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Necessary and Sufficient Conditions for Oscillation of Second Order Autonomous Neutral Equations with Distributed Delay

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Absract.

In this paper the autonomous neutral equation with distributed delay

$$\frac{d^2}{dt^2}[x(t)+\delta_1\int_0^{\tau}x(t-s)dr_1(s)]+\delta_2\int_0^{\tau}x(t-s)dr_2(s)=0,$$

where $\delta_i=\pm1$, i=1,2, is considered. It is proved that the necessary and sufficient condition for all solutions of this equations to oscillate is that the corresponding characteristic equation

$$z^2(1+\delta_1\int_0^{\tau}e^{-zs}dr_1(s))+\delta_2\int_0^{\tau}e^{-zs}dr_2(s)=0$$

should have no real root.

1.Introduction.

To the problem of obtaining necessary and sufficient conditions for oscillation of all solutions of second and higher order neutral differential equations the papers [1]-[5] are devoted. The neutral equations considered are with a finite number of concentrated delays. The most general results were obtained in [1] and [4], in [1] systems of equations being investigated. The only result in this direction for neutral equations with distributed delay is the work [6] which concerns first order equations. In the present paper the equation

$$\frac{d^2}{dt^2}[x(t)+\delta_1\int_0^{\tau}x(t-s)dr_1(s)]+\delta_2\int_0^{\tau}x(t-s)dr_2(s)=0, \tag{1}$$

is investigated. It is proved that the necessary and sufficient condition for all solutions of (1) to oscillate is that the characteristic equation of (1)

$$Q(z)=z^2(1+\delta_1\int_0^{\tau}e^{-zs}dr_1(s))+\delta_2\int_0^{\tau}e^{-zs}dr_2(s)=0 \tag{2}$$

should have no real root. The result is a generalization of the work [3].

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2. Preliminary notes.

We shall say that conditions (A) are met if the following conditions hold;

$$A1. \quad f \in C([-\tau, \infty), R)$$

$$A2. \quad f(t) + \delta_1 \int_0^{\tau} f(t-s)dr_1(s) \in C^2([0, +\infty), R)$$

Definition 1. The function $x(t)$ is a *solution* of (1) if conditions (A) are met. $x(t)$ satifies (1) for $t \in [0, \infty)$ and $x(t) = \phi(t)$ for $t \in [-\tau, 0]$, where the initial function $\phi \in C([-\tau, 0], R)$.

We shall say that conditions (B) are met if the following conditions hold:

$$B1. \quad r_i(0) \text{ and } r_i(\tau) > 0, i = 1, 2$$

$$B2. \quad r_i(s) \text{ are nondecreasing in } [0, \tau], i = 1, 2$$

$$B3. \quad r_1(s) \text{ is continuous at the point } s = 0$$

Remark 1. Without loss of generality we may assume that the functions $r_i(s), i = 1, 2$ are continuous from the right.

Introduce the following notation

$$\tau_i = \inf\{s | r_i(v) = r_i(\tau) \text{ for } v \in [s, \tau]\}, i = 1, 2$$

In view of Remark 1 it is clear that $r_i(\tau_i) = r_i(\tau), i = 1, 2$.

Definition 2. The solution $x(t)$ of (1) is said to *oscillate* if the set of its zeros is unbounded from above. Otherwise it is said to be *non-oscillating*.

Definition 3. The function f is said eventually to enjoy the property K if there exists t_0 such that for $t > t_0$ the function f enjoys the property K.

LEMMA 1. *Let conditions (B) hold and $\delta_2 = -1$. Then equation (1) has at least one nonoscillating solution.*

PROOF. Since $Q(0) = -r_2(\tau) < 0$ and $\lim_{z \to +\infty} Q(z) = +\infty$, then the characteristic equation (2) has a real root λ_0. Then the solution of (1) $x(t) = e^{\lambda_0 t}$ is nonoscillating. ∎

LEMMA 2. *Let $\delta_2 = 1$. For equation (1), let conditions (B) hold. Then, if $x(t)$ is a solution of (1), then the functions $\alpha x(t-\beta)$, $\int_0^\tau x(t-s)dr_1(s)$, $\int_{t-\alpha}^{t-\beta} x(s)ds$ and $\int_{t-\alpha}^{\infty} x(s)ds$ $\left((x(t) \in L^1[t_0, \infty) \text{ and } \lim_{t\to\infty} x(t) = 0\right)$ are also solutions of (1) for $\alpha, \beta \in R$, where $x(t) \in L^1[t, \infty)$ and $\lim_{t\to\infty} x(t) = 0$.*

PROOF. The assertion of the lemma follows immeadiately from the linearity and autonomy of equation (1). ∎

Denote by M_4 the set of all solutions of (1) which are at least 4 times continuously differentiable and such that

$$(-1)^\nu w^{(\nu)}(t) > 0, \quad \nu = 0,1,2,3,4$$
$$\lim_{t\to\infty} w^{(\nu)}(t) = 0, \quad \nu = 0,1,2,3$$

Denote by N_4 the set of all solutions of (1) which are at least 4 times continuously differentiable and such that

$$w^{(\nu)}(t) > 0, \quad \nu = 0,1,2,3,4$$
$$\lim_{t\to\infty} w^{(\nu)}(t) = 0, \quad \nu = 0,1,2,3$$

LEMMA 3. *Let $x(t)$ be a nonoscillating solution of equation (1). Then (1) has a nonoscillating solution $w(t)$ belonging either to the set M_4 or to the set N_4.*

PROOF. Without loss of generality we may assume that $x(t) > 0$ eventually. Let

$$z(t) = x(t) + \delta_1 \int_0^\tau x(t-s)dr_1(s) \tag{3}$$
$$w(t) = z(t) + \delta \int_0^\tau z(t-s)dr_1(s). \tag{4}$$

