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Existence Results for Cone Saddle Points by Using Vector Variational-like Inequalities

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Abstract

This paper is concerned with existence theorems for cone saddle-points of vector-valued functions. By means of vector variational-like inequalities, we first characterize a vector-valued saddle-point problem, and then obtain the existence result for cone invex and Fréchet differentiable vector-valued functions. In Section 1, we introduce historical background on this field and our motivation for this study briefly. In Section 2, we introduce formulations of vector-valued saddle-point problem and vector variational-like inequality problem. Next, we introduce some elementary concepts related to our results. In Section 3, we show a relationship between a vector-valued saddle-point problem and a vector variational-like inequality problem, and we prove an existence result of a vector-valued saddle-point problem.

1 Introduction

Studies on vector-valued minimax theorems or vector-valued saddle-point problems have been extended widely; see [12] and references cited therein. Existence results for cone saddle-points are based on some fixed point theorems or scalar minimax theorems; see [10, 11]. Recently, this kind of problems has been solved by a different approach in [7], in which the equivalence to a vector variational inequality problem has been established, and then an existence theorem for weak saddle-points of a vector-valued function is shown by using this property. However, the setting of their

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papers is limited to finite dimensional Euclidean spaces and the objective vector-valued functions with two arguments are convex with respect to the first variable in the sense of vector-ordering with the non-negative cone. In this paper, we consider its generalization to vector problems involving the concept of cone invexity under the general setting on normed spaces, and then we show an existence theorem with weaker condition than in [7, 8].

2 Preliminary and terminology

Let K be a nonempty subset of a normed space X, E a nonempty subset of a normed space Y and Z a normed space. We denote the topological interior and closure of a set S by $\operatorname{int} S$ and $\operatorname{cl} S$, respectively, and the complementary set of S by S^c. Let C be a solid pointed convex cone subset of Z, i.e.,

(i) $\operatorname{int} C \neq \emptyset$,

(ii) $C \cap (-C) = \{0_Z\}$,

(iii) $tz_1 + sz_2 \in C$ for all $z_1, z_2 \in C$ and $t, s \geq 0$,

where 0_Z stands for the origin of Z. Given a vector-valued function $F : X \times Y \to Z$, the vector-valued saddle-point problem, denoted by (P), is to find a pair $x_0 \in K$ and $y_0 \in E$ such that

$$(\mathrm{P}) \quad \begin{cases} F(x_0, y_0) - F(x, y_0) \notin \operatorname{int} C \text{ for all } x \in K, \\ F(x_0, y) - F(x_0, y_0) \notin \operatorname{int} C \text{ for all } y \in E. \end{cases}$$

Definition 2.1. A point $(x_0, y_0) \in K \times E$ is said to be a weak C-saddle-point of function F on $K \times E$, if it is a solution of problem (P).

Now, in order to consider a vector variational-like inequality problem with a close relation to problem (P), we define the following multifunction $T : K \to 2^E$,

$$T(x) := \{y \in E : F(x, v) - F(x, y) \notin \operatorname{int} C \text{ for all } v \in E\}. \tag{1}$$

Lemma 2.2. *Let K be a nonempty closed set in X, E a nonempty closed set in Y and T a multifunction defined by (1). Assume that the function F in (1) is continuous on $K \times E$ and we take two sequences $\{x_n\}$, $\{y_n\}$ such that $x_n \in K$ and $y_n \in T(x_n)$ for each $n \in N$. If $x_n \to x$ and $y_n \to y$, then $y \in T(x)$.*

Proof. Let $\{x_n\} \subset K$ with $x_n \to x$, $y_n \in T(x_n)$ with $y_n \to y$ and $v \in E$. Since F is continuous on $K \times E$, $(F(x_n, v) - F(x_n, y_n)) \to (F(x, v) - F(x, y))$. Since $(\operatorname{int} C)^c$ is closed, $(F(x, v) - F(x, y)) \in (\operatorname{int} C)^c$. □

We see that $C' := (\operatorname{int} C) \cup \{0_Z\}$ is correct, i.e.,

$$(\operatorname{cl} C') + C' \backslash \{C' \cap (-C')\} \subset C'.$$

Now, if E is compact and $v \mapsto F(x,v)$ is continuous on E for each $x \in K$, then for any $x \in K$, $\cup_{x\in E} F(x,v)$ is nonempty compact, so we have $T(x) \neq \emptyset$ for all $x \in K$; see Theorem 2.6 in [9].

