.$

(3.6) Let $\lambda \in X$. Then

$$\begin{aligned} q_*\mathcal{L}_{\mathfrak{X}}(p\lambda) &\simeq q_*q^*\mathcal{L}_{\mathfrak{X}_1}(p\lambda) \quad \text{(cf. [CPS], (2.7), [K], (1.8))} \\ &\simeq \mathcal{L}_{\mathfrak{X}_1}(p\lambda) \otimes_{\mathcal{O}_{\mathfrak{X}_1}} q_*\mathcal{O}_{\mathfrak{X}} \quad \text{by the projection formula.} \end{aligned} \tag{1}$$

Under the identification define $\tilde{\sigma}_{p\lambda} \in \mathbf{Mod}_{\mathfrak{X}_1}(q_*\mathcal{L}_{\mathfrak{X}}(p\lambda), \mathcal{L}_{\mathfrak{X}_1}(p\lambda))$ by the commutative diagram

$$\begin{array}{ccc} q_*\mathcal{L}_{\mathfrak{X}}(p\lambda) & \xrightarrow{\tilde{\sigma}_{p\lambda}} & \mathcal{L}_{\mathfrak{X}_1}(p\lambda) \\ \wr\| & & \|\wr \\ \mathcal{L}_{\mathfrak{X}_1}(p\lambda) \otimes_{\mathcal{O}_{\mathfrak{X}_1}} q_*\mathcal{O}_{\mathfrak{X}} & \xrightarrow[\mathcal{L}_{\mathfrak{X}_1}(p\lambda)\otimes_{\mathcal{O}_{\mathfrak{X}_1}}\tilde{\sigma}]{} & \mathcal{L}_{\mathfrak{X}_1}(p\lambda) \otimes_{\mathcal{O}_{\mathfrak{X}_1}} \mathcal{O}_{\mathfrak{X}_1}. \end{array} \tag{2}$$

Recall the isomorphism $F \in \mathbf{Sch}_K(\mathfrak{X}_1, \mathfrak{X})$ from (3.2)(8). Generalizing (3.2)(14) one has an isomorphism

(3) $$(F^{-1})_*\mathcal{L}_{\mathfrak{X}}(\lambda) \longrightarrow \mathcal{L}_{\mathfrak{X}_1}(p\lambda) \quad \text{via} \quad a \longmapsto a^p.$$

Proposition 3.7 *Let $\lambda \in X$ and $\mathfrak{V} \in \{Bw_0B/G_1B, G/G_1B\}$.*

(i) *$\tilde{\sigma}_{p\lambda}(\mathfrak{V})$ is a T-equivariant surjection. In particular, $\tilde{\sigma}_{p\lambda}(\mathfrak{V})$ annihilates all ν-weight spaces with $\nu \in X \setminus pX$.*

(ii) *If $\alpha \in S$, then for any $r \in \mathbf{N}$ with $p \mid r$,*

$$\tilde{\sigma}_{p\lambda}(\mathfrak{V}) \circ X_{-\alpha}^{(r)} = X_{-\alpha}^{(r)} \circ \tilde{\sigma}_{p\lambda}(\mathfrak{V}) \quad on \quad (q_*\mathcal{L}_{G/B}(p\lambda))(\mathfrak{V}).$$

(iii) *$\forall a \in ((F^{-1})_*\mathcal{L}_{G/B}(\lambda))(\mathfrak{V}) = \mathcal{L}_{G/B}(\lambda)(q^{-1}\mathfrak{V})$,*

$$\tilde{\sigma}_{p\lambda}(\mathfrak{V})(a^p) = a^p.$$

Proof. As in (3.4)(12) we have only to check the assertions on $\mathfrak{V} = \mathfrak{V}_0 = Bw_0B/G_1B$.

(i) As $\tilde{\sigma}$ is a split epi, so is $\tilde{\sigma}_{p\lambda}$, hence the sujectivity. The second assertion follows from the identification $\mathcal{L}_{\mathfrak{X}_1}(p\lambda)(\mathfrak{V}_0) \simeq K[y_\beta^p]_{\beta \in R^+}$ with each y_β^p having weight $pw_0(\lambda - \beta)$.