Then

$$\ddot{z} = -\int_0^\tau x(t-s)dr_2(s) \tag{5}$$
$$\ddot{w} = -\int_0^\tau z(t-s)dr_2(s) \tag{6}$$
$$w^{(4)} = -\int_0^\tau \ddot{z}(t-s)dr_2(s) \tag{7}$$

From the fact that $x(t) > 0$ eventually it follows that $\ddot{z}(t) < 0$ eventually and $w^{(4)}(t) > 0$ eventually. Hence the functions $z(t)$ and $w(t)$ are eventually monotonic. From $\ddot{z}(t) < 0$, it follows that $\dot{z}(t)$ is an eventually decreasing function. Then either

$$\lim_{t\to\infty} \dot{z}(t) = -\infty \tag{8}$$

or there exists the finite limit

$$\lim_{t\to\infty} \dot{z}(t) = L \tag{9}$$

Let (8) hold. Then $\lim_{t\to\infty} z(t) = -\infty$. Consequently, $\lim_{t\to\infty} \ddot{w}(t) = +\infty$, and then $\lim_{t\to\infty} w(t) = \lim_{t\to\infty} \dot{w}(t) = +\infty$. Differentiating (6) and taking into account that

$\lim_{t\to\infty} \dot{z}(t) = -\infty$, we obtain that $\lim_{t\to\infty} w^{(3)}(t) = +\infty$. Thus we proved that if (8) holds, equation (1) has a solution $w(t) \in N_4$.

Let (9) hold. We shall prove that $L = 0$. Suppose that this is not true. Let $L > 0$. Since $\dot{z}(t)$ is an eventually decreasing function, then $\dot{z}(t) > L$ eventually. Hence $\lim_{t\to\infty} z(t) = +\infty$,whence we obtain $\lim_{t\to\infty} \ddot{w}(t) = -\infty$. Consequently, $\lim_{t\to\infty} \dot{w}(t) = -\infty$. On the other hand, differentiating (4) and taking into account that $\dot{z}(t)$ is a bounded function,we obtain that $\dot{w}(t)$ is a bounded function. The contradiction obtained shows that $L \leq 0$. Analogously the case $L < 0$ is excluded. Thus we proved that $\lim_{t\to\infty} \dot{z}(t) = 0$. Consequently, $\lim_{t\to\infty} \dot{w}(t) = 0$. Since $\dot{z}(t)$ is an eventually decreasing function and $\lim_{t\to\infty} \dot{z}(t) = 0$, then $\dot{z}(t) > 0$ eventually. Hence $z(t)$ is an eventually increasing function. Differentiating (6) and that taking into account that $\lim_{t\to\infty} \dot{z}(t) = 0$, $\dot{z}(t) > 0$ eventually, we obtain that $\lim_{t\to\infty} w^{(3)}(t) = 0$. Since $w^{(4)}(t) > 0$ eventually, then $w^{(3)}(t)$ is an eventually increasing function. Hence $w^{(3)}(t) < 0$ eventually. In order to show that $w(t) \in M_4$, it remains to prove that $\lim_{t\to\infty} w(t) = \lim_{t\to\infty} \ddot{w}(t) = 0$ and $w(t) > 0, \dot{w}(t) < 0$ and $\ddot{w}(t) > 0$ eventually. Suppose that $\lim_{t\to\infty} z(t) \neq 0$. From the fact that $z(t)$ is an eventually increasing function, it follows that there exist positive constants α and β such that either $z(t) > \alpha$ eventually, or $z(t) < -\beta$ eventually in dependence on the sign of $\lim_{t\to\infty} z(t)$. Then from (6) it is immediately seen that there exist positive conatants α' and β' such that $\ddot{w}(t) < -\alpha'$ or $\ddot{w}(t) > \beta'$ respectively. But then $\lim_{t\to\infty} \dot{w}(t) = -\infty$ or $\lim_{t\to\infty} \dot{w}(t) = +\infty$ respectively, which contradicts $\lim_{t\to\infty} \dot{w}(t) = 0$. Hence $\lim_{t\to\infty} z(t) = 0$ and then $z(t) < 0$ eventually. From $\lim_{t\to\infty} z(t) = 0$,(4) and (6), it follows that $\lim_{t\to\infty} w(t) = \lim_{t\to\infty} \ddot{w}(t) = 0$. This immediately implies that $w(t) > 0, \dot{w}(t) < 0, \ddot{w}(t) > 0$. Thus we proved that $w(t) \in M_4$ and Lemma 3 is proved. ∎

LEMMA 4. *For equation (1) let conditions (B) hold. Let $\delta_1 = -1$ and*

$$r_1(\tau_1^-) \neq r_1(\tau_1^+). \tag{10}$$

Then a necessary condition for the characteristic equation of (1) to have no real root is $\tau_1 < \tau_2$.

PROOF. Suppose that this is not true. Let $\tau_1 \geq \tau_2$. We shall show that (2) has a real root. Since $Q(0) > 0$,then it suffices to show that $\lim_{z\to-\infty} Q(z) = -\infty$. Let $z < 0$. From the definition of the Riemann-Stieltjes integral and from (10) it follows that

$$\int_0^T e^{-zs} dr_1(s) \geq e^{-z\tau_1} [r_1(\tau_1^+) - r_1(\tau_1^+)]$$

Let $r_1(\tau_1^+) - r_1(\tau_1^-) = \delta$. Then

$$\int_0^T e^{-zs} dr_1(s) \geq \delta e^{-z\tau_1} \tag{11}$$

On the other hand,

$$\int_0^T e^{-zs} dr_2(s) = \int_0^{\tau_2} e^{-zs} dr_2(s) \leq e^{-z\tau_2} r_2(\tau_2)$$

From the last inequality and (11) we obtain

$$Q(z) < z^2(1 - \delta e^{-z\tau_1}) + e^{-z\tau_2} r_2(\tau_2)$$

for $z < 0$. Then $\lim_{z \to -\infty} Q(z) = -\infty$. Thus Lemma 4 is proved. ∎

Remark 2. By arguments analogous to the above case it is immediately seen that if the characteristic equation $Q(z) = 0$ has no real root, then $\lim_{a \to -\infty} Q(z) = \lim_{a \to +\infty} Q(z) = +\infty$. Consequently, $\inf_R Q(z) > 0$.

