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Bellman equations for discrete time two–parameter optimal stopping problems

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Abstract

We study Bellman equations associated with two-parameter optimal stopping problems for discrete time bi–Markov processes. The existence and the uniqueness of a solution of the Bellman equation for our problem are investigated by using the concept of the bi–excessive function.

Keywords : Bellman equation * bi–excessive function * bi–Markov process * strategy * tactic * two–parameter optimal stopping problem

1 Introduction

Throughout this paper we consider the stochastic processes indexed by $\mathbf{N}^2$. Let $\mathbf{T} = \mathbf{N}^2$. The index set $\mathbf{T}$ is extended to its one–point compactification $\mathbf{T} \cup \{\infty\}$ endowed with the following partial order : for all $z = (s,t), z' = (s',t') \in \mathbf{T}$,

$$\begin{aligned} z \leq z' &\quad \text{if and only if} \quad s \leq s', t \leq t', \\ z < z' &\quad \text{if and only if} \quad s < s', t < t', \\ z \leq \infty &\quad \text{for all } z \in \mathbf{T}. \end{aligned}$$

For $i = 1,2$, let $X^i = (\Omega^i, \mathcal{F}^i, \mathcal{F}^i_t, X^i(t), P^i_x)$ be a time homogeneous Markov chain with a state space $(E^i, \mathcal{B}^i)$. We assume that X^1 and X^2 are mutually independent.

We define a bi–Markov process introduced in Mazziotto [8] , that is, the family of a two–parameter process taking values in $E = E^1 \times E^2$

$$X(z) = (X^1(s), X^2(t)) \quad z = (s,t) \in \mathbf{T}$$

on the probability space $(\Omega = \Omega^1 \times \Omega^2, \mathcal{F} = \mathcal{F}^1 \otimes \mathcal{F}^2, P_{(x,y)} = P^1_x \otimes P^2_y, (x,y) \in E)$ endowed with the smallest two–parameter filtration $\{\mathcal{F}_z, z \in \mathbf{T}\}$ which contains $\{\mathcal{F}^1_s \otimes \mathcal{F}^2_t, (s,t) \in \mathbf{T}\}$ and satisfies the conditions

$\mathcal{F} = \sigma(\cup_z \mathcal{F}_z)$,
$\{\mathcal{F}_z, z \in \mathbf{T}\}$ is complete.

A strategy is the family of stopping points $\{\sigma_t, t \geq 0\}$ satisfying the conditions :

$\sigma_0 = z$,
$\sigma_{t+1} = \sigma_t + e_1$ or $\sigma_t + e_2$,
σ_{t+1} is measurable with respect to $\mathcal{F}_{\sigma_t}$,

where $e_1 = (1, 0)$, $e_2 = (0, 1)$ and $\mathcal{F}_{\sigma_t} = \{A \in \mathcal{F} | A \cap \{\sigma_t \leq z\} \in \mathcal{F}_z, \forall z\}$.

A tactic is the pair (σ_t, τ) of a strategy $\{\sigma_t\}$ and a stopping time τ with respect to $\mathcal{F}_{\sigma_t}$.

We shall denote by $\mathbf{B}$, $\mathbf{B}(A^-)$ and $\mathbf{B}(A^+)$ the set of all $\mathcal{B} = \mathcal{B}^1 \otimes \mathcal{B}^2$–measurable functions taking on values in $(-\infty, +\infty]$, the functions f in $\mathbf{B}$ which satisfy the conditions

$$A^- \ : \ E_{(x,y)}[\sup_z f^-(X(z))] < \infty, \quad (x, y) \in E,$$
$$A^+ \ : \ E_{(x,y)}[\sup_z f^+(X(z))] < \infty, \quad (x, y) \in E,$$

respectively, and also write $\mathbf{B}(A^-, A^+) = \mathbf{B}(A^-) \cap \mathbf{B}(A^+)$, $\mathbf{L}(A^-) = \mathbf{L} \cap \mathbf{B}(A^-)$, $\mathbf{L}(A^+) = \mathbf{L} \cap \mathbf{B}(A^+)$, and $\mathbf{L}(A^-, A^+) = \mathbf{L}(A^-) \cap \mathbf{L}(A^+)$ where $\mathbf{L}$ is the set of all functions $f \in \mathbf{B}$ with $E_{(x,y)}[f^-(X(e_i))] < \infty$, $(x, y) \in E$, $i = 1, 2$.

Let $\bar{\Sigma}$be the set of all tactics with $P_{(x,y)}(\tau \leq \infty) = 1$, $(x, y) \in E$, Σ the set of all tactics with $P_{(x,y)}(\tau < \infty) = 1$, $(x, y) \in E$.

The two–parameter optimal stopping problem studied in this paper is to find $(\sigma_t^*, \tau^*) \in \Sigma(\text{resp.}\bar{\Sigma})$ such that

$$S(x, y) = E_{(x,y)}[g(X(\sigma_{\tau^*}^*))] = \sup_{(\sigma_t, \tau) \in \Sigma} E_{(x,y)}[g(X(\sigma_\tau))]$$
$$\bar{S}(x, y) = E_{(x,y)}[g(X(\sigma_{\tau^*}^*))] = \sup_{(\sigma_t, \tau) \in \bar{\Sigma}} E_{(x,y)}[g(X(\sigma_\tau))]$$

where $g(X(\infty)) = \limsup_{z \to \infty} g(X(z))$. We shall call S and $\bar{S}$ the optimal value function.