(ii) and (iii) follow from (3.5).

(3.8) Assume $\lambda \in X^+$. Let $(m_j)_j$ be a K-basis of $H^0(\lambda)$, and let

(1) $$\Delta_{H^0(\lambda)} : m_j \longmapsto \sum_k m_k \otimes a_{jk}, \quad a_{jk} \in K[G]$$

be the G-module structure on $H^0(\lambda)$. If

$$H_1^0(p\lambda) = \mathcal{L}_{\mathfrak{X}_1}(p\lambda)(\mathfrak{X}_1) = \mathrm{ind}_{G_1B}^G(p\lambda),$$

then $(m_j^p)_j$ forms a K-basis of $H_1^0(p\lambda)$ by (3.6)(3) with the G-module structure given by

(2) $$\Delta_{H_1^0(p\lambda)} : m_j^p \longmapsto \sum_k m_k^p \otimes a_{jk}^p.$$

Define a K-linear bijection $[-1](\mathfrak{X}_1) : H^0_1(p\lambda) \to H^0(\lambda)$ via

(3) $$m_j^p \longmapsto m_j.$$

Define likewise a K-linear bijection $[-1](\mathfrak{V}_0) : \mathcal{L}_{\mathfrak{X}_1}(p\lambda)(\mathfrak{V}_0) \to (q_*\mathcal{L}_{\mathfrak{X}}(\lambda))(\mathfrak{V}_0)$ by the commutative diagram

(4) $$\begin{array}{ccc} \mathcal{L}_{\mathfrak{X}_1}(p\lambda)(\mathfrak{V}_0) & \xrightarrow{[-1](\mathfrak{V}_0)} & (q_*\mathcal{L}_{\mathfrak{X}}(\lambda))(\mathfrak{V}_0) \\ \| & & \| \\ \mathbf{Sch}_K(w_0U^+, p\lambda)^{U_1^+} & & \mathbf{Sch}_K(w_0U^+, \lambda) \\ \wr| & & |\wr \\ K[y_\beta^p]_{\beta\in R^+} & \longrightarrow & K[y_\beta]_{\beta\in R^+} \\ \prod y_\beta^{pn_\beta} & \longmapsto & \prod y_\beta^{n_\beta}. \end{array}$$

Set

(5) $$\sigma_{p\lambda}[-1] = [-1](\mathfrak{X}_1) \circ \tilde{\sigma}_{p\lambda}(\mathfrak{X}_1) \in \mathbf{Mod}_K(H^0(p\lambda), H^0(\lambda))$$

and

(6) $$\begin{aligned} \sigma^0_{p\lambda}[-1] &= [-1](\mathfrak{V}_0) \circ \tilde{\sigma}_{p\lambda}(\mathfrak{V}_0) \\ &\in \mathbf{Mod}_K(\mathcal{L}_{\mathfrak{X}}(p\lambda)(q^{-1}\mathfrak{V}_0), \mathcal{L}_{\mathfrak{X}}(\lambda)(q^{-1}\mathfrak{V}_0)). \end{aligned}$$

One has a commutative diagram in $\mathbf{Mod}_K$

(7) $$\begin{array}{ccc} H^0(p\lambda) & \xrightarrow{\sigma_{p\lambda}[-1]} & H^0(\lambda) \\ \text{res}\downarrow & & \downarrow\text{res} \\ \mathcal{L}_{\mathfrak{X}}(p\lambda)(q^{-1}\mathfrak{V}_0) & \xrightarrow[\sigma^0_{p\lambda}[-1]]{} & \mathcal{L}_{\mathfrak{X}}(\lambda)(q^{-1}\mathfrak{V}_0). \end{array}$$