Remark 3. Condition (10) is necessary only in the case when $\tau_1 = \tau_2$. If $\tau_1 > \tau_2$, then $Q(z) = 0$ has a real root even if $r_1(s)$ is continuous at the point τ_1. Choose $\epsilon > 0$ so that $\tau_1 - \epsilon > \tau_2$. Then

$$\int_0^\tau e^{-zs} dr_1(s) \geq \int_{\tau_1 - \epsilon}^{\tau_1} dr_1(s)$$
$$\geq e^{-z(\tau_1 - \epsilon)}[r_1(\tau_1) - r_1(\tau_1 - \epsilon)].$$

From the definition of τ_1 it follows that $\delta_\epsilon = r_1(\tau_1) - r_1(\tau_1 - \epsilon) > 0$ for any $\epsilon > 0$. Thus we obtain that

$$\int_0^\tau e^{-zs} dr_1(s) \geq \delta_\epsilon e^{-z(\tau_1 - \epsilon)}.$$

Arguing further as in the proof of Lemma 4, it is shown that $\lim_{z \to -\infty} Q(z) = -\infty$, i.e. the equation $Q(z) = 0$ has a real root. If $\tau_1 = \tau_2$ and $r_1(s)$ is continuous at the point τ_1, then nothing definite can be said about whether the equation $Q(z) = 0$ has or does not have real roots. We shall illustrate this fact by the following eamples.

Let $\tau = 2$.

$$r_1(s) = \begin{cases} 2s - s^2 & ,0 \leq s \leq 1 \\ 1 & ,1 < s \leq 2 \end{cases} \qquad r_2(s) = \begin{cases} s & ,0 \leq s \leq 1 \\ 1 & ,1 < s \leq 2. \end{cases}$$

Then $\tau_1 = \tau_2$ and straightforward calculations yield

$$Q(z) = \begin{cases} z^2 - 2z + 2 - 2e^{-z} + \frac{1}{z} - \frac{1}{z}e^{-z}, & z \neq 0 \\ 1, & z = 0 \end{cases}$$

and $\lim_{z \to -\infty} Q(z) = -\infty, \lim_{z \to \infty} Q(z) = +\infty$. Consequently the characteristic equation (2) has a real root.

For the same τ and $r_1(s)$ consider the function

$$r_2(s) = \begin{cases} 0 & ,0 \leq s < 1 \\ \kappa & ,1 \leq s \leq 2 \quad (\kappa > 0). \end{cases}$$

Let $\kappa > 2$. Then $\lim_{z \to -\infty} Q(z) = \lim_{z \to +\infty} Q(z) = +\infty$. It is easy to check that for sufficiently large $\kappa, Q(z) > 0$ for all z. Consequently the equation $Q(z) = 0$ has no real root.

LEMMA 5. *For equation (1) let condition (B) hold. Let $\delta_1 = -1$ and $\tau_1 < \tau_2$. Then*
a) if $x(t) \in M_4$, then there exists a solution $w(t)$ of equation (1) such that $w(t) \in M_4$ and the set

$$\Lambda(w) = \{\lambda > 0 | -\ddot{w}(t) + \lambda^2 w(t) \leq 0\} \neq \phi$$

b) if $x(t) \in N_4$, then there exists a solution $w(t)$ of equation (1) such that $w(t) \in N_4$ and the set $\Lambda(w) \neq \phi$.

PROOF. a) Let

$$w(t) = -[x(t) - \int_0^\tau x(t-s)dr_1(s)] \tag{12}$$

from Lemma 2, it follows that $w(t)$ is a solution of (1). It is immediately checked that $w(t) \in M_4$. From $x(t) \in M_4$ it follows that $x(t)$ is an eventually decreasing function. Using this fact and (12), we obtain the estimate

$$w(t) < \int_0^\tau x(t-s)dr_1(s) = \int_0^{\tau_1} x(t-s)dr_1(s) \leq x(t-\tau_1)r_1(\tau_1).$$

From the fact that $x(t)$ is a solution of (1), and from (12), it follows that

$$-\ddot{w}(t) + \int_0^\tau x(t-s)dr_2(s) == 0. \tag{13}$$

As above,we have the estimate

$$0 \geq -\ddot{w}(t) + \int_{\tau_1}^\tau x(t-s)dr_2(s) \geq -\ddot{w}(t) + x(t-\tau_1)[r_2(\tau) - r_2(\tau_1)].$$

Then from both estimates it follows that

$$-\ddot{w}(t) + \frac{r_2(\tau) - r_2(\tau_1)}{r_1(\tau_1)} w(t) < 0.$$

From the definition of τ_1 and τ_2 it follows that $r_2(\tau) = r_2(\tau_2) > r_2(\tau_2 - \epsilon)$ for any $\epsilon > 0$. Then

$$\frac{r_2(\tau) - r_2(\tau_1)}{r_1(\tau_1)} > 0$$

and

$$\left(\frac{r_2(\tau) - r_2(\tau_1)}{r_1(\tau_1)}\right)^{\frac{1}{2}} \in \Lambda(w).$$

b) Let $z(t) = -[x(t) - \int_0^\tau x(t-s)dr_1(s)]$. As in the proof of a) $z(t)$ is a solution of (1) and $z(t) \in N_4$. Then $z(t) > 0$ eventually and especially. Hence

$$x(t) - \int_0^\tau x(t-s)dr_1(s) < 0.$$

From conditions B1 and B3, it follows that for any $\epsilon > 0$ we can choose δ_ϵ such that for $s < \delta_\epsilon, r_1(s) < \epsilon$. Let $\epsilon < 1$ and let $\delta < \delta_\epsilon$. Then $1 - r_1(\delta) > 0$. Using this inequality and the fact that $x(t)$ is an eventually increasing function ($x(t) \in N_4$), we obtain

$$x(t) < \int_0^\tau x(t-s)dr_1(s) = \int_0^\delta x(t-s)dr_1(s) + \int_\delta^\tau x(t-s)dr_1(s)$$
$$< x(t)r_1(\delta) + x(t-\delta)[r_1(\tau) - r_1(\delta)].$$