These problems have been studied by several authors (see Krengel and Sucheston [3], Lawler and Vanderbei [4], Mandelbaum [6], Mandelbaum and Vanderbei [7]). Mandelbaum and Vanderbei [7] introduced the concept which is called the multi–excessive function. Mazziotto [8] also introduced the concept which is called the bi–excessive function, and developed the potential theory associated to the continuous time bi–Markov processes.

By the way, it is well known that the Bellman equation associated to the two–parameter optimal stopping problem is the following type :

$$f(x, y) = \max\{g(x, y), T^1 f(x, y), T^2 f(x, y)\}$$

$$= \max\{g(x,y), \max_{i=1,2} T^i f(x,y)\}. \tag{1}$$

Here T^i be a transition operator of X^i, then,

$$T^1 f(x,y) = E_{(x,y)}[f(X(1,0))], \tag{2}$$
$$T^2 f(x,y) = E_{(x,y)}[f(X(0,1))]. \tag{3}$$

Our aim in this paper is to study the existence of the solution of (1) by using the successive approximation and the relation between a solution of (1) and the optimal value function, and also to give the sufficient condition in order that (1) has a unique solution.

As for classical one–parameter optimal stopping problems, the excessive functions play an important role in studying the properties on the optimal value functions. Shiryayev [10] has given the excessive characterization of the optimal values functions. In this paper we shall also give the bi–excessive characterization of the values S and $\bar{S}$ in accordance with the line of Shiryayev [10].

2 Bi–excessive functions and optimal value functions

In this section we shall give some results of bi–excessive functions and smallest bi-excessive majorants.

Let $\{X(z), \mathcal{F}_z, P_{(x,y)}\}_{z\in\mathbf{T}}$ be a bi–Markov process with the state space $(E, \mathcal{B})$ introduced in section 1.

Definition 2.1 *A function $f \in \mathbf{B}$ is said to be a bi–excessive function (with respect to T^1 and T^2) if for all $(x,y) \in E$ and $i = 1,2$, $T^i f(x,y)$ defined by* (2) *and* (3) *is well defined and $T^i f(x,y) \le f(x,y)$.*

Let $\{f_n\}$ be a nondecreasing sequence of bi–excessive functions of $\mathbf{L}$. Then $\lim_{n\to\infty} f_n$ is also bi-excessive.

Definition 2.2 *A bi–excessive functionf $\in \mathbf{B}$ is said to be the smallest bi–excessive majorant of $g \in \mathbf{B}$ if $f \ge g$ and for any bi–excessive function h such that $h \ge g$, $f \le h$.*

Definition 2.3 *Let a function f be a solution of the equation* (1)*. A tactic (σ_t, τ) is said to be an admissible tactic associated with f if (σ_t, τ) has the following properties :*

$$\begin{aligned} \sigma_0 &= (0,0), \\ \sigma_{t+1} &= \sigma_t + e_i \qquad \text{if} \quad X(\sigma_t) \in A^i, \\ \tau &= \inf\{t \ge 0 : X(\sigma_t) \in B\}, \end{aligned}$$

where $B = \{f = g\}$, $A^1 = \{f = T^1 f\} \setminus B$ and $A^2 = \{f = T^2 f\} \setminus (A^1 \cup B)$.

Here is a fundamental result obtained by Mandelbaum and Vanderbei [7].

Lemma 2.1 *Let $g \in \mathbf{B}$ and V the smallest bi-excessive majorant of g. Then*

$$V = \max\{g, T^1V, T^2V\}.$$

Let the operator Q be defined by

$$Qg = \max\{g, T^1g, T^2g\}.$$

Then the function $V = \lim_{n\to\infty} Q^n g$ is the smallest bi-excessive majorant of g.

Lemma 2.2 *Let $g \in \mathbf{B}$, f a solution of the equation (1) and (σ_t, τ) an admissible tactic associated with f. Put*

$$\tau_\epsilon = \inf\{t \geq 0 : f(X(\sigma_t)) \leq g(X(\sigma_t)) + \epsilon\}, \quad \epsilon \geq 0.$$

Then, if $(x,y) \in E$ is such that $f(x,y) < \infty$, for any $t \in \mathbf{N}$,

$$E_{(x,y)}[f(X(\sigma_{\tau_\epsilon \wedge t}))] = f(x,y).$$

Proof. τ_ϵ is a stopping time with respect to $\mathcal{F}_{\sigma_t}$. Then we have

$$\begin{aligned} f(x,y) &= E_{(x,y)}[f(X(0))] \\ &= E_{(x,y)}[f(X(0))1_{\{\tau_\epsilon=0\}} + f(X(0))1_{\{\tau_\epsilon>0\}}] \\ &= E_{(x,y)}[f(X(0))1_{\{\tau_\epsilon=0\}} + f(X(\sigma_1))1_{\{\tau_\epsilon>0\}}], \end{aligned}$$

since $f(X(0)) = E_{(x,y)}[f(X(\sigma_1))|\mathcal{F}_0]$ on $\{\tau_\epsilon > 0\}$. Similar considerations show that

$$\begin{aligned} f(x,y) &= E_{(x,y)}[f(X(\sigma_{\tau_\epsilon}))1_{\{\tau_\epsilon\leq 1\}} + f(X(\sigma_1))1_{\tau_\epsilon>1}] \\ &= E_{(x,y)}[f(X(\sigma_{\tau_\epsilon}))1_{\{\tau_\epsilon\leq 1\}} + f(X(\sigma_2))1_{\{\tau_\epsilon>1\}}] \\ &\vdots \\ &= E_{(x,y)}[f(X(\sigma_{\tau_\epsilon}))1_{\{\tau_\epsilon\leq t\}} + f(X(\sigma_t))1_{\{\tau_\epsilon>t\}}] \\ &= E_{(x,y)}[f(X(\sigma_{\tau_\epsilon \wedge t}))] \end{aligned}$$

□

We define the operator G by

$$Gf = \max\{g, T^1f, T^2f\}.$$

Lemma 2.3 *Let $g \in \mathbf{B}(A^+)$ and $\varphi(x,y) = E_{(x,y)}[\sup_z g(X(z))]$. Then $G^{n+1}\varphi(x,y) \leq G^n\varphi(x,y)$, and $\tilde{V} = \lim_{n\to\infty} G^n\varphi$ satisfies the equation (1).*

This lemma is obtained by the same arguments as in Shiryayev [10, Chapter 2 Lemma 9].