From (3.7) one obtains

Corollary 3.9 *Let $\lambda \in X^+$ and $\alpha \in S$.*

(i) *Both $\sigma_{p\lambda}[-1]$ and $\sigma^0_{p\lambda}[-1]$ annihilate the ν-weight spaces for all $\nu \in X \setminus pX$ while both send each $p\eta$-weight space onto the η-weight space for $\eta \in X$.*

(ii) $\forall r \in \mathbf{N}$, *one has*

$$X_{-\alpha}^{(r)} \circ \sigma_{p\lambda}[-1] = \sigma_{p\lambda}[-1] \circ X_{-\alpha}^{(pr)} \quad \text{and} \quad X_{-\alpha}^{(r)} \circ \sigma_{p\lambda}^{0}[-1] = \sigma_{p\lambda}^{0}[-1] \circ X_{-\alpha}^{(pr)}.$$

(iii) $\forall m \in H^0(\lambda)$ *and* $v \in \mathcal{L}_{G/B}(\lambda)(Bw_0B/B)$,

$$\sigma_{p\lambda}[-1](m^p) = m \quad \text{and} \quad \sigma_{p\lambda}^{0}[-1](v^p) = v.$$

4 Proof of the conjecture

Lemma 4.1 *Let* $\lambda \in X^+$ *and recall from* (2.9)(4) *an injection* $h_\lambda \in B\mathbf{Mod}(H^0(\lambda), H_B^0(w_0\lambda))$. *There is an isomorphism*

$$\psi \in B\mathbf{Mod}(\mathcal{L}_{G/B}(\lambda)(Bw_0B/B), H_B^0(w_0\lambda)) \quad \text{such that} \quad \psi \circ res = h_\lambda.$$

In particular, $\mathcal{L}_{G/B}(\lambda)(Bw_0B/B)$ *is injective in* $B\mathbf{Mod}$.

Proof. Put for simplicity $\mathfrak{V}_0^q = q^{-1}\mathfrak{V} = Bw_0B/B = w_0U^+B/B$. Define a K-linear isomorphism

$$\psi : \mathbf{Sch}_K(w_0U^+, \lambda) \longrightarrow \mathbf{Sch}_K(U, w_0\lambda) \quad \text{via} \quad f \longmapsto f(?w_0)$$

with inverse $g \mapsto g(?w_0^{-1})$. One then checks $\psi \in B\mathbf{Mod}$.

On the other hand, the Frobenius reciprocity yields

$$B\mathbf{Mod}(H^0(\lambda), H_B^0(w_0\lambda)) \simeq T\mathbf{Mod}(H^0(\lambda), w_0\lambda) \simeq K,$$

hence, or directly, the assertion follows.

(4.2) Let $\mathbf{grA}$ (resp. $\mathbf{grA}_B$) be the category of $\mathbf{N}$-graded K-algebras whose homogeneous parts are G- (resp. B-) modules with G- (resp. B-) equivariant multiplication.

Let $\lambda, \mu \in X^+$ and set

(8) $$\mathfrak{A} = \coprod_{m \geq 0} \mathfrak{A}_m \quad \text{with} \quad \mathfrak{A}_m = H^0(m\lambda) \otimes_K H^0(m\mu).$$

Under the cup product

(9) $$\mathfrak{A} \in \mathbf{grA}.$$

Define also a B-module

$$(10) \qquad \mathfrak{B}_m = H^0_B(mw_0\lambda) \otimes_K H^0(m\mu) \simeq \mathcal{L}_{\mathfrak{X}}(m\lambda)(\mathfrak{W}^q_0) \otimes_K H^0(m\mu) \quad \text{by (4.1)}.$$

If $\mathfrak{B} = \coprod_{m\geq 0} \mathfrak{B}_m$, the cup product makes

$$(11) \qquad \mathfrak{B} \in \mathbf{grA}_B.$$

Under the restriction from $\mathfrak{X}$ to $\mathfrak{W}^q_0$ on the first factor one has

$$(12) \qquad \mathfrak{A} \leq \mathfrak{B} \quad \text{in} \quad \mathbf{grA}_B.$$