We denote the space of continuously linear operators from X to Z by $\mathcal{L}(X,Z)$. Let $F'(x,y)$ stand for the Fréchet derivative of F, refer to Definition 2.3, with respect to the first variable at $(x,y) \in X \times Y$, i.e.,

$$F'(x,y) : X \to \mathcal{L}(X,Z).$$

With respect to problem (P), we introduce the following vector variational-like inequality problem with respect to $\eta : K \times K \to X$, denoted by (Q): find a pair $x_0 \in K$ and $y_0 \in T(x_0)$ such that

$$\text{(Q)} \quad F'(x_0,y_0)(\eta(x,x_0)) \notin -\operatorname{int} C \text{ for all } x \in K.$$

Under certain condition, problem (Q) has a close relation to (P). We will consider the condition and relationship in the next section.

Definition 2.3. Let D be an open set in X and f a vector-valued function from D to Z. If for given $x \in D$, there exists $f'(x) \in \mathcal{L}(X,Z)$, which is said to be Fréchet derivative of f at x, such that for any $\varepsilon > 0$ we can choose $\delta > 0$ satisfying the following condition:

$$\|f(x+h) - f(x) - f'(x)h\| \leq \varepsilon\|h\| \text{ for every } h \in \{x \in X : \|x\| < \delta\},$$

then f is said to be Fréchet differentiable at x, where $\|x\|$ stands for the norm of x. If for given $S \subset D$, f is Fréchet differentiable at each $x \in S$, then f is said to be Fréchet differentiable on S.

Definition 2.4. Let K be a convex set in X, C a pointed convex cone in Z and f a vector-valued function from X to Z. A function f is said to be C-convex on K if

$$\lambda f(x_1) + (1-\lambda)f(x_2) - f(\lambda x_1 + (1-\lambda)x_2) \in C \text{ for all } x_1, x_2 \in K \text{ and } \lambda \in [0,1].$$

Remark 1. Let K be a convex set in X, C a pointed closed convex cone in Z, and assume that a vector-valued function $f : X \to Z$ is Fréchet differentiable on K. Then f is C-convex on K if and only if

$$f(x_2) - f(x_1) - f'(x_1)(x_2 - x_1) \in C \text{ for all } x_1, x_2 \in K,$$

where $f'(x_1)$ stands for Fréchet derivative of f at $x_1 \in K$.

Definition 2.5. ([4].) Let K be a convex set in X and C a pointed convex cone in Z. Let $f : X \to Z$ be Fréchet differentiable on K and η a function from $K \times K$ to X. If f satisfies

$$f(x_2) - f(x_1) - f'(x_1)(\eta(x_2, x_1)) \in C \text{ for all } x_1, x_2 \in K,$$

where $f'(x_1)$ stands for Fréchet derivative of f at $x_1 \in K$, then f is said to be C-invex on K with respect to η.

If C is a pointed closed convex cone in Z, then each Fréchet differentiable C-convex function is C-invex, by with $\eta(x_2, x_1) = x_2 - x_1$.

Definition 2.6. ([12].) Let K be a convex set in X and C a pointed convex cone in Z. A function $f : K \to Z$ is said to be C-quasiconvex if it satisfies one of the following two equivalent conditions:

(i) for each $x, y \in K$ and $\lambda \in [0, 1]$,

$$f(\lambda x + (1 - \lambda)y) \in z - C, \text{ for all } z \in C(f(x), f(y)),$$

where $C(f(x), f(y))$ is the set of upper bounds of $f(x)$ and $f(y)$, i.e.,

$$C(f(x), f(y)) := \{z \in Z | z \in f(x) + C \text{ and } z \in f(y) + C\};$$

(ii) for each $z \in Z$,

$$A(z) := \{x \in K | f(x) \in z - C\}$$

is convex or empty.