Lemma 2.4 *Let $g \in \mathbf{B}(A^+)$, V its smallest bi-excessive majorant and (σ_t, τ) an admissible tactic associated with V. If*

$$\limsup g(X(z)) \geq \limsup V(X(z)), \tag{4}$$

then for any $\epsilon > 0$,

$$P_{(x,y)}(\tau_\epsilon < \infty) = 1$$

where $\tau_\epsilon = \inf\{t \geq 0 : V(X(\sigma_t)) \leq g(X(\sigma_t)) + \epsilon\}$.

Noting the condition (4), this lemma is obtained by the same arguments as in Shiryayev [10, Chapter 2 Lemma 8].

Remark 2.1 *If the reward process $\{g(X(z))\}$ satisfies*

$$\sup_{z \geq w} E_{(x,y)}[\sup_{p \geq w} g(X(p)) | \mathcal{F}_z] \leq \sup_{p \geq w} g(X(p)),$$

then the condition (4) is satisfied.

We state another condition in order that the condition (4) be satisfied. Suppose that our filtration $\mathcal{F}_z$ defined in section 1 satisfy the Vitali condition (see Nevev [9, Chapter V Proposition V − 1 − 3] *). Then it is known that*

$$\limsup_z E[Y|\mathcal{F}_z] = \liminf_z E[Y|\mathcal{F}_z] = E[Y|\sigma(\cup_z \mathcal{F}_z)] \quad a.s.$$

for an integrable random variable Y (see Neveu [9, Chapter V Proposition V − 1 − 3] *). Using this fact and $\mathcal{F} = \sigma(\cup_z \mathcal{F}_z)$, we can prove that the condition (4) is satisfied.*

Lemma 2.5 (i) *Let $g \in \mathbf{B}(A^+)$, $\tilde{V} = \lim_n G^n \varphi$ and (σ_t, τ) an admissible tactic associated with $\tilde{V}$. If*

$$\limsup g(X(z)) \geq \limsup \tilde{V}(X(z)),$$

then, for any $\epsilon > 0$,

$$\tilde{V}(x,y) \leq E_{(x,y)}[\tilde{V}(X(\sigma_{\tilde{\tau}_\epsilon}))],$$

where

$$\tilde{\tau}_\epsilon = \inf\{t \geq 0 : \tilde{V}(X(\sigma_t)) \leq g(X(\sigma_t)) + \epsilon\}, \quad \epsilon \geq 0.$$

(ii) *Let $g \in \mathbf{B}(A^-, A^+)$. If*

$$\limsup g(X(z)) \geq \limsup \tilde{V}(X(z)),$$

then,

$$\tilde{V}(x,y) = E_{(x,y)}[\tilde{V}(X(\sigma_{\tilde{\tau}_\epsilon}))],$$

and $\tilde{V} = V$, where V is the smallest bi-excessive majorant of g.

Proof.

(i) By Lemma 2.1 and Lemma 2.3, we can take an admissible tactic (σ_t, τ) associated with $\tilde{V}$, and then

$$\begin{aligned}\tilde{V}(x,y) &= E_{(x,y)}[\tilde{V}(X(\sigma_{\tilde{\tau}_\epsilon\wedge t}))] \\ &= E_{(x,y)}[\tilde{V}(X(\sigma_{\tilde{\tau}_\epsilon\wedge t}))1_{\{\tilde{\tau}_\epsilon\le t\}} + \tilde{V}(X(\sigma_{\tilde{\tau}_\epsilon\wedge t}))1_{\{\tilde{\tau}_\epsilon> t\}}]. \qquad (5)\end{aligned}$$

By Lemma 2.3, we have, for $(x,y)\in E$,

$$\begin{aligned}V(X(\sigma_t)) &\le G^n(X(\sigma_t)) \\ &\le \varphi(X(\sigma_t)) \\ &= E_{X(\sigma_t)}[\sup_z g(X(z))] \\ &\le E_{(x,y)}[\sup_z g^+(X(z))|\mathcal{F}_{\sigma_t}].\end{aligned}$$

From which, we obtain

$$\begin{aligned}&E_{(x,y)}[\tilde{V}(X(\sigma_{\tilde{\tau}_\epsilon\wedge t}))1_{\{\tilde{\tau}_\epsilon> t\}}] \\ =\ & E_{(x,y)}[E_{(x,y)}[\tilde{V}(X(\sigma_t))|\mathcal{F}_{\sigma_t}]1_{\{\tilde{\tau}_\epsilon> t\}}] \\ \le\ & E_{(x,y)}[\sup_z g^+(X(z))1_{\{\tilde{\tau}_\epsilon> t\}}].\end{aligned}$$