Consequently,

$$x(t)[1 - r_1(\delta)] < x(t-\delta)[r_1(\tau) - r_1(\delta)]$$
$$x(t) < \frac{r_2(\tau) - r_2(\tau_1)}{r_1(\tau_1)} x(t-\delta)$$

Choose the positive integer κ so that $\kappa\delta > \tau_2$. Then from the above inequality it follows that

$$x(t) < \left(\frac{r_1(\tau) - r_1(\delta)}{1 - r_1(\delta)}\right)^\kappa \cdot x(t - \kappa\delta) < \left(\frac{r_1(\tau) - r_1(\delta)}{1 - r_1(\delta)}\right)^\kappa \cdot x(t - \tau_2).$$

Since $x(t)$ is a solution of (1) and $x(t) \in N_4$, $\ddot{x}(t) > 0$ eventually. Then the following inequality holds

$$-\int_0^\tau \ddot{x}(t-s)dr_1(s) + \int_0^\tau x(t-s)dr_2(s) < 0.$$

Using the fact that $x(t)$ and $\ddot{x}(t)$ are eventually increasing functions,we obtain

$$-\int_0^\tau \ddot{x}(t-s)dr_1(s) + \int_0^\tau x(t-s)dr_2(s)$$
$$= -\int_0^\tau \ddot{x}(t-s)dr_1(s) + \int_0^{\tau_2} x(t-s)dr_2(s)$$
$$\geq \ddot{x}(t)r_1(\tau) + x(t-\tau_2)r_2(\tau_2)$$

Consequently,

$$-\ddot{x}(t) + x(t-\tau_2)\frac{r_2(\tau_2)}{r_1(\tau_1)} < 0.$$

From the last inequality and from the inequality

$$x(t) < \left(\frac{r_1(\tau) - r_1(\delta)}{1 - r_1(\delta)}\right)^\kappa x(t-\tau_2),$$

we obtain that

$$-\ddot{x}(t) + \left(\frac{r_1(\tau) - r_1(\delta)}{1 - r_1(\delta)}\right)^\kappa \cdot \frac{r_2(\tau_2)}{r_1(\tau_1)} x(t) < 0.$$

Hence

$$[\left(\frac{1 - r_1(\delta)}{r_1(\tau) - r_1(\delta)}\right)^\kappa \cdot \frac{r_2(\tau_2)}{r_1(\tau_1)}] \in \Lambda(x).$$

Thus Lemma 5 is proved. ∎

LEMMA 6 [4]. *For equation (1) let conditions (B) hold.*

(a) Let $x(t)$ be a solution of (1), $x(t) \in M_4$ and $\Lambda(x) \neq 0$. Then, if for a given ω there exists $M > 0$ such that

$$(-1)^{\kappa} x^{(\kappa)}(t) > M(-1)^{\kappa} x^{(\kappa)}(t-\omega), \kappa = 0,1,2$$

then the positive number $\lambda_0 = \frac{1}{\omega}\log\frac{1}{M}$ is an upper bound of $\Lambda(x)$.

(b) Let $x(T)$ be a solution of (1), $x(t) \in N_4$ and $\Lambda(x) \neq \phi$. Then, if for a given $\omega > 0$ there exists $M > 0$ such that

$$x^{(\kappa)}(t) < M x^{(\kappa)}(t-\omega), \kappa = 0,1,2$$

then the positive number $\lambda_0 = \frac{1}{\omega}\log\frac{1}{M}$ is an upper bound of $\Lambda(x)$.

LEMMA 7. *For equation (1) let conditions (B) hold. $\delta_1 = -1$ and $\tau_1 < \tau_2$. Then*

(a) if $x(t) \in M_4$ and $\Lambda(x) \neq \phi$, then the set $\Lambda(x)$ has an upper bound independent of x.

(b) if $x(t) \in N_4$ and $\Lambda(x) \neq \phi$,then the set $\Lambda(x)$ has an upper bound independent of x.

PROOF. a). Define $w(t)$ as in (12). Then $w(t) \in M_4$ and (13) is met. Using the fact that $x(t), -\dot{x}(t), \ddot{x}(t)$ are eventually decreasing functions,from (12) we obtain the estimates

$$\begin{aligned} 0 &< w(t) < r_1(\tau_1)x(t-\tau_1) \\ 0 &< -\dot{w}(t) < -r_1(\tau_1)\dot{x}(t-\tau_1) \\ 0 &< \ddot{w}(t) < r_1(\tau_1)\ddot{x}(t-\tau_1). \end{aligned} \tag{14}$$

Set $\rho = \frac{1}{2}(\tau_2 - \tau_1)$. Then from (12), we obtain the inequality

$$-\ddot{w}(t) + \int_{\tau_1+\rho}^{\tau} x(t-s)dr_2(s) \leq 0.$$

From this inequality, using the fact that $x(t)$ is an eventually decreasing function, it follows that

$$-\ddot{w}(t) + [r_2(t) - r_2(\tau_1+\rho)]x(t-(\tau_1+\rho)) \leq 0.$$

Let $\gamma = r_2(\tau) - r_2(\tau_1+\rho)$. Then we obtain the inequality

$$\begin{aligned} -\ddot{w}(t) + \gamma x(t-(\tau_1+\rho)) &\leq 0 \\ w^{(3)}(t) - \gamma\dot{x}(t-(\tau_1+\rho)) &\leq 0 \\ -w^{(4)}(t) + \gamma\ddot{x}(t-(\tau_1+\rho)) &\leq 0. \end{aligned} \tag{15}$$