3 Existence result of cone saddle-points

Theorem 3.1. *Let K be a convex set in X, E a subset in Y and C a solid pointed convex cone in Z. Let $F : X \times Y \to Z$ be Fréchet differentiable with respect to the first variable on K. If the map $x \mapsto F(x, y)$ is C-invex on K with respect to $\eta : K \times K \to X$ for each $y \in E$, then each solution of problem (Q) is also a solution of (P).*

Proof. Let $(x_0, y_0) \in K \times E$ be a solution of problem (Q). This means that

$$F'(x_0, y_0)(\eta(x, x_0)) \notin -\text{int}\, C \text{ for all } x \in K, \tag{2}$$

and $y_0 \in T(x_0)$. The C-invexity of F means that

$$F(x, y_0) - F(x_0, y_0) - F'(x_0, y_0)(\eta(x, x_0)) \in C \text{ for all } x \in K. \tag{3}$$

Conditions (2) and (3) imply that

$$F(x_0, y_0) - F(x, y_0) \notin \operatorname{int} C \text{ for all } x \in K,$$

and $y_0 \in T(x_0)$ means that

$$F(x_0, y) - F(x_0, y_0) \notin \operatorname{int} C \text{ for all } y \in E.$$

□

Remark 2. If C is closed and $x \mapsto F(x, y)$ is Fréchet differentiable C-convex on K for each $y \in E$, then Theorem 3.1 is reduced to Theorem 2.1 in [8]. Moreover, if X, Y and Z are restricted to finite-dimensional Euclidean spaces, respectively, and if the ordering cone C is the non-negative cone, then Theorem 2.1 in [8] is reduced to Theorem 2.2 in [7].

The following classical Fan-KKM lemma is a powerful tool for vector variational and variational-like problems and so on.

Lemma 3.2. ([5].)*Let V be a nonempty subset in a topological vector space $\mathcal{X}$ and $G : V \to 2^{\mathcal{X}}$ a given multifunction. Assume that $G(x)$ is a nonempty closed set in $\mathcal{X}$ for each $x \in V$. If G is a KKM map, i.e., the convex hull of every finite subset $\{x_1, \ldots, x_n\}$ of V is contained in the corresponding union $\bigcup_{i=1}^{n} G(x_i)$, and $G(x_0)$ is compact for some $x_0 \in V$, then $\bigcap_{x \in V} G(x) \neq \emptyset$.*

Theorem 3.3. *Let K be a nonempty closed convex set in X, E a nonempty compact subset in Y and C a solid pointed convex cone in Z. Let $F : X \times Y \to Z$ be jointly continuous on $K \times E$ and Fréchet differentiable with respect to the first variable on K. Assume that the map $x \mapsto F(x, y)$ is C-invex on K with respect to $\eta : K \times K \to X$ for each $y \in E$, and that F' and η satisfy the following three conditions:*

(i) *$u \mapsto F'(x, y)\eta(u, x)$ is C-quasiconvex on K for each $x \in K$, $y \in E$,*

(ii) *$(x, y) \mapsto F'(x, y)\eta(u, x)$ is jointly continuous on $K \times E$ for each $u \in K$, and*

(iii) *$\eta(x, x) = 0_X$ for each $x \in K$, where 0_X stands for the origin of X.*

Moreover, assume that the following two conditions:

(iv) *For each $u \in K$, the set $F'(x, T(x))(\eta(u, x))$ is included in either $-\operatorname{int} C$ or $(-\operatorname{int} C)^c$, where $T(x)$ is defined in (1).*

(v) *There exist a nonempty compact subset B of X and $\hat{x} \in (B \cap K)$ such that for any pair $x \in (K \cap B^c)$ and $y \in T(x)$,*

$$F'(x, y)(\eta(\hat{x}, x)) \in -\operatorname{int} C.$$

Then problem (P) has at least one solution.

Proof. By the assumptions, $T(x) \neq \emptyset$ for any $x \in K$. Define a multifunction $G : K \to 2^X$ by

$$G(u) := \{x \in K : F'(x,y)(\eta(u,x)) \notin -\operatorname{int} C,\ y \in T(x)\},\ u \in K. \tag{4}$$

In order to prove the theorem, by Theorem 3.1, it is sufficient to show that problem (Q) has at least one solution pair $x_0 \in K$ and $y_0 \in T(x_0)$, and so we only have to show the following three conditions, by Lemma 3.2,

(a) G is a KKM-map,

(b) $G(x)$ is nonempty closed for each $x \in K$, and

(c) there exists $\hat{x} \in K$ such that $G(\hat{x})$ is compact.