By using the same arguments as that of Lemma 2.2, we can get

$$P(\tilde{\tau}_\epsilon<\infty)=1. \qquad (6)$$

By (6) and Fatou's lemma,

$$\begin{aligned}&\limsup_t E_{(x,y)}[\tilde{V}(X(\sigma_{\tilde{\tau}_\epsilon\wedge t}))1_{\{\tilde{\tau}_\epsilon> t\}}] \\ \le\ & \limsup_t E_{(x,y)}[\sup_z g^+(X(z))1_{\{\tilde{\tau}_\epsilon> t\}}] \\ \le\ & E_{(x,y)}[\limsup_t \sup_z g^+(X(z))1_{\{\tilde{\tau}_\epsilon> t\}}] \\ =\ & 0.\end{aligned}$$

Therefore

$$\begin{aligned}\tilde{V}(x,y) &\le \limsup_t E_{(x,y)}[\tilde{V}(X(\sigma_{\tilde{\tau}_\epsilon\wedge t}))1_{\{\tilde{\tau}_\epsilon\le t\}}] + \limsup_t E_{(x,y)}[\tilde{V}(X(\sigma_{\tilde{\tau}_\epsilon\wedge t}))1_{\{\tilde{\tau}_\epsilon> t\}}] \\ &\le E_{(x,y)}[\tilde{V}(X(\sigma_{\tilde{\tau}_\epsilon}))1_{\{\tilde{\tau}_\epsilon<\infty\}}] \\ &= E_{(x,y)}[\tilde{V}(X(\sigma_{\tilde{\tau}_\epsilon}))].\end{aligned}$$

(ii) If $g \in \mathbf{B}(A^-)$, by using the same arguments as that of (i), we can get

$$\liminf_t E_{(x,y)}[\tilde{V}(X(\sigma_{\tilde{\tau}_\epsilon \wedge t}))1_{\{\tilde{\tau}_\epsilon > t\}}] \geq 0.$$

Hence if $g \in \mathbf{B}(A^-, A^+)$,

$$\lim_t E_{(x,y)}[\tilde{V}(X(\sigma_{\tilde{\tau}_\epsilon \wedge t}))1_{\{\tilde{\tau}_\epsilon > t\}}] = 0.$$

By (5), we have

$$\tilde{V}(x,y) = E_{(x,y)}[\tilde{V}(X(\sigma_{\tilde{\tau}_\epsilon}))].$$

By the definition of $\tilde{\tau}_\epsilon$,

$$\begin{aligned} \tilde{V}(x,y) &= E_{(x,y)}[\tilde{V}(X(\sigma_{\tilde{\tau}_\epsilon}))] \\ &\leq E_{(x,y)}[g(X(\sigma_{\tilde{\tau}_\epsilon}))] + \epsilon \\ &\leq E_{(x,y)}[V(X(\sigma_{\tilde{\tau}_\epsilon}))] + \epsilon \\ &\leq V(x,y) + \epsilon, \end{aligned}$$

and then $\tilde{V} \leq V$. On the other hand, by Lemma 2.1 and Lemma 2.3, we have $V \leq \tilde{V}$. □

The following theorem gives the bi–excessive characterization of S and $\bar{S}$ under the condition A^-.

THEOREM 2.1 *Let* $g \in \mathbf{L}(A^-)$. *Then*
(i) S *is the smallest bi–excessive majorant of* g.
(ii) $S = \bar{S}$.
(iii) $S = \max\{g, T^1 S, T^2 S\}$.
(iv) $S = \lim_{n\to\infty} Q^n g = \lim_{b\to\infty} \lim_{n\to\infty} Q^n g^b$ *where* $g^b(x,y) = \min\{g(x,y), b\}$.

Proof. Let V be the smallest bi–excessive majorant of g and (σ_t, τ) an admissible tactic associated with V. Since $\limsup_z V(X(z)) \geq \limsup_z g(X(z))$, for any $\mathcal{F}_{\sigma_t}$–stopping time η,

$$E_{(x,y)}[g(X(\sigma_\eta))] \leq E_{(x,y)}[V(X(\sigma_\eta))] \leq V(x,y).$$

Therefore we have

$$S(x,y) \leq \bar{S}(x,y) \leq V(x,y). \tag{7}$$

Let Q be the operator introduced in Lemma 2.1 and S_n be the optimal value function for an n–stage two–parameter optimal stopping problem :

$$S_n(x,y) = \sup_{(\sigma_t,\tau)\in\Sigma(n)} E_{(x,y)}[g(X(\sigma_\tau))]$$

where $\Sigma(n) = \{(\sigma_t, \tau) | \tau \le n, \ E_{(x,y)}[g^-(X(\sigma_\tau))] < \infty\}$.

Then it is well-known that

$$\begin{aligned} S_n(x,y) &= Q^n g(x,y), \\ S_{n+1}(x,y) &= \max\{g(x,y), T^1 S_n(x,y), T^2 S_n(x,y)\}. \end{aligned}$$

Therefore we can define the function S^* by

$$S^* = \lim_n S_n,$$

and from the assumption $g \in \mathbf{L}(A^-)$, we get

$$\begin{aligned} S^*(x,y) &\le S(x,y), \\ S^*(x,y) &= \max\{g(x,y), T^1 S^*(x,y), T^2 S^*(x,y)\}. \end{aligned} \tag{8}$$