The last two inequalities of (15) are obtained from (13) just as the first one. Set $\alpha = \frac{1}{8}(\tau_1 - \tau_2)$ and integrate (15) from $t-\alpha$ to t. We obtain

$$-\dot{w}(t) + \dot{w}(t-\alpha) + \gamma\int_{t-\alpha}^{t} x(s-(\tau_1+\rho))ds \leq 0.$$

Using the fact that $\dot{w}(t) < 0$ eventually and that $x(t)$ is an eventaully decreasing function, we obtain the inequality

$$\begin{aligned} \dot{w}(t-\alpha) + \gamma x(t-(\tau_1+\rho))\alpha &< 0 \\ -\dot{w}(t) &> \gamma\alpha x(t-(\tau_1+\rho)+\alpha). \end{aligned}$$

In the same way further two inequalities are derived and we obtain

$$\begin{aligned} -\dot{w}(t) &> \gamma\alpha x(t-(\tau_1+\rho)+\alpha) \\ \ddot{w}(t) &> -\gamma\alpha \dot{x}(t-(\tau_1+\rho)+\alpha) \\ -w^{(3)}(t) &> \gamma\alpha \ddot{x}(t-(\tau_1+\rho)+\alpha). \end{aligned} \tag{16}$$

Analogously, from inequalities (16) we obtain the inequalities

$$\begin{aligned} w(t) &> \gamma\alpha^2 x(t-(\tau_1+\rho)+2\alpha) \\ -\dot{w}(t) &> -\gamma\alpha^2 \dot{x}(t-(\tau_1+\rho)+2\alpha) \\ \ddot{w}(t) &> \gamma\alpha^2 \ddot{x}(t-(\tau_1+\rho)+2\alpha). \end{aligned} \tag{17}$$

From the first inequality of (14) and the first inequality of (17) we obtain

$$\gamma\alpha^2 x(t-(\tau_1+\rho)+2\alpha) < r_1(\tau_1)x(t-\tau_1).$$

Then

$$\begin{aligned} x(t) &> \frac{\gamma\alpha^2}{r_1(\tau_1)} x\left(t-\frac{1}{4}(\tau_2-\tau_1)\right) \\ -\dot{x}(t) &> -\frac{\gamma\alpha^2}{r_1(\tau_1)} \dot{x}\left(t-\frac{1}{4}(\tau_2-\tau_1)\right) \\ \ddot{x}(t) &> \frac{\gamma\alpha^2}{r_1(\tau_1)} \ddot{x}\left(t-\frac{1}{4}(\tau_2-\tau_1)\right) \end{aligned} \tag{18}$$

The last two inequalities in (18) are obtained as the first one. From inequalities (18) and from Lemma 6, it follows that the positive number $\lambda_0 = \frac{4}{\tau_2-\tau_1}\log\frac{r_1(\tau_1)}{\gamma\alpha^2}$, where $\alpha = \frac{1}{8}(\tau_2-\tau_1)$ and $\gamma = r_2(\tau) - r_2(\frac{\tau_1+\tau_2}{2})$, is an upper bound of $\Lambda(x)$ which is independent of the concrete $x \in M_4$.

b) Let $x(t) \in N_4$. Define $w(t)$ as in (12). Then $w(t) \in N_4$ and the following equalities are valid

$$w^{(\nu)}(t) = -[x^{(\nu)}(t) - \int_0^\tau x^{(\nu)}(t-s)dr_1(s)], \nu = 0,1,2.$$

From $w(t) \in N_4$ it follows that $w^{(\nu)}(t) > 0, \nu = 0,1,2$. Hence

$$-x^{(\nu)}(t) + \int_0^\tau x^{(\nu)}(t-s)dr_1(s) > 0.$$

Using the fact that $x^{(\nu)}(t)$ are increasing functions, just as in the proof of Lemma 5b), we obtain that the following inequalities hold

$$x^{(\nu)}(t) < \frac{r_1(\tau) - r_1(\delta_1)}{1 - r_1(\delta)} x^{(\nu)}(t - \delta), \nu = 0, 1, 2,$$

where δ is chosen as in Lemma 5b).

From the last inequalities and Lemma 6 it follows that the positive number $\lambda_0 = \frac{1}{\delta} \log \frac{r_1(\tau) - r_1(\delta)}{1 - r_1(\delta)}$ is an upper bound of $\Lambda(x)$ which is independent of the concrete $x \in N_4$. ∎

Remark 4. Lemma 7 claims that the number $\lambda_0 = \frac{4}{\tau_2 - \tau_1} \log \frac{r_1(\tau_1)}{\gamma\alpha^2}$, an upper bound of $\Lambda(x)$, is positive. The first can be established in the following way. From inequality (18) and from the fact that $x(t)$ is a dexreasing function when $x(t) \in M_4$, it follows that

$$x(t) > \frac{\gamma\alpha^2}{r_1(\tau_1)} x\left(t - \frac{1}{4}(\tau_2 - \tau_1)\right) > \frac{\gamma\alpha^2}{r_1(\tau_1)} x(t).$$

Consequently, $\frac{\gamma\alpha^2}{r_1(\tau_1)} < 1$ and then $\log \frac{r_1(\tau_1)}{\gamma\alpha^2} > 0$.

In the same way the case $x(t) \in N_4$ is considered.