First, we prove condition (a). Suppose to the contrary to our claim that there exist $x_1, \ldots, x_m \in K$ and $\alpha_1, \ldots, \alpha_m \geq 0$ such that

$$\bar{x} := \sum_{i=1}^{m} \alpha_i x_i \notin \bigcup_{i=1}^{m} G(x_i),\ \sum_{i=1}^{m} \alpha_i = 1.$$

Then, $\bar{x} \in K$ and $\bar{x} \notin G(x_i)$ for all $i = 1, \ldots, m$ and hence for $y \in T(\bar{x})$,

$$F'(\bar{x}, y)(\eta(x_i, \bar{x})) \in -\operatorname{int} C \text{ for all } i = 1, \ldots, m.$$

Then, by assumptions (i) and (iii), we have,

$$F'(\bar{x}, y)\left(\eta\left(\sum_{i=1}^{m} \alpha_i x_i, \bar{x}\right)\right) \in -\operatorname{int} C,$$

and so

$$0_Z \in -\operatorname{int} C.$$

This is a contradiction. Thus, we deduce that G is a KKM-map.

Next, we show that condition (b) holds. Let $u \in K$. Since $T(u) \neq \emptyset$ and $\eta(u,u) = 0_X$, we have $G(u) \neq \emptyset$. Let $\{x_n\} \subset G(u)$ such that $x_n \to x \in K$. Since $x_n \in G(u)$ for all $n \in N$,

$$F'(x_n, y_n)(\eta(u, x_n)) \in (-\operatorname{int} C)^c \text{ for some } y_n \in T(x_n).$$

As $\{y_n\}$ in compact set E, without loss of generality, we can assume that there exists $y \in E$ such that $y_n \to y$. Since F is continuous on both variables, by Lemma 2.2, $y \in T(x)$. Since assumption (ii) and the closedness of $(-\operatorname{int} C)^c$, we have

$$F'(x_n, y_n)(\eta(u, x_n)) \to F'(x, y)(\eta(u, x)) \in (-\operatorname{int} C)^c.$$

Hence $x \in G(u)$, i.e., $G(u)$ is a closed set.

Finally in order to prove condition (c), we show that $G(\hat{x})$ is a compact set for $\hat{x} \in B$ in assumption (v). Since $G(\hat{x})$ is closed and B is compact, it is sufficient to show $G(\hat{x}) \subset B$. Suppose to the contrary that there exists $\bar{x} \in G(\hat{x})$ such that $\bar{x} \notin B$. Since $\bar{x} \in G(\hat{x})$,

$$F'(\bar{x}, \bar{y})(\eta(\hat{x}, \bar{x})) \notin -\mathrm{int}\, C \text{ for some } \bar{y} \in T(\bar{x}).$$

However, this is a contradiction to assumption (v).

Consequently, it follows that $\bigcap_{x \in K} G(x) \neq \emptyset$. Thus, there exists $x_0 \in \bigcap_{x \in K} G(x)$ and then, by condition (iv), for any $x \in K$ and $y \in T(x_0)$ in (1),

$$F'(x_0, y)(\eta(x, x_0)) \notin -\mathrm{int}\, C.$$

Therefore there exists $x_0 \in K$ and $y_0 \in T(x_0)$ such that

$$F'(x_0, y_0)(\eta(x, x_0)) \notin -\mathrm{int}\, C, \tag{5}$$

for all $x \in K$. Therefore, there exists at least one solution of problem (Q). □

Remark 3. The assumption (iv) in Theorem 3.3, or something like that similar condition, is crucial but it has not been assumed in Theorem 2.3 in [7] and Theorem 2.3 in [8]. Without the assumption (iv) the vector $y_0 \in T(x_0)$ satisfying condition (5) in the proof of Theorem 3.3 might depend upon $x \in K$, and hence there does not necessarily exist (x_0, y_0) with $y_0 \in T(x_0)$ guaranteeing condition (5) for all vector $x \in K$.