In the notation of (3.8) define $\sigma \in \mathbf{Mod}_K(\mathfrak{B}, \mathfrak{B})$ by

$$(13) \qquad \sigma|_{\mathfrak{B}_m} = \begin{cases} \sigma^0_{m\lambda}[-1] \otimes_K \sigma_{m\mu}[-1] & \text{if } p \mid m \\ 0 & \text{otherwise.} \end{cases}$$

Upon restriction to $\mathfrak{A}$ one has

$$(14) \qquad \sigma|_{\mathfrak{A}_m} = \begin{cases} \sigma_{m\lambda}[-1] \otimes_K \sigma_{m\mu}[-1] & \text{if } p \mid m \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 4.3 *σ is a Frobenius splitting of $\mathfrak{B}$ that stabilizes $\mathfrak{A}$. In particular, $\mathfrak{A}$ and $\mathfrak{B}$ are both reduced.*

Proof. By (3.9)(iii) σ is a left inverse to the p-th power map on $\mathfrak{B}$, hence $\mathfrak{B}$ is reduced by Ramanathan's lemma (3.1). Moreover, by construction one sees for each $a \in H^0_B(rw_0\lambda)$ and $b \in H^0(ps\lambda)$ with $r, s \in \mathbf{N}$,

$$\sigma^0_{p(r+s)\lambda}[-1](a^p b) = a\sigma^0_{ps\lambda}[-1](b) \quad \text{in} \quad \mathfrak{B}_{r+s}.$$

Hence σ is a Frobenius splitting of $\mathfrak{B}$.

(4.4) Let $\nu \in X$, and set

$$(1) \qquad \mathfrak{A}(\nu) = \coprod_{m\geq 0} \mathcal{F}_{m\nu}(\mathfrak{A}_m) \quad \text{and} \quad \mathfrak{A}^+(\nu) = \coprod_{m\geq 0} \mathcal{F}^+_{m\nu}(\mathfrak{A}_m).$$

By (2.2)(iii)

$$(2) \qquad \mathfrak{A}(\nu) \leq \mathfrak{A} \quad \text{in} \quad \mathbf{grA} \quad \text{with} \quad \mathfrak{A}^+(\nu) \trianglelefteq \mathfrak{A}(\nu).$$

If $\mathfrak{A}^0(\nu) = \mathfrak{A}(\nu)/\mathfrak{A}^+(\nu) = \coprod_{m\geq 0}\mathcal{F}^0_{m\nu}(\mathfrak{A}_m)$, then

(3) $$\mathfrak{A}^0(\nu) \in \mathbf{grA}^\nu.$$

Define likewise

(4) $$\mathfrak{B}(\nu) = \coprod_{m\geq 0} \mathcal{F}_{m\nu}(\mathfrak{B}_m), \quad \mathfrak{B}^+(\nu) = \coprod_{m\geq 0} \mathcal{F}^+_{m\nu}(\mathfrak{B}_m),$$

and set

(5) $$\mathfrak{B}^0(\nu) = \mathfrak{B}(\nu)/\mathfrak{B}^+(\nu) = \coprod_{m\geq 0} \mathcal{F}^0_{m\nu}(\mathfrak{B}_m) \in \mathbf{grA}^\nu_B.$$

Then

(6) $$\mathfrak{A}^0(\nu) \leq \mathfrak{B}^0(\nu) \quad \text{in} \quad \mathbf{grA}^\nu_B.$$

Also by (4.1) and (2.3)(4)

(7) $$\mathfrak{B}^0(\nu) \text{ is injective in } B\mathbf{Mod}.$$

Proposition 4.5 *The Frobenius splitting σ of $\mathfrak{B}$ stabilizes $\mathfrak{B}(\nu)$, $\mathfrak{A}(\nu)$, $\mathfrak{B}^+(\nu)$, and $\mathfrak{A}^+(\nu)$ for all $\nu \in X$. Hence σ induces a Frobenius splitting $\sigma^0(\nu)$ of $\mathfrak{B}^0(\nu)$ that stabilizes $\mathfrak{A}^0(\nu)$.*

Proof. We will show σ stabilizes $\mathfrak{B}(\nu)$. The rest follows likewise.

