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ON SOME CIRCLES IN PSEUDO-RIEMANNIAN MANIFOLDS

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§1. Introduction.

Let $\widetilde{M}$ be a Riemannian manifold. A totally umbilical submanifold M of $\widetilde{M}$ with parallel mean curvature vector field is said to be an *extrinsic sphere* [2][1]).

One-dimensional extrinsic spheres are the curves c to be called *circles*, which were considered under the name of geodesic circles or curvature circles characterized by the following differential equations

$$\nabla_X \nabla_X X + \langle \nabla_X X, \nabla_X X \rangle X = 0,$$

where $\langle , \rangle$ is the metric, ∇ is covariant differentiation along c and X is the unit tangent vector field of c. For a circle c, $k := \langle \nabla_X X, \nabla_X X \rangle^{\frac{1}{2}}$ is a non-negative constant which is called the *curvature* of c. Especially $k = 0$, a circle c is a *geodesic*. The following theorems are well-known:

Theorem A([2]). *Let M (dim $M \geq 2$) be a connected Riemannian submanifold of a Riemannian manifold $\widetilde{M}$. For some $k > 0$, the following conditions are equivalent:*
(1) Every circle of radius k in M is a circle in $\widetilde{M}$,
(2) M is an extrinsic sphere in $\widetilde{M}$.

On the other hand, if the development of $c(s)$ in the tangent Möbius space is a circle, then $c(s)$ is called a *conformal circle* (cf. [1], [3]). Then the equation of the conformal circle is given by

$$\nabla_X \nabla_X X + \left(\langle \nabla_X X, \nabla_X X \rangle + \frac{1}{n-2} \langle SX, X \rangle \right) X - \frac{1}{n-2} SX = 0, \tag{1.1}$$

where S is the Ricci operator of M (dim $M = n \geq 3$). Remark that (1.1) is represented by the Riemannian metric and the Riemannian connection. Also they showed in [1] that, when every circle in M is a conformal circle in $\widetilde{M}$, M is totally umbilical in $\widetilde{M}$.

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1) Numbers in brackets refer to the references at the end of the paper.

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In this paper , we will consider some similar theorems by the *Q-circle* with respect to a tensor field Q of type (1,1) and conformal circle on a pseudo-Riemannian manifold.

§2. Preliminaries.

First of all, we recall the general theory of pseudo-Riemannian submanifolds immersed into a pseudo-Riemannian manifold to fix our notations. Let M be an n-dimensional pseudo-Riemannian manifold isometrically immersed into an m-dimensional pseudo-Riemannian manifolld $\widetilde{M}$. Then M is called a *pseudo-Riemannian submanifold* of $\widetilde{M}$. By $\langle , \rangle$, we mean the metric tensor field of $\widetilde{M}$ as well as the metric induced on M. A non-zero vector x of M is said to be *null* if $\langle x, x\rangle=0$ and *unit* if $\langle x, x\rangle=+1$ or -1. We denote by $\widetilde{\nabla}$ the covariant differentiation of $\widetilde{M}$ and by ∇ the covariant differentiation of M determined by the induced metric on M. Then we have Gauss' formula

$$\widetilde{\nabla}_X Y = \nabla_X Y + B(X, Y), \tag{2.1}$$

where X and Y are vector fields tangent to M and B is the second fundamental form of M. Weingarten's formula is

$$\widetilde{\nabla}_X \xi = -A_\xi X + \nabla^\perp_X \xi, \tag{2.2}$$

where X (resp. ξ) is a vector field tangent (resp. normal) to M and $\nabla^\perp$ is the covariant differentiation with respect to the induced connection in the normal bundle of M in $\widetilde{M}$ and A_ξ is the shape operator of M. We have the relation

$$\langle A_\xi X, Y\rangle = \langle B(X, Y), \xi\rangle.$$

For the second fundamental form B, we define an normal bundle-valued tensor field $\overline{\nabla} B$ as

$$(\overline{\nabla} B)(Y, Z, X) = \nabla^\perp_X B(Y, Z) - B(\nabla_X Y, Z) - B(Y, \nabla_X Z), \tag{2.3}$$

where X, Y and Z are tangent vector fields of M. The mean curvature vector field H of M is defined by

$$H := \frac{1}{n}\sum_{i=1}^{n}\langle e_i, e_i\rangle B(e_i, e_i),$$

where $\{ e_1, \cdots, e_n \}$ is an orthonormal frame at each point of M. H is said to be *parallel* if $\nabla^\perp_X H = 0$ holds for any tangent vector field X of M. If the second fundamental form B satisfies

$$B(X, Y) = \langle X, Y\rangle H,$$

for any tangent vector fields X and Y of M, then M is said to be *totally umbilical submanifold* of $\widetilde{M}$. A totally umbilical submanifold with parallel mean curvature vector field is called an *extrinsic sphere.*

§3. Circles in pseudo-Riemannian manifolds.