By Lemma 2.1, we have

$$S^* = V. \tag{9}$$

Therefore, by (7), (8) and (9), we have

$$S = \bar{S} = S^* = V.$$

□

THEOREM 2.2 *Let $g \in \mathbf{L}(A^-, A^+)$, V its smallest bi-excessive majorant and (σ_t, τ) an admissible tactic associated with V. If*

$$\limsup g(X(z)) \ge \limsup V(X(z)),$$

then

(i) *for any $\epsilon > 0$, $(\sigma_t, \tau_\epsilon)$ is ϵ-optimal in Σ, that is,*

$$S(x,y) \le E_{(x,y)}[g(X(\sigma_{\tau_\epsilon}))] + \epsilon.$$

(ii) *(σ_t, τ) is optimal in $\bar{\Sigma}$.*

(iii) *if E^1 and E^2 are finite, then $P_{(x,y)}(\tau < \infty) = 1$.*

Proof. The assertion (i) follows from Lemma 2.4, Lemma 2.5 (ii) and Theorem 2.1. By Lemma 2.2 and Theorem 2.1 (iii),

$$E_{(x,y)}[V(X(\sigma_{\tau \wedge t}))] = V(x,y) = S(x,y).$$

$$
\begin{aligned}
& E_{(x,y)}[V(X(\sigma_{\tau\wedge t}))] \\
= \; & E_{(x,y)}[V(X(\sigma_\tau))1_{\{\tau<t\}} + V(X(\sigma_t))1_{\{t\le\tau<\infty\}} + V(X(\sigma_t))1_{\{\tau=\infty\}}] \\
\le \; & E_{(x,y)}[g(X(\sigma_\tau))1_{\{\tau<t\}} + \sup_z g^+(X(z))1_{\{t\le\tau<\infty\}} + V(X(\sigma_t))1_{\{\tau=\infty\}}].
\end{aligned}
$$

By virtue of Fatou's lemma, we have

$$V(x,y) \le E_{(x,y)}[g(X(\sigma_\tau))],$$

from which we obtain (ii).

At last we can obtain the assertion (iii) by using the same arguments as in Shiryayev[10, Chapter 2 Theorem 4].

□

Next we shall give the regular characterization of S and $\bar{S}$ under the condition A^+.

Definition 2.4 *A function $f \in \mathbf{B}$ is said to be a regular function if for any $(\sigma_t, \tau) \in \bar{\Sigma}$, $(x,y) \in E$, $E_{(x,y)}[f(X(\sigma_\tau))]$ is well defined, and for any strategy $\{\sigma_t\}$, $\mathcal{F}_{\sigma_t}$–stopping times τ_1 and τ_2 with $\tau_1 \ge \tau_2$,*

$$E_{(x,y)}[f(X(\sigma_{\tau_1}))] \le E_{(x,y)}[f(X(\sigma_{\tau_2}))].$$

A regular function $f \in \mathbf{B}$ is said to be the smallest regular majorant of $g \in \mathbf{B}$ if $f \ge g$ and for any regular function h such that $h \ge g$, $f \le h$.

Lemma 2.6 *Let $f \in \mathbf{L}(A^-)$ be bi–excessive. Then f is a regular function.*

Proof. Noting that σ_{t+1} is $\mathcal{F}_{\sigma_t}$–measurable, we have

$$
\begin{aligned}
E_{(x,y)}[f(X(\sigma_{t+1}))|\mathcal{F}_{\sigma_t}] &= E_{(x,y)}[\sum_{i=1}^{2} f(X(\sigma_t + e_i))1_{A_i}|\mathcal{F}_{\sigma_t}] \\
&= \sum_i 1_{A_i} E_{X(\sigma_t)}[f(X(e_i))] \\
&= \sum_i 1_{A_i} T^i f(X(\sigma_t)) \\
&\le \sum_i 1_{A_i} f(X(\sigma_t)) \\
&= f(X(\sigma_t)),
\end{aligned}
$$

where $A_i = \{\sigma_{t+1} = \sigma_t + e_i\} \in \mathcal{F}_{\sigma_t}$. Therefore $\{f(X(\sigma_t)), \mathcal{F}_{\sigma_t}\}$ is a one–parameter supermartingale. By the assumption $f \in \mathbf{L}(A^-)$ and the martingale convergence theorem, there exists an integrable variable $Y (= V(X(\infty)))$ such that

$$\lim_{t\to\infty} f(X(\sigma_t)) = Y.$$

Then by virtue of Fatou's lemma, we have for any s,

$$f(X(\sigma_s)) \geq E_{(x,y)}[Y|\mathcal{F}_{\sigma_s}].$$

Applying the optional sampling theorem for one–parameter stochastic process, we conclude the proof.

□

THEOREM 2.3 *Let $g \in \mathbf{B}(A^+)$. If for any $a \leq 0$*

$$\limsup g_a(X(z)) \geq \limsup V_a(X(z))$$

where $g_a(x,y) = \max\{g(x,y),\ a\}$ and V_a is the smallest bi–excessive majorant of g_a, then

(i) *S is the smallest regular majorant of g.*

(ii) *$S = \bar{S}$.*

(iii) *$S = \max\{g, T^1 S, T^2 S\}$.*

(iv) *$S = \lim_{b\to\infty} \lim_{a\to-\infty} \lim_{n\to\infty} Q^n g_a^b$ where $g_a^b(x,y) = \min\{\max\{g(x,y),a\},b\}$.*

Proof. The proof is given by the same lines as that in Shiryayev [10]. Here we shall given an outline of the proof.

Put

$$\begin{aligned} S_a(x,y) &= \sup_{\Sigma} E_{(x,y)}[g_a(X(\sigma_\tau))], \\ S_*(x,y) &= \lim_{a\to-\infty} S_a(x,y). \end{aligned}$$

By our assumption, then we have

$$\begin{aligned} &S_* \geq \bar{S} \geq S \geq g, \\ &S_a = \max\{g_a, T^1 S_a, T^2 S_a\}, \end{aligned}$$

and therefore

$$S_* = \max\{g, T^1 S_*, T^2 S_*\}.$$

By Lemma 2.6, S_* is a regular majorant of g.