3. Main Results

THEOREM 1. *For equation (1) let conditions (B) hold,$\delta_2 = 1, \delta_1 = 1$. Then each solution oscillates.*

PROOF. Suppose that the equation has at least one nonoscillating solution $x(t)$. Without loss of generarity we may assume that $x(t) > 0$ evetually. By Lemma 3 equation (1) has a nonoscillaing solution $w(t)$ belonging to the set M_4 or to the set N_4. In both cases $w(t)$ and $\ddot{w}(t)$ are eventually positive functions. Then eventually the following inequality holds

$$\ddot{w}(t) + \int_0^T \ddot{w}(t - s) dr_2(s) + \int_0^T w(t - s) dr_2(s) > 0.$$

Hence $w(t)$ cannot be a solution of (1). From the contradiction obtained it follows that each solution of (1) oscillates. Thus Theorem 1 is proved. ∎

THEOREM 2. *For equation (1) let conditions (B) hold,$\delta_2 = 1, \delta_1 = -1$. Moreover,let condition (10) hold. Then the necessary and sufficient condition for each solution of (1) to oscillate is that the characteristic equation (2) should have no real root.*

PROOF. In order to prove the theorem it suffices to prove that equation (1) has a nonoscillating solution if and only if (2) has at least one real root. If (2) has a real root z_0, then $x(t) = e^{z_0 t}$ is a nonoscillating solution. We shall prove that if (1) has a nonoscilating solution $y(t)$, then the characteristic equation (2) has a real root. Suppose that this is not true,i.e. (2) has no real root. From the fact that $y(t)$ is a nonoscillating solution,by

Lemma 3, it follows that equation (1) has a solution $x(t)$ belonging to the set M_4 or to the set N_4. Let $x(t) \in M_4$. From the assumption that the characteristic equation (2) has no real root and from (10), by Lemma 4, it follows that $\tau_1 < \tau_2$. Therefore, the conditions are met,under which Lemma 5 and Lemma 7 are valid. By Lemma 5a) without loss of generality we may assume that $\Lambda(x) \neq \phi$, and let $\lambda' \in \Lambda(x)$. By Lemma 7a) the set $\Lambda(x)$ is bounded from above, and let λ_0 be an upper bound of $\Lambda(x)$ (independent of x). Let $\lambda \geq \lambda'$ and $\lambda \in \Lambda(x)$. Set

$$z(t) = F_1 x = -[x(t) - \int_0^{\tau} x(t-s)dr_1(s)]$$
$$w(t) = F_2 z = -\lambda \dot{z}(t) + \ddot{z}(t)$$

$$u(t) = F_3 w = \frac{d}{dt}[w(t) - \int_0^{\tau} w(t-s)dr_1(s)] + \int_0^{\tau}\int_{t-\tau}^{t-s} w(\nu)d\nu dr_2(s) + \lambda^2 \int_{t-\tau}^{t} w(s)ds. \quad (19)$$

It is aesy to check that $z(t), w(t), u(t)$ are solution of (1) and belong to the set M_4. We shall show that $(\lambda^2 + m_0)^{\frac{1}{2}} \in \Lambda(u)$, where

$$m_0 = \frac{m}{e^{\lambda_0 \tau}[1 + r_1(\tau) + \frac{r_2(\tau)}{\lambda'^2}]}, m = \inf_R Q(z). \quad (20)$$

For this purpose,we have to prove that $-\ddot{u}(t) + (\lambda^2 + m_0)u(t) \leq 0$.
Let $\phi(t) = -e^{\lambda t}\dot{w}(t)$. From $w(t) \in M_4$ it follows that $\phi(t) > 0$ eventually.

$$\begin{aligned}
\dot{\phi}(t) &= e^{\lambda t}[-\ddot{w}(t) - \lambda \dot{w}(t)] \\
&= e^{\lambda t}[-z^{(4)}(t) + \lambda^2 \ddot{z}(t)] \\
&= e^{\lambda t}[\frac{d^2}{dt^2}[\frac{d^2}{dt^2}(x(t) - \int_0^{\tau} x(t-s)dr_1(s)] \\
&\qquad - \lambda^2 \frac{d^2}{dt^2}[x(t) - \int_0^{\tau} x(t-s)dr_1(s)]\} \\
&= e^{\lambda t}[-\int_0^{\tau} \ddot{x}(t-s)dr_2(s) + \lambda^2 \int_0^{\tau} x(t-s)dr_2(s)] \\
&= e^{\lambda t}\int_0^{\tau}[-\ddot{x}(t-s) + \lambda^2 x(t-s)]dr_2(s) \leq 0.
\end{aligned}$$

The last inequality folows from the fact taht $\lambda \in \Lambda(x)$. Thus we showed that $\phi(t)$ is a nonincreasing function. From the definition of $\phi(t)$ it follows that $\dot{w}(T) = -e^{-\lambda t}\phi(t)$. Integrating this equality from t to t_1 and passing to the limit as $t_1 \to +\infty$, we obtain

$$w(t) = \int_0^{\infty} e^{-\lambda s}\phi(s)ds \leq \frac{1}{\lambda}e^{-\lambda t}\phi(t).$$

Then for any $\omega < \tau$ the following estimate is valid.

$$\int_{t-\tau}^{t-\omega} w(s)ds \le \frac{1}{\lambda}\int_{t-\tau}^{t-\omega} \phi(s)ds \le \frac{1}{\lambda^2}e^{-\lambda t}\phi(t-\tau)(e^{\lambda\tau} - e^{\lambda\omega})$$

Hence we have the inequality

$$\int_{t-\tau}^{t-\omega} w(s)ds \le \frac{1}{\lambda^2}e^{-\lambda\tau}\phi(t-\tau)(e^{\lambda\tau} - e^{\lambda\omega}). \tag{21}$$

Differentialing twice (19),we obtain

$$\begin{aligned}\ddot{u}(t) &= \frac{d}{dt}\{\frac{d^2}{dt^2}[w(t) - \int_0^\tau w(t-s)dr_1(s)]\} + \int_0^\tau [\dot{w}(t-s) - \dot{w}(t-\tau)]dr_2(s) \\ &\quad + \lambda^2[\dot{w}(t) - \dot{w}(t-\omega)] \\ &= -\frac{d}{dt}[\int_0^\tau w(t-s)dr_2(s)] + \int_0^\tau \dot{w}(t-s)dr_2(s) - r_2(\tau)\dot{w}(t-\tau) \\ &\quad + \lambda^2[\dot{w}(t) - \dot{w}(t-\tau)].\end{aligned}$$

Therefore

$$\ddot{u}(t) = -r_2(\tau)\dot{w}(t-\tau) + \lambda^2[\dot{w}(t) - \dot{w}(t-\tau)].$$