As σ vanishes on $\mathfrak{B}_m$ if $p \nmid m$, it is enough to show

(1) $$\sigma(\mathcal{F}_{pm\nu}\mathfrak{B}_{pm}) \subseteq \mathcal{F}_{m\nu}(\mathfrak{B}_m) \quad \forall m \in \mathbf{N}.$$

By (3.9)(i) one has for all η and $\eta' \in X$

(2) $$\sigma(H^0_B(pmw_0\lambda)_\eta \otimes_K H^0(pm\mu)_{\eta'}) = \begin{cases} H^0_B(mw_0\lambda)_{\frac{1}{p}\eta} \otimes_K H^0(m\mu)_{\frac{1}{p}\eta'} & \text{if } \eta, \eta' \in pX \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, for each $\alpha \in S$, $r \in \mathbf{N}$, and $x \in H^0_B(pmw_0\lambda)_{p\eta}$, $y \in H^0(pm\mu)_{p\eta'}$ with $\eta, \eta' \in X$ one has

$$(3)\qquad X_{-\alpha}^{(r)} \cdot \sigma(x \otimes y) = \sum_{i=0}^{r} (X_{-\alpha}^{(i)} \otimes X_{-\alpha}^{(r-i)}) \sigma(x \otimes y) \quad \text{(cf. (1.3)(7))}$$

$$= \sigma(\sum_{i=0}^{r} (X_{-\alpha}^{(pi)} \otimes X_{-\alpha}^{(p(r-i))})(x \otimes y)) \quad \text{by (3.9)(ii)}$$

$$= \sigma(\sum_{i=0}^{pr} (X_{-\alpha}^{(i)} \otimes X_{-\alpha}^{(pr-i)})(x \otimes y)) \quad \text{by (2)}$$

$$= \sigma(X_{-\alpha}^{(pr)} \cdot (x \otimes y))$$

$$\in \sigma(\mathcal{F}_{pm\nu} \mathfrak{B}_{pm}).$$

Hence $\sigma(\mathcal{F}_{pm\nu}\mathfrak{B}_{pm}) \leq \mathfrak{B}_m$ in $\mathrm{Dist}(U_{-\alpha})$**Mod** for each $\alpha \in S$, so $\sigma(\mathcal{F}_{pm\nu}\mathfrak{B}_{pm}) \leq \mathfrak{B}_m$ in $U_{-\alpha}$**Mod** (cf. (1.3)(5)). Consequently, together with (2) one gets

$$(4)\qquad \sigma(\mathcal{F}_{pm\nu}\mathfrak{B}_{pm}) \leq \mathfrak{B}_m \quad \text{in} \quad B\mathbf{Mod},$$

hence by (2) again

$$(5)\qquad \sigma(\mathcal{F}_{pm\nu}\mathfrak{B}_{pm}) \leq \mathcal{F}_{m\nu}\mathfrak{B}_m,$$

as desired.

Theorem 4.6 *Let $\lambda, \mu \in X^+$. The G-module $H^0(\lambda) \otimes_K H^0(\mu)$ admits a good filtration.*

Proof. Put $M = H^0(\lambda) \otimes_K H^0(\mu)$. Consider the G-filtration of M by $\mathcal{F}_\nu^0(M)$, $\nu \in X$ (2.2)(iii). It suffices by (2.7)(iv) to show

$$(1)\qquad \mathcal{F}_\nu^0(M) = \mathcal{F}_\nu E(\mathcal{F}_\nu^0(M)) \quad \forall \nu \in X.$$

