

On vector equilibrium problems: Remarks on a general existence theorem and applications

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Abstract

We focus our attention on generalized vector equilibrium problems. In particular, we formulate a general and unified existence theorem, present an analysis for the assumptions used in this result, and give some applications to vector variational inequalities, vector complementarity problems and vector optimization.

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1 Introduction and preliminaries

In the present paper, we investigate the following form of generalized vector equilibrium problems:

Find $\bar{x} \in C$ such as to satisfy $\varphi(\bar{x}, y) \not\subseteq K(\bar{x})$ for all $y \in C$, (GVEP)

where

- X and E are topological vector spaces,
- C is a nonempty closed convex subset of X ,
- $\varphi : C \times C \rightarrow 2^E$ is a set-valued map, and

- K is a set-valued map from C to E .

In particular, we formulate a general existence result, present an analysis for the assumptions used, and give some applications.

Let us first recall and introduce some concepts about convexity and semi-continuity for set-valued vector maps. We denote by $\mathcal{F}(C)$ the set of all finite subsets of C .

1.1 Generalized set-valued convexity

Definition 1.1. We say that $\psi : C \times X \rightarrow 2^E$ is diagonally quasi convex in its first argument relatively to L , in short L -diagonally quasi convex in x , if for any A in $\mathcal{F}(C)$ and any y in $co(A)$, we have $\psi(A, y) \not\subseteq L(y)$.

Let us consider the scalar case, that is when $E = \mathbb{R}$ and ψ is a numeric valued function. If we set $L :=]-\infty, \gamma[$ (resp. $L :=]\gamma, +\infty[$) for all $x \in C$, where $\gamma \in \overline{\mathbb{R}}$, then the L -diagonal quasi convexity of ψ in x collapses to the following:

$$\forall A \in \mathcal{F}(C) \forall y \in co(A) : \min_{x \in A} \psi(x, y) \leq \gamma \text{ (resp. } \max_{x \in A} \psi(x, y) \geq \gamma),$$

which is the γ -diagonal quasi convexity (resp. concavity) considered by Zhou and Chen in [20].

Let us also recall the definition of some concepts of convexity for set-valued maps related to moving cones in E . Let $(P(y))_{y \in C}$ be a family of proper cones in E with $int P(y) \neq \emptyset$ for all $y \in C$.

Definition 1.2. [14] $\psi : C \times C \rightarrow 2$ is said to be right P_y -convex if, for all $x_1, x_2, y \in C$ and all $\lambda \in [0, 1]$, one has

$$\psi(\lambda x_1 + (1 - \lambda)x_2, y) \subseteq \lambda \psi(x_1, y) + (1 - \lambda) \psi(x_2, y) - P(y).$$

Definition 1.3. [3] ψ is said to be right P_y -quasiconvex¹ if, for all $x_1, x_2, y \in C$ and all $\lambda \in [0, 1]$, one has either

$$\psi(\lambda x_1 + (1 - \lambda)x_2, y) \subseteq \psi(x_1, y) - P(y)$$

or

$$\psi(\lambda x_1 + (1 - \lambda)x_2, y) \subseteq \psi(x_2, y) - P(y).$$

¹In [3], it is called P_y -quasiconvex-like, and the left P_y -quasiconvexity below is called P_y -quasiconvex in [13]. When P is constant, it is called type-(v) P -properly quasiconvex in [9].

It should be mentioned that a right P_y -convex set-valued map is not necessarily right P_y -quasiconvex. When F is single-valued and P is constant, we recover the P -convexity and the properly quasi- P -convexity of [8], respectively. All these concepts reduce to the ordinary convexity and quasiconvexity when considering the scalar case.

Furthermore, Konnov and Yao [13] work with following set-valued concept.

Definition 1.4. [13] ψ is said to be left P_y -quasiconvex if, for all $x_1, x_2, y \in C$ and all $\lambda \in [0, 1]$, one has either

$$\begin{aligned} \psi(x_1, y) &\subseteq \psi(\lambda x_1 + (1 - \lambda)x_2, y) + P(y) \\ &\text{or} \\ \psi(x_2, y) &\subseteq \psi(\lambda x_1 + (1 - \lambda)x_2, y) + P(y). \end{aligned}$$

1.2 Transfer semicontinuity

Definition 1.5. We say that φ is K -transfer semicontinuous in y if for any $(x, y) \in C \times X$ with $\varphi(x, y) \subset K(y)$, there exist an element $x' \in C$ and an open $V \subset X$ containing y such that $\varphi(x', v) \subset K(v)$ for all $v \in V$.