Next we shall show that $S_* \leq S$. Let $\{A_t, \xi\}$ is an admissible tactic associated with S_a, we put

$$\begin{aligned} \eta^a &= \inf\{t \geq 0 | S_*(X(A_t)) \leq g_a(X(A_t)) + \epsilon\} \\ \tau^a &= \inf\{t \geq 0 | S_a(X(A_t)) \leq g_a(X(A_t)) + \epsilon\} \\ \tau &= \inf\{t \geq 0 | S_*(X(A_t)) \leq g(X(A_t)) + \epsilon\} \end{aligned}$$

By Lemma 2.4 and Lemma 2.5, we have

$$P_{(x,y)}(\tau^a < \infty) = 1$$
$$S_a(x,y) = E_{(x,y)}[S_a(X(A_{\tau^a}))]$$

By using the same arguments as that in Shiryayev [10] we can get

$$P_{(x,y)}(\tau < \infty) = 1$$
$$S_*(x,y) \leq E_{(x,y)}[S_*(X(A_\tau))]$$

Then

$$\begin{aligned} S_*(x,y) &\leq E_{(x,y)}[S_*(X(A_\tau))] \\ &\leq E_{(x,y)}[g(X(A_\tau))] \\ &\leq S(x,y) + \epsilon \end{aligned}$$

Therefore we have $S_* \leq S$. At last we shall show that $\bar{S}$ is the smallest regular majorant. Let f be any regular majorant of g. Then

$$\begin{aligned} f &\geq g, \\ f(x,y) &\geq E_{(x,y)}[f(X(\sigma_\tau))]. \end{aligned}$$

Hence

$$f(x,y) \geq E_{(x,y)}[f(X(\sigma_\tau))] \geq E_{(x,y)}[g(X(\sigma_\tau))],$$

and then $f \geq \bar{S}$. Therefore $\bar{S}$ is the smallest regular majorant of g.

□

3 Uniqueness conditions of the equation (1)

In this section we shall give the sufficient condition in order that the equation (1) has a unique solution.

Bellman equations for the case of classical one–parameter optimal stopping problems are the following type :

$$f = \max\{g, Tf\}. \tag{10}$$

Grigelionis and Shiryayev [2] and Grigelionis [1] gave the uniqueness conditions of the solution of the Bellman equation (10).

In contrast to the two–parameter optimal stopping problem, the main difference is the existence of a nonlinear (degenerate) operator in (1) :

$$\max_{i=1,2} T^i \tag{11}$$

which appears in the stochastic continuous control problem.

Here, by regarding each operator T^i as an operator defined on the whole space E, we shall show that, under the condition given by Grigelionis [1], the equation (1) has a unique solution.

We put for $G \in \mathcal{B}$,

$$\begin{aligned} \hat{T}_G f(x,y) &= \max\{T^1 1_G f(x,y), T^2 1_G f(x,y)\} \\ \rho_n(G) &= \sup_{(x,y)\in G} (\hat{T}_G)^n 1(x,y). \end{aligned}$$

Theorem 3.1 *Let f_1 and f_2 be two solutions of* (1) *belonging to the class* **L**, *such that*

$$\sup_{(x,y)} |f_1(x,y) - f_2(x,y)| < \infty.$$

If there exists a set $G \in \mathcal{B}$ such that

$$\begin{aligned} &\rho_n(G) < 1 \qquad \textit{for some} \quad n \\ &f_1(x,y) = f_2(x,y) \quad \forall (x,y) \in E \setminus G, \end{aligned}$$

then $f_1 = f_2$ on E.

The proof of Theorem 3.1 is just the same as that of Grigelionis [1].

Corollary 3.1 *If* $\sup_{(x,y)\in E} \max\{T^1 1_{E^1}(x,y), T^2 1_{E^2}(x,y)\} < 1$, *then the solution of* (1) *is unique in the class of measurable bounded functions.*

Proof. Let f_1 and f_2 be two solutions of (14). Put $r(x,y) = |f_1(x,y) - f_2(x,y)|$. Then, by using the similar arguments as in Grigelionis [1, Theorem 1], we can get

$$r(x,y) \leq (\hat{T}_E)^n r(x,y) \qquad \text{for each} \quad n.$$

By assumption,

$$\begin{aligned} \hat{T}_E 1(x,y) &= \max\{T^1 1_E(x,y), T^2 1_E(x,y)\} \\ &= \max\{T^1 1_{E^1}(x,y), T^2 1_{E^2}(x,y)\} \\ &< 1, \end{aligned}$$

from which $\rho_n(E) < 1$ for each n. Hence we have

$$\sup_{(x,y)} r(x,y) \leq \rho_n(E) \cdot \sup_{(x,y)} r(x,y).$$

Therefore we obtain $f_1(x,y) = f_2(x,y)$.

□

Next we shall give another uniqueness condition.
Put

$$M(t,(x,y),A) = \frac{1}{t}\sum_{k=1}^{t}(T^1 + T^2)^k 1_A(x,y)$$

for $t \in \mathbf{N}$, $(x,y) \in E$ and $A \in \mathcal{B}$.

Suppose that there exists a finite measure M on E such that , for any bounded measurable function f on E,

$$\int_E f(q)M(t,p,dq) \longrightarrow \int_E f(q)M(dq) \tag{12}$$

as $t \to \infty$ for all $p \in E$.