Example 1. Let $X = R$, $Y = R$ and $Z = R^2$, and let $K := [0,1]$, $E := [0,1]$, $C := \{0_Z\} \cup \{(z_1, z_2) : z_1 + z_2 > 0\}$, $\eta(u, x) := u - x$ and $f(x, y) := \big((1-y)(2x-1) + \frac{1}{4}y(1-y)((1-2x)^2 - 1), \frac{1}{4}y(1-y)((2x-1)^2 - 1) + y(1-2x)\big)^t$.

In this setting, it is easily seen that all assumptions in Theorem 3.3 except (iv) are satisfied and that f has no C-saddle-point on $K \times E$.

Remark 4. If C is closed and the map $x \mapsto F(x, y)$ is Fréchet differentiable C-convex on K for each $y \in E$, then Theorem 3.3 is reduced to Theorem 2.3 in [8]. Moreover, if X, Y and Z are restricted to finite-dimensional Euclidean spaces, respectively, and if the ordering cone C is the non-negative cone, then Theorem 2.3 in [8] is reduced to Theorem 2.3 in [7].

Corollary 3.4. *Assume that K is a nonempty compact convex set in X, and that the other conditions are the same as Theorem 3.3 except removing assumption (v). Then problem (P) has at least one solution.*

Example 2. Let $K := [1,2]\times[1,2]$, $E := [0,2]\times[0,2]$ and $Z := R^2$ with the ordering cone $C := \{(z_1,z_2) \in R^2 : z_1, z_2 \geq 0\}$. Let $F(x,y) := (2x_1^3(y_1-1)^2, -3x_2^3y_2^2)^t$ and $\eta(u,x) := \frac{1}{3}\left(\frac{u_1^3-x_1^3}{x_1^2}, \frac{u_2^3-x_2^3}{x_2^2}\right)^t$. Then all conditions of Corollary 3.4 are satisfied, and then problem (P) has at least one solution. However, F is not C-convex on K with respect to the first variable; this example shows that Theorem 3.3 is a generalization of the result in [8].

Corollary 3.5. *If $T(x)$ is single-valued in Theorem 3.3, and the other conditions are the same except removing assumption (iv), then problem (P) has at least one solution.*

Corollary 3.6. *We can replace the condition (i) of Theorem 3.3 by the following condition*

(i)' *$\eta(\cdot,x)$ is affine on K for each $x\in K$, i.e., $\eta(tu_1+(1-t)u_2,x) = t\eta(u_1,x) + (1-t)\eta(u_2,x)$ for any $u_1,u_2\in K$ and $t\in R$.*

Proof. By Theorem 3.3, it is sufficient to show that $F'(x,y)\eta(\cdot,x)$ is affine for each $x\in K$ and $y\in E$, because any affine function is also C-quasiconvex. Since $F'(x,y)$ is a linear operator for each $x\in K$ and $y\in E$, we see that $F'(x,y)\eta(\cdot,x)$ is affine. □

Remark 5. By the following Proposition, we see that if the ordering cone C is closed, then Corollary 3.6 above and Theorem 2.3 in [8] are equivalent with each other.

Proposition 3.7. *Let K be a nonempty convex set in X, C a pointed convex cone in Z and f a vector-valued function from X to Z. Assume that f is Fréchet differentiable and C-invex on K with respect to some η satisfying the following two conditions:*

(i) *$\eta(\cdot,x)$ is affine on K for each $x\in K$,*

(ii) *$\eta(x,x) = 0_X$ for each $x\in K$.*

Then f is also C-convex on K.

Proof. Since f is C-invex with respect to η and C is a convex cone, we have for any $x,y\in K$ and $\lambda\in[0,1]$

$$\lambda f(x) + (1-\lambda)f(y) - f(u) - (\lambda f'(u)(\eta(x,u)) + (1-\lambda)f(u)(\eta(y,u))) \in C,$$

where $u := \lambda x + (1-\lambda)y$. Then, by condition (i), we have

$$\lambda f(x) + (1-\lambda)f(y) - f(u) - f'(u)(\eta(u,u)) \in C.$$

Hence, by condition (ii), we have

$$\lambda f(x) + (1-\lambda)f(y) - f(\lambda x + (1-\lambda)y) \in C.$$

□

Remark 6. In this paper, we can replace a normed space Y by a topological vector space.

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