An equivalent statement is as follows: for each $y \in X$ satisfying

$$\forall x \in C \exists y_\alpha \rightarrow y \text{ in } X : \varphi(x, y_\alpha) \not\subseteq K(y_\alpha),$$

we have

$$\varphi(x, y) \not\subseteq K(y) \quad \forall x \in C.$$

Definition 1.6. φ is said to be transfer u.s.c. (resp. l.s.c.) in y if, for any $(x, y) \in C \times X$ and any open $N \subset E$ with $\varphi(x, y) \subset N$ (resp. $\varphi(x, y) \cap N \neq \emptyset$) there exist an element $x' \in C$ and an open $V \subset X$ such that $\varphi(x', v) \subset N$ (resp. $\varphi(x', v) \cap N \neq \emptyset$) for all $v \in V$.

Note here that an u.s.c. (resp. l.s.c.) set-valued bi-map in one of its arguments is obviously transfer u.s.c. (resp. l.s.c.). Moreover, the relationship between the two concepts of transfer continuity, introduced above, is the following.

Proposition 1.1. If φ is transfer u.s.c. in y with compact values and if K has an open graph, then φ is K -transfer semicontinuous in y .

Proof. Let $y \in X$ such that for any $x \in C$ there is a net (y_α) converging to y in X with $\varphi(x, y_\alpha) \not\subseteq K(y_\alpha)$. Therefore, for every $x \in C$, there exists $z_\alpha \in \varphi(x, y_\alpha)$ with $z_\alpha \notin K(y_\alpha)$. Passing to a subnet, if necessarily, we can certainly assume that, for every $x \in C$ there exists $z \in \varphi(x, y)$ such that $z_\alpha \rightarrow z$. Otherwise, there exists $x' \in C$ such that, for every $z \in \varphi(x', y)$, there is an open $N_z \subset E$ containing z and satisfying, for every $y'_\alpha \rightarrow y$ in X and for every $z'_\alpha \in \varphi(x', y'_\alpha)$, $z'_\alpha \notin N_z$ for α sufficiently large. Since $\varphi(x', y) \subset \bigcup_{z \in \varphi(x', y)} N_z$ and $\varphi(x', y)$ is compact, $\varphi(x', y) \subset N = \bigcup_{i=1}^{i=n} N_{z_i}$ for some finite subset $\{z_1, \dots, z_n\}$ in $\varphi(x', y)$. Hence, there exists $x'' \in C$ such that, for every $y'_\alpha \rightarrow y$ in X and for α large enough, $\varphi(x'', y'_\alpha) \subset N$ because of the transfer upper semicontinuity of φ . But, we can also write, for some $y_\alpha \rightarrow y$ in X and some $z_\alpha \in \varphi(x'', y_\alpha)$, $z_\alpha \notin N$ for α large enough. It follows that $\varphi(x'', y_\alpha) \not\subseteq N$ for some $y_\alpha \rightarrow y$ in X , which is a contradiction. The same argument is still applicable to show the assertion on compact subsets of X . This completes our proof. ■

2 The existence theorem

For the proof of the main result, we need the following KKM lemma, which is a particular case of the extended version of Fan-KKM theorem [11, Theorem 2.1].

Lemma 2.1. *Assume that $F, G : C \rightarrow 2^X$ satisfy*

- (0i) $G(x) \subset F(x)$ for all $x \in C$,
- (i) G is a KKM map,
- (ii) F is transfer closed-valued,
- (iii) there is a compact subset B in X such that for each $A \in \mathcal{F}(C)$ there exists a compact convex B_A in X containing A such that

$$\bigcap_{x \in B_A \cap C} \overline{G(x)} \cap B_A \subset B.$$

Then $\bigcap_{x \in C} B \cap F(x) \neq \emptyset$.

Recall that G is a KKM map when $co(A) \subset \bigcup_{x \in A} G(x)$ for any subset $A \in \mathcal{F}(C)$, and that F is transfer closed-valued whenever $y \in X$ and $x \in C$ such that $y \notin F(x)$ there exists $x' \in C$ such that $y \notin F(x')$.

Now we are in position to state the following existence theorem for (GVEP) in topological vector spaces.

Theorem 2.1. *Suppose that*

(A0) $\psi(x, y) \not\subseteq L(y) \implies \varphi(x, y) \not\subseteq K(y) \forall x, y \in C$;

(A1) ψ is L -diagonally quasi-convex in x ;

(A2) φ is K -transfer semicontinuous in y ;

(A3) *there is a nonempty compact subset B in X such