Let M be a n-dimensional pseudo-Riemannian manifold. A regular curve $c = c(s)$ is said to be a *unit speed curve* in M when $\langle X, X\rangle = +1$ or -1 for the tangent vector field $X = c'(s)$. A *circle* of M as a unit speed curve is defined by the differential equation

$$\nabla_X\nabla_X X + \langle\nabla_X X, \nabla_X X\rangle\langle X, X\rangle X = 0,$$

where ∇_X is the covariant derivative along c. On the other hand, a *conformal circle* of M is defined by

$$\nabla_X\nabla_X X + \Big(\langle\nabla_X X, \nabla_X X\rangle\langle X, X\rangle + \frac{1}{n-2}\langle SX, X\rangle\Big)X - \frac{1}{n-2}\langle X, X\rangle SX = 0,$$

where S is the Ricci operator of M.

Let Q be an arbitrary tensor field of type (1,1) on M. We call $c(s)$ a *Q-circle* if the unit tangent vector field X of $c(s)$ satisfies

$$\nabla_X\nabla_X X + \Big(\langle\nabla_X X, \nabla_X X\rangle\langle X, X\rangle + \langle QX, X\rangle\Big)X - \langle X, X\rangle QX = 0.$$

Concerning ordinary differential equations on M, we have the following lemma:

Lemma 3.1. *Let p be a point of M and x, $y \in T_pM$ be orthogonal such that $\langle x, x\rangle = \epsilon = +1$ or -1. Then there exists a real number $r > 0$ and a unique solution σ, X, Y of the following differential equations:*

$$\begin{aligned}
&\frac{d\sigma}{dt} = X,\\
&\nabla_X X = Y,\\
&\nabla_X Y = (-\epsilon\langle Y, Y\rangle - \langle QX, X\rangle)X + \langle X, X\rangle QX \qquad \text{on } (-r, r),\\
&\sigma(0) = p, X(0) = x, Y(0) = y.
\end{aligned}$$

Moreover σ is a unit speed curve.

Proof. From the theory of ordinary differential equatons, it follows that there exists a real number $r > 0$ and a unique regular curve σ such that the above differential equations has a unique solution $\sigma(t), X(t), Y(t)$ on $(-r, r)$ with the initial conditions $\sigma(0) = p, X(0) = x, Y(0) = y$. Put

$$\begin{aligned}
\lambda(t) &:= \langle X(t), X(t)\rangle - \epsilon,\\
\mu(t) &:= \langle X(t), Y(t)\rangle.
\end{aligned}$$

Therefore, we have

$$\frac{d\lambda}{dt} = \frac{d}{dt}\langle X, X\rangle = 2\langle X, Y\rangle = 2\mu,$$
$$\frac{d\mu}{dt} = \frac{d}{dt}\langle X, Y\rangle = \langle Y, Y\rangle + \langle Y, \nabla_X Y\rangle = \langle Y, Y\rangle - \epsilon\langle Y, Y\rangle\langle X, X\rangle = -\epsilon\langle Y, Y\rangle\lambda,$$
$$\lambda(0) = \mu(0) = 0$$

Thus λ and μ satisfy the following linear homogeneous differential equations with the given functions 0, 2, $-\epsilon\langle Y, Y\rangle$:

$$\text{(A)} \qquad \frac{d}{dt}\begin{pmatrix}\lambda \\ \mu\end{pmatrix} = \begin{pmatrix} 0 & 2 \\ -\epsilon\langle Y, Y\rangle & 0\end{pmatrix}\begin{pmatrix}\lambda \\ \mu\end{pmatrix}.$$

It is clear that $\begin{pmatrix}\overline{\lambda} \\ \overline{\mu}\end{pmatrix} \equiv \begin{pmatrix}0 \\ 0\end{pmatrix}$ is a solution of (A) with the initial conditions $\begin{pmatrix}\overline{\lambda}(0) \\ \overline{\mu}(0)\end{pmatrix} = \begin{pmatrix}0 \\ 0\end{pmatrix}$. By the uniqueness theorem of ordinary differential equation theory, we obtain $\lambda \equiv \overline{\lambda} \equiv 0$ and $\mu \equiv \overline{\mu} \equiv 0$. Therefore, we have $\langle X, X\rangle = \epsilon$ along σ. □

§4. Main theorems.