Substituing $\dot{w}(t) = -e^{-\lambda t}\phi(t)$ into the last inequality,we obtain

$$-\ddot{u}(t) = r_2(\tau)e^{-\lambda(t-\tau)}\phi(t-\tau) + \lambda^2[e^{-\lambda t\phi(t)} - e^{-\lambda(t-\tau)}\phi(t-\tau)]. \tag{22}$$

From (19), taking into account that $\dot{w}(t) = -e^{-\lambda t}\phi(t)$ and inequality (21), we obtain the esitimate

$$\begin{aligned}u(t) &\le -e^{-\lambda t}\phi(t) + \int_0^\tau e^{-\lambda(t-s)}\phi(t-s)dr_1(s) \\ &\qquad + \frac{1}{\lambda^2}e^{-\lambda t}\phi(t-\tau)\int_0^\tau (e^{\lambda\tau} - e^{\lambda s})dr_2(s) \\ &\quad + e^{-\lambda t}\phi(t-\tau)(e^{\lambda\tau} - 1)\end{aligned} \tag{23}$$

Then from (22) and (23) we obtain

$$\begin{aligned}-\ddot{u}(t) + \lambda^2 u(t) &\le \lambda^2\int_0^\tau \phi(t-s)dr_1(s) \\ &\qquad - e^{-\lambda t}\phi(t-\tau)\int_0^\tau e^{\lambda s}dr_2(s) - \lambda^2 e^{-\lambda t}\phi(t-\tau) \\ &\le e^{-\lambda t}\phi(t-\tau)[-\lambda^2 + \lambda^2\int_0^\tau e^{\lambda s}dr_1(s) - \int_0^\tau e^{\lambda s}dr_2(s)].\end{aligned}$$

From the definition of m it follows that $Q(\lambda) \geq m$. Hence

$$\lambda^2 - \lambda^2 \int_0^\tau e^{-\lambda s} dr_1(s) + \int_0^\tau e^{-\lambda s} dr_2(s) \geq m$$
$$\lambda^2 + \lambda^2 \int_0^\tau e^{-\lambda s} dr_1(s) - \int_0^\tau e^{-\lambda s} dr_2(s) \leq -m.$$

This inequality holds for all real λ. Replacing λ by $-\lambda$,we obtain the inequality

$$-\lambda^2 + \lambda^2 \int_0^\tau e^{\lambda s} dr_1(s) - \int_0^\tau e^{\lambda s} dr_2(s) \leq -m.$$

Using this inequality,we obtain the estimate

$$-\ddot{u}(t) + \lambda^2 u(t) \leq e^{-\lambda t}\phi(t-\tau)(-m).$$

Hence

$$-\ddot{u}(t) + (\lambda^2 + m_0)u(t) \leq e^{-\lambda t}\phi(t-\tau)(-m) + m_0 u(t). \tag{24}$$

From (23), taking into account that $\phi(t) > 0$ eventually and that $\phi(t)$ is an eventually nonincreasing function, we obtain

$$\begin{aligned} u(t) &\leq \int_0^\tau e^{-\lambda(t-s)}\phi(t-s)dr_1(s) + \frac{1}{\lambda^2}e^{-\lambda t}\phi(t-\tau)e^{\lambda\tau}r_2(\tau) \\ &\quad + e^{-\lambda t}\phi(t-\tau)e^{\lambda\tau} \\ &\leq e^{-\lambda t}\phi(t-\tau)e^{\lambda_0\tau}[1 + r_1(\tau) + \frac{r_2(\tau)}{\lambda'^2}] \\ &= e^{-\lambda t}\phi(t-\tau)\frac{m}{m_0}. \end{aligned}$$

Then from (24) it follows that

$$-\ddot{u}(t) + (\lambda^2 + m_0)u(t) \leq 0.$$

Consequently, $(\lambda^2 + m_0)^{\frac{1}{2}} \in \Lambda(u)$. Set $x_0 = x, x_1 = Fx = F_3(F_2(F_1x)), x_2 = Fx_1, ..., x_n = Fx_{n-1}$. It is easy to check that $x_n \in M_4$ for any positive integer n.

$$\lambda \in \Lambda(x_0), (\lambda^2 + m_0)^{\frac{1}{2}} \in \Lambda(u) = \Lambda(x_1).$$

Therwfore,$(\lambda^2 + 2m_0)^{\frac{1}{2}} \in \Lambda(x_2)$. For any positive integer n we have $(\lambda^2 + nm_0)^{\frac{1}{2}} \in \Lambda(x_n)$ and since $m_0 > 0$, then $\lim_{n\to\infty}(\lambda^2 + nm_0)^{\frac{1}{2}} = +\infty$ which contradicts the fact that λ_0 is an upper bound of $\Lambda(x_n)$ for any positive integer n.

Let $x(t) \in N_4$. From Lemma 4 it follows that $\tau_1 < \tau_2$. Without loss of generality we may assume,by Lemma 5b), that the set $\Lambda(x) \neq \phi$. Let $\lambda'' \in \Lambda(x)$. By Lemma 7b) there exists $\lambda_0 > 0$ such that $\Lambda(x)$ is bounded above (λ_0 is independent of x). For $\lambda \geq \lambda''$ and $\lambda \in \Lambda(x)$ consider the functions

$$\begin{aligned} z(t) &= -[x(t) - \int_0^\tau x(t-s)dr_1(s)] \\ w(t) &= \lambda \dot{z}(t) + \ddot{z}(t) \\ u(t) &= -\frac{d}{dt}[w(t) - \int_0^\tau w(t-s)dr_1(s)] \\ &\quad + \int_0^\tau \int_{t-s}^{t+\tau} w(v)dvdr_2(s) + \lambda^2 \int_t^{t+\tau} w(s)ds \end{aligned} \tag{25}$$

It is immediately verified that the functions $z(t), w(t), u(t)$ are solutions of (1) and belong to the set N_4. Let

$$m_0 = \frac{m}{\int_0^\tau e^{-\lambda'' s}dr_1(s) + e^{\lambda_0 \tau}(1 + \frac{r_2(\tau)}{\lambda''})}. \tag{26}$$

We shall shows that $(\lambda^2 + m_0)^{\frac{1}{2}} \in \Lambda(u)$. For this purpose it suffices to prove that $-\ddot{u}(t) + (\lambda^2 + m_0)u(t) \neq 0$.