For $\lambda = (\lambda_1, \lambda_2)$ satisfying

$$\lambda_i \;:\; E \to [0,1]$$
$$\lambda_1(x,y) + \lambda_2(x,y) = 1,$$

we define a linear operator T_λ by

$$T_\lambda f(x,y) = \sum_{i=1}^{2} \lambda_i(x,y) T^i f(x,y). \tag{13}$$

Then Mandelbaum and Vanderbei [7] gave the following characterization of the bi–excessive function.

Proposition 3.1 *A function $f \in \mathbf{B}$ is bi–excessive with respect to T^1 and T^2 if and only if it is excessive with respect to the operator T_λ defined by (13) for all λ, that is,*

$$T_\lambda f(x,y) \le f(x,y)$$

for all $(x,y) \in E$ and $\lambda = (\lambda_1, \lambda_2)$.

$$\frac{1}{t}\sum_{k=1}^{t}(T_\lambda)^k f \le \frac{1}{t}\sum_{k=1}^{t}(T^1 + T^2)^k f.$$

We consider the following equation :

$$f = \max\{g, T_\lambda f\}. \tag{14}$$

Then we obtain the same result as that in Grigelionis and Shiryayev [2] under the condition (12).

Theorem 3.2 *Let f_1 and f_2 be two solutions of* (14) *belonging to the class* L, *such that*

$$\sup_{(x,y)} |f_1(x,y) - f_2(x,y)| < \infty.$$

If there exists a set $\Lambda \in \mathcal{B}$ such that

$$M(\Lambda) < 1$$
$$f_1(x,y) = f_2(x,y) \quad \forall (x,y) \in E \setminus \Lambda,$$

then $f_1 = f_2$ on E.

Proof. Let f_1 and f_2 be two solutions of (14). Put $r(x,y) = |f_1(x,y) - f_2(x,y)|$. Then, by using the similar arguments as in Grigelionis [1, Theorem 1], we can get

$$r(x,y) \le (T_\lambda)^t r(x,y) \qquad \text{for each} \quad t.$$

$$r \le \frac{1}{t}\sum_{k=1}^{t} (T_\lambda)^k r \le \frac{1}{t}\sum_{k=1}^{t} (T^1 + T^2)^k r.$$

By assumption,

$$r(x,y) \le \int_E r(q) M(t,(x,y),dq) \longrightarrow \int_E r(q) M(dq)$$

as $t \to \infty$. Hence we have

$$\sup_{(x,y)} r(x,y) \le \int_\Lambda r(q) M(dq) \le M(\Lambda) \cdot \sup_{(x,y)} r(x,y).$$

Therefore we obtain $f_1(x,y) = f_2(x,y)$. □

4 Solutions of the equation (1)

In this section we shall discuss the expression of the Bellman equations (1).

At first we give the boundary condition at ∞ which is the sufficient condition in order that a solution of (1) be equal to the optimal value function S.

Proposition 4.1 *Let $g \in \mathrm{L}(A^-, A^+)$ and f be a solution of* (1) *such that $f \in \mathrm{L}(A^+)$. A sufficient condition for this solution to coincide with the optimal value function S is that f satisfy the following condition :*

$$\limsup_z g(X(z)) \ge \limsup_z f(X(z)). \tag{15}$$

Proof. Let (σ_t, τ) be an admissible tactic associated with f and for $\epsilon > 0$, $\tau_\epsilon = \inf\{t \geq 0 | f(X(\sigma_t)) \leq g(X(\sigma_t)) + \epsilon\}$.

Then, by using the same arguments as that in Lemma 2.4, we have $P_{(x,y)}(\tau_\epsilon < \infty) = 1$. The assumption $f \in \mathbf{L}(A^+)$ implies that, for any $(x, y) \in E$, $f(x, y) < \infty$. Hence by Lemma 2.2

$$\begin{aligned} S(x,y) &\geq E_{(x,y)}[g(X(\sigma_{\tau_\epsilon}))] \\ &\geq E_{(x,y)}[f(X(\sigma_{\tau_\epsilon}))] - \epsilon \\ &= f(x,y) - \epsilon \end{aligned}$$

Therefore $S \geq f$. On the other hand, let V be the smallest bi–excessive majorant of g. By Lemma 2.1, then $f \geq V$. From which

$$\limsup_z g(X(z)) \geq \limsup_z V(X(z)).$$

By Theorem 2.1, $S = V$. Therefore we get $S \leq f$. □

Remark 4.1 *In the case of the one–parameter optimal stopping problem, the condition of the type* (15) *is necessary and sufficient (see Shiryayev* [10]*).*

We shall conclude this section by discussing the two–parameter version of the solution of the Bellman equation studied by Lazrieva [5].

Let $g \in \mathbf{L}(A^-)$ and C be $\mathcal{B}$–measurable function with $E_{(x,y)}[|\limsup_z C(X(z))|] < \infty$. We define the function S_C by

$$S_C(x,y) = \sup_{\Sigma}\{\int_{\{\tau<\infty\}} g(X(\sigma_\tau))dP_{(x,y)} + \int_{\{\tau=\infty\}} \limsup_z C(X(z))dP_{(x,y)}\}.$$

Theorem 4.1 *We assume that*

$$\limsup_z \bar{g}(X(z)) \geq \limsup_z \bar{V}(X(z))$$

where $\bar{g}(x,y) = \max\{g(x,y), E_{(x,y)}[\limsup_z C(X(z))]\}$ *and* $\bar{V}$ *is the smallest bi–excessive majorant of* $\bar{g}$.