Let $M(\dim M \geq 2)$ be a pseudo-Riemannian submanifold in a pseudo-Riemannian manifold $\widetilde{M}$. First of all, we consider the case that every Q-circle c in M is a $\widetilde{Q}$-circle $f \circ c$ in $\widetilde{M}$, where f is the isometric immersion. By assumption, the curve c satisfies the following two equations

$$\text{(4.1)} \qquad \widetilde{\nabla}_X\widetilde{\nabla}_X X + \Big(\langle\widetilde{\nabla}_X X, \widetilde{\nabla}_X X\rangle\langle X, X\rangle + \langle\widetilde{Q}X, X\rangle\Big)X - \langle X, X\rangle\widetilde{Q}X = 0,$$

$$\text{(4.2)} \qquad \nabla_X\nabla_X X + \Big(\langle\nabla_X X, \nabla_X X\rangle\langle X, X\rangle + \langle QX, X\rangle\Big)X - \langle X, X\rangle QX = 0,$$

where $\widetilde{Q}$ (resp. Q) is a tensor field of type (1,1) on $\widetilde{M}$ (resp. M) and $X = \frac{dc}{ds}$. From (2.1) , (2.2) , (2.3) and (4.2), it follows that

$$\begin{aligned} \widetilde{\nabla}_X\widetilde{\nabla}_X X &= \nabla_X\nabla_X X + B(X, \nabla_X X) - A_{B(X,X)}X + \nabla^{\perp}_X B(X, X) \\ &= -\langle\nabla_X X, \nabla_X X\rangle\langle X, X\rangle X - \langle QX, X\rangle X + \langle X, X\rangle QX \\ &\quad + 3B(X, \nabla_X X) - A_{B(X,X)}X + \overline{\nabla}B(X, X, X). \end{aligned}$$

Substituting this equation into (4.1), we have

$$\text{(4.3)} \qquad \begin{aligned} &A_{B(X,X)}X - \langle B(X, X), B(X, X)\rangle\langle X, X\rangle X \\ &- \langle\widetilde{Q}X - QX, X\rangle X + \langle X, X\rangle(\widetilde{Q}X - QX) \\ &- 3B(X, \nabla_X X) - \overline{\nabla}B(X, X, X) = 0. \end{aligned}$$

For the component normal to M in (4.3), we obtain

$$\overline{\nabla}B(X,X,X)+3B(X,\nabla_X X)-\langle X,X\rangle(\widetilde{Q}X)^{\perp}=0, \tag{4.4}$$

where $(\widetilde{Q}X)^{\perp}$ denotes the normal part of $\widetilde{Q}X$.

Let p be an arbitrary point of M and x and y any orthonormal vectors in T_pM. From Lemma 3.1, there exists a Q-circle c_1 of M such that

$$c_1(0)=p,\ c_1'(0)=x \ \text{ and } \ (\nabla_{c_1'}c_1')(0)=ky,$$

where k is a positive constant. Since $f\circ c_1$ is a $\widetilde{Q}$-circle of $\widetilde{M}$, we get from (4.4)

$$\overline{\nabla}B(x,x,x)+3kB(x,y)-\langle x,x\rangle(\widetilde{Q}x)^{\perp}=0. \tag{4.5}$$

From Lemma 3.1, there also exists a Q-circle c_2 of M such that

$$c_2(0)=p,\ c_2'(0)=x \ \text{ and } \ (\nabla_{c_2'}c_2')(0)=-ky.$$

Thus we get

$$\overline{\nabla}B(x,x,x)-3kB(x,y)-\langle x,x\rangle(\widetilde{Q}x)^{\perp}=0. \tag{4.6}$$

Making use of (4.5) and (4.6), we have

$$B(x,y)=0, \tag{4.7}$$

where x and y are orthonormal.

Let $\{e_1,\cdots,e_n\}$ be an orthonormal frame at each point of M. Let $\langle e_i,e_i\rangle=\epsilon_i\ (=\pm1)$ and $\langle e_j,e_j\rangle=\epsilon_j\ (=\pm1)$ $(1\le i\ne j\le n)$. Here, we divide the situation into two cases where $\epsilon_i=\epsilon_j$ (Case 1) and $\epsilon_i=-\epsilon_j$ (Case 2).