Let $\phi(t) = e^{-\lambda t}\dot{w}(t)$. From $w(t) \in N_4$ it follows that $\phi(t) > 0$ eventually.

$$\begin{aligned} \dot{\phi}(t) &= -e^{-\lambda t}[\ddot{w}(t) + \lambda \dot{w}(t)] \\ &= -e^{-\lambda t}[-z^{(4)}(t) + \lambda^2 \ddot{z}(t)] \\ &= -e^{-\lambda t}\{\frac{d^2}{dt^2}[\frac{d^2}{dt^2}(x(t) - \int_0^\tau x(t-s)dr_1(s)] \\ &\qquad - \lambda^2 \frac{d^2}{dt^2}[x(t) - \int_0^\tau x(t-s)dr_1(s)]\} \\ &= -e^{-\lambda t}[-\int_0^\tau \ddot{x}(t-s)dr_2(s) + \lambda^2 \int_0^\tau x(t-s)dr_2(s)] \\ &= -e^{-\lambda t}\int_0^\tau [-\ddot{x}(t-s) + \lambda^2 x(t-s)]dr_2(s) \geq 0 \end{aligned}$$

The last inequality follows from the fact that $\lambda \in \Lambda(x)$. Hence the function $\phi(t)$ is eventually nondecreasing. From the definition of $\phi(t)$ we obtain

$$\dot{w}(t) = e^{\lambda t}\phi(t). \tag{27}$$

As in [5], we extend the definition of the functions $w^{(k)}(t), k = 0,1,2$ so that they should be continuous, positive and increasing in $(-\infty, \infty)$ and $\lim_{t \to +\infty} w^{(k)}(t) = 0, k = 0,1$ be valid. Then, in view of (27), we get

$$\begin{aligned} w(t) &= \int_{-\infty}^t \dot{w}(s)ds = \int_{-\infty}^t e^{\lambda s}\phi(s)ds \\ &\leq \phi(t)\int_{-\infty}^t e^{\lambda s}ds = \frac{1}{\lambda}e^{\lambda t}\phi(t). \end{aligned}$$

From this inequality we obtain the estimate

$$\int_{t-\omega}^{t+\tau} w(s)ds \leq \frac{1}{\lambda}\int_{t-\omega}^{t+\tau} e^{\lambda s}\phi(s)ds \leq \frac{1}{\lambda^2}\phi(t+\tau)[e^{\lambda\tau} - e^{-\lambda\omega}]e^{\lambda t}.$$

Hence

$$\int_{t-\omega}^{t+\tau} w(s)ds \leq \frac{1}{\lambda^2}\phi(t+\tau)[e^{\lambda\tau} - e^{-\lambda\omega}]e^{\lambda t}. \tag{28}$$

Just as in the proof of the case $x(t) \in M_4$ of (25), using (27),(28) and the fact that $\phi(t)$ is an eventually nobdecreasing function, we obtain the inequality

$$\begin{aligned}
&-\ddot{u}(t) + \lambda^2 u(t) \\
&\quad \leq e^{\lambda t}\phi(t+\tau)[-\lambda^2 + \lambda^2\int_0^\tau e^{-\lambda s}dr_1(s) - \int_0^\tau e^{-\lambda s}dr_2(s)] \\
&\quad \leq e^{\lambda t}\phi(t-\tau)(-m).
\end{aligned}$$

The last inequality follows from the inequality $-Q(\lambda) \leq -m$, where $Q(z) = 0$ is the characteristic equation of (1). From (25) there follows the estimate for the function $u(t)$

$$\begin{aligned}
u(t) &\leq -e^{\lambda t}\phi(t) + \int_0^\tau e^{\lambda(t-s)}\phi(t-s)dr_2(s) \\
&\qquad + \frac{1}{\lambda^2}\int_0^\tau e^{\lambda t}\phi(t+\tau)(e^{\lambda\tau} - e^{-\lambda s})dr_2(s) \\
&\qquad + e^{\lambda t}\phi(t+\tau)e^{\lambda\tau} \\
&\leq \int_0^\tau e^{\lambda(t-s)}\phi(t-s)dr_1(s) \\
&\qquad + \frac{1}{\lambda^2}\int_0^\tau e^{\lambda t}\phi(t+\tau)e^{\lambda\tau}dr_2(s) + e^{\lambda t}\phi(t+\tau)e^{\lambda\tau} \\
&\leq e^{\lambda t}\phi(t+\tau)[\int_0^\tau e^{-\lambda s}dr_1(s) + \frac{1}{\lambda^2}e^{\lambda\tau}r_2(\tau) + e^{\lambda\tau}] \\
&\leq e^{\lambda t}\phi(t+\tau)[\int_0^\tau e^{-\lambda'' s}dr_1(s) + e^{\lambda_0\tau}(1 + \frac{r_2(\tau)}{\lambda''^2})] \\
&= e^{\lambda t}\phi(t+\tau)\frac{m}{m_0}.
\end{aligned}$$

Consequently,

$$\begin{aligned}
&-\ddot{u}(t) + (\lambda^2 + m_0)u(t) = -\ddot{u}(t) + \lambda^2 u(t) + m_0 u(t) \\
&\leq e^{\lambda t}\phi(t+\tau)(-m) + m_0 e^{\lambda t}\phi(t+\tau)\frac{m}{m_0} \\
&= 0.
\end{aligned}$$

Thus we prove that $(\lambda^2 + m_0)^{\frac{1}{2}} \in \Lambda(u)$. We complete the proof of Theorem 2 ust as the proof on the case $x(t) \in M_4$. ∎

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