(i) *We have for any* $(x, y) \in E$

$$S_C(x,y) = \sup_{\Sigma}\{\int_{\{\tau<\infty\}} \bar{g}(X(\sigma_\tau))dP_{(x,y)} + \int_{\{\tau=\infty\}} \limsup_z \bar{g}(X(z))dP_{(x,y)}\}.$$

(ii) S_C *satisfies the equation*

$$S_C = \max\{g, T^1 S_C, T^2 S_C\}.$$

Proof.
(i) By assumption, $\bar{g} \in \mathbf{L}(A^-)$. and

$$\limsup_z \bar{g}(X(z)) = \limsup_z C(X(z)).$$

Then, by Theorem 2.1,

$$\begin{aligned}
& \sup_{\Sigma}\{\int_{\{\tau<\infty\}} \bar{g}(X(\sigma_\tau))dP_{(x,y)} + \int_{\{\tau=\infty\}} \limsup_z \bar{g}(X(z))dP_{(x,y)}\} \\
= \; & \sup_{\Sigma} E_{(x,y)}[\bar{g}(X(\sigma_\tau))] \\
= \; & \sup_{\Sigma}\{\int_{\{g(X(\sigma_\tau))\geq E_{X(\sigma_\tau)}[\limsup_z C(X(z))]\}} g(X(\sigma_\tau))dP_{(x,y)} \\
& + \int_{\{g(\sigma_\tau)<E_{X(\sigma_\tau)}[\limsup_z C(X(z))]\}} E_{X(\sigma_\tau)}[\limsup_z C(X(z))]dP_{(x,y)}\} \\
= \; & \sup_{\Sigma}\{\int_{\{g(X(\sigma_\tau))\geq E_{X(\sigma_\tau)}[\limsup_z C(X(z))]\}} g(X(\sigma_\tau))dP_{(x,y)} \\
& + \int_{\{g(X(\sigma_\tau))<E_{X(\sigma_\tau)}[\limsup_z C(X(z))]\}} \limsup_z C(X(z))dP_{(x,y)}\} \\
= \; & \sup_{\Sigma}\{\int_{\{\eta_\tau<\infty\}} g(X(\sigma_{\eta_\tau}))dP_{(x,y)} + \int_{\{\eta_\tau=\infty\}} \limsup_z C(X(z))dP_{(x,y)}\} \\
\leq \; & \sup_{\Sigma}\{\int_{\{\tau<\infty\}} g(X(\sigma_\tau))dP_{(x,y)} + \int_{\{\tau=\infty\}} \limsup_z C(X(z))dP_{(x,y)}\} \\
= \; & S_C(x,y),
\end{aligned}$$

where

$$\eta_\tau = \begin{cases} \tau & \text{on } \{g(X(\sigma_\tau)) \geq E_{X(\sigma_\tau)}[\limsup_z C(X(z))]\} \\ \infty & \text{on } \{g(X(\sigma_\tau)) < E_{X(\sigma_\tau)}[\limsup_z C(X(z))]\} \end{cases}.$$

Therefore

$$\begin{aligned}
S_C(x,y) \; & \geq \; \sup_{\Sigma}\{\int_{\{\tau<\infty\}} \bar{g}(X(\sigma_\tau))dP_{(x,y)} + \int_{\{\tau=\infty\}} \limsup_z \bar{g}(X(z))dP_{(x,y)}\} \\
& \geq \; \sup_{\Sigma}\{\int_{\{\tau<\infty\}} \bar{g}(X(\sigma_\tau))dP_{(x,y)} + \int_{\{\tau=\infty\}} \limsup_z C(X(z))dP_{(x,y)}\} \\
& = \; S_C(x,y)
\end{aligned}$$

(ii) Let S^* be the optimal value function for $\bar{g}$. By Theorem 2.1, we have

$$S^* = \max\{\bar{g}, T^1 S^*, T^2 S^*\}.$$

From the definition of $\bar{g}$,

$$\begin{aligned}
S^*(x,y) \; & \geq \; T^i E_{(x,y)}[\limsup C(X(z))] \\
T^i S^*(x,y) \; & \geq \; T^i E_{(x,y)}[\limsup_z C(X(z))] = E_{(x,y)}[\limsup_z C(X(z))]
\end{aligned}$$

Therefore we get

$$\begin{aligned} S_C &= S^* \\ &= \max\{g, T^1 S^*, T^2 S^*\} \\ &= \max\{g, T^1 S_C, T^2 S_C\}. \end{aligned}$$

□

THEOREM 4.2 *Let* $g \in \mathbf{L}(A^-, A^+)$ *and* $f \in \mathbf{L}(A^+)$ *be a solution of* (1) *such that* $f_{(x,y)} \geq E_{(x,y)}[\limsup_z f(X(z))]$. *Then*

$$f(x,y) = \sup_{\Sigma}\{\int_{\{\tau<\infty\}} g(X(\sigma_\tau))dP_{(x,y)} + \int_{\{\tau=\infty\}} \limsup f(X(z))dP_{(x,y)}\}.$$

Proof. We put

$$\bar{g}(x,y) = \max\{g(x,y), E_{(x,y)}[\limsup_z f(X(z))]\}.$$

Then, by assumption, f satisfies the equation

$$f = \max\{\bar{g}, T^1 f, T^2 f\}.$$

Noting that $\limsup_z f(X(z)) = \limsup_z \bar{g}(X(z))$, by Proposition 4.1,

$$f(x,y) = \sup_{\Sigma} E_{(x,y)}[\bar{g}(X(\sigma_\tau))].$$

Therefore we get the assertion. □

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