Case 1. Let $v=\frac{1}{\sqrt{2}}(e_i+e_j)$ and $w=\frac{1}{\sqrt{2}}(e_i-e_j)$. Then we can find easily that v and w are orthonormal vectors in $T_p(M)$. So, we have from (4.7),

$$B(e_i,e_i)=B(e_j,e_j).$$

Case 2. Let $v=\sqrt{2}e_i+e_j$ and $w=e_i+\sqrt{2}e_j$. Then also, we can find that v and w are non-null orthonormal vectors in $T_p(M)$. So we have from (4.7),

$$B(e_i,e_i)=-B(e_j,e_j).$$

It follows from Case 1 and Case 2 that

$$\epsilon_i B(e_i,e_i)=\epsilon_j B(e_j,e_j) \quad (1\le i\ne j\le n). \tag{4.8}$$

Let $X=\sum_{i=1}^{n}X^ie_i$ and $Y=\sum_{j=1}^{n}Y^je_j$. Then, by virtue of (4.7) and (4.8), we have

$$\begin{aligned} B(X,Y)&=\sum_{i,j=1}^{n}X^iY^jB(e_i,e_j)\\ &=\sum_{i=1}^{n}X^iY^iB(e_i,e_i)=H\sum_{i=1}^{n}\epsilon_iX^iY^i\\ &=\langle X,Y\rangle H, \end{aligned}$$

for arbitrary tangent vectors X and Y in $T_p(M)$. Thus we have the following.

Theorem 4.1. *Let $M(\dim M \geq 2)$ be a pseudo-Riemannian submanifold in a pseudo-Riemannian manifold $\widetilde{M}$. If every Q-circle in M is a $\widetilde{Q}$-circle in $\widetilde{M}$, then M is totally umbilical in $\widetilde{M}$.*

If, for any tangent vector field X of M, $\widetilde{Q}X$ is tangent to M, then we call M a *$\widetilde{Q}$-invariant submanifold.* By Theorem 4.1, we see that M is totally umbilical in $\widetilde{M}$. Thus from (4.3), we obtain

$$\langle \widetilde{Q}X - QX, X\rangle X - \langle X, X\rangle(\widetilde{Q}X - QX) + \nabla^{\perp}_{X} H = 0,$$

where H is the mean curvature vector field of M. By taking the tangential (resp. normal) part of the above equation, we get

$$\langle X, X\rangle((\widetilde{Q}X)^{\top} - QX) = \langle \widetilde{Q}X - QX, X\rangle X, \tag{4.9}$$

$$\nabla^{\perp}_{X} H = \langle X, X\rangle(\widetilde{Q}X)^{\perp}, \tag{4.10}$$

where $(\widetilde{Q}X)^{\top}$ denotes the tangential part of $\widetilde{Q}X$. From (4.9) and (4.10), we have the followings.

Proposition 4.2. *Let $M(\dim M \geq 2)$ be a pseudo-Riemannian submanifold of a pseudo-Riemannian manifold $\widetilde{M}$. Assume that every Q-circle in M is a $\widetilde{Q}$-circle in $\widetilde{M}$. Then M is a $\widetilde{Q}$-invariant submanifold if and only if M is an extrinsic sphere in $\widetilde{M}$.*

Let $\widetilde{Q}^{\top}$ be the tensor field of type (1,1) on M defined by $\widetilde{Q}^{\top}X := (\widetilde{Q}X)^{\top}$.

Proposition 4.3. *Let M and $\widetilde{M}$ be as in Proposition 4.2. Assume that every Q-circle in M is a $\widetilde{Q}$-circle in $\widetilde{M}$. Then $\widetilde{Q}^{\top} = \sigma I$ if and only if $Q = \lambda I$, where σ (resp. λ) is a smooth function on $\widetilde{M}$ (resp. M) and I an identity map of TM.*

On the other hand, in the case where Q-circle is a conformal one, we can state as follows:

Corollary 4.4. *Let M ($\dim M \geq 3$) be a pseudo-Riemannian submanifold of a pseudo-Riemannian manifold $\widetilde{M}$ and every conformal circle in M be a conformal circle in $\widetilde{M}$. Then M is an Ricci-invariant submanifold if and only if M is an extrinsic sphere in $\widetilde{M}$.*

Corollary 4.5. *Let M and $\widetilde{M}$ be as in Corollary 4.4. Assume that every conformal circle in M is a conformal circle in $\widetilde{M}$. If $\widetilde{M}$ is an Einstein manifold, then M is an Einstein manifold and an extrinsic sphere in $\widetilde{M}$.*

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