For that, as $\mathcal{F}_\nu^0(M) = \mathfrak{A}^0(\nu)_1$, it will be enough to show more generally

$$(2)\qquad \mathfrak{A}^0(\nu) = \mathcal{F}E(\mathfrak{A}^0(\nu)).$$

One has in $\mathbf{grA}_B^\nu$

$$(3)\qquad \mathfrak{A}^0(\nu) \leq \mathcal{F}E(\mathfrak{A}^0(\nu)) \leq E_B(\mathfrak{A}^0(\nu)) \quad \text{by (2.9)}$$

$$\leq E_B(\mathfrak{B}^0(\nu)) \quad \text{as } \mathfrak{A}^0(\nu) \leq \mathfrak{B}^0(\nu) \text{ in } \mathbf{grA}_B^\nu$$

$$= \mathfrak{B}^0(\nu) \quad \text{as } \mathfrak{B}^0(\nu) \text{ is already injective by (4.4)(7).}$$

As $\mathfrak{A}^0(\nu)$ is reduced by (4.5), if (2) failed, then (2.12) would imply

$$\exists m \in \mathbf{N}^+ \text{ and } a \in \mathcal{F}E(\mathfrak{A}^0(\nu))_m \setminus \mathfrak{A}^0(\nu)_m : a^p \in \mathfrak{A}^0(\nu)_{pm}. \tag{4}$$

Then by (4.5) one would have

$$a = \sigma^0(\nu)(a^p) \in \sigma^0(\nu)(\mathfrak{A}^0(\nu)_{pm}) = \mathfrak{A}^0(\nu)_m, \tag{5}$$

absurd.

References

[AM] Atiyah, M.F. and Macdonald, I.G., *Introduction to Commutative Algebra*, Reading 1969 (Addison-Wesley)

[CPS] Cline, E., Parshall, B. and Scott, L., *A Mackey imprimitivity theory for algebraic groups*, Math. Z. **182** (1983), 447–471

[DG] Demazure, M. and Gabriel, P., *Introduction to Algebraic Geometry and Algebraic Groups*, Amsterdam 1980 (North-Holland)

[D1] Donkin, S., *Rational Representations of Algebraic Groups*, LNM **1140** Berlin (Springer-Verlag)

[D2] Donkin, S., *Good filtrations of rational modules for reductive groups*, Proc. Symp. Pure Math. **47**, 69-80 Arcata 1987 (AMS)

[F] Fulton, W., *Algebraic Curves*, Reading 1981 (Benjamin)

[H] Humphreys, J.E., *Linear Algebraic Groups*, GTM **21**, New York 1975 (Springer-Verlag)

[J] Jantzen, J.C., *Representations of algebraic groups*, Orlando 1987 (Academic Press)

[K] Kaneda M., *The Frobenius morphism of Schubert schemes*, J. Alg. (to appear)

[M1] Mathieu, O., *Construction d'un groupe de Kac-Moody et applications*, Comp. Math. **69** (1989), 37-60

[M2] Mathieu, O., *Filtrations of G-modules*, Ann. Sci. ENS **23** (1990), 625-644

[MR] Mehta, V.B. and Ramanathan, A., *Frobenius splitting and cohomology vanishing for Schubert varieties*, Ann. Math. **122** (1985), 27–40

[Mi] Miyanishi M., Daisuukikagaku, Tokyo 1990 (Shoukabou)

[P] Paradowski, J., *Filtrations of modules over the quantum algebra*, (to appear)

[R] Ramanathan, A., *Equations defining Schubert varieties and Frobenius splitting of diagonals*, IHES Pub. Math. **65** (1987), 61–90

[RR] Ramanan, S. and Ramanathan, A., *Projective normality of flag varieties and Schubert varieties*, Inv. Math. **79** (1985), 217–224

[S] Springer, T.A., *Linear Algebraic Groups*, PM **9**, Boston 1981 (Birkhäuser)

[vdK] van der Kallen, W., *Frobenius Splittings and B-modules*, Tata Inst. Fund. Res. **84**, Berlin 1993 (Springer-Verlag)

[W] Wang J.-p., Sheaf cohomology of G/B and tensor products of Weyl modules, J. Alg. **77** (1982), 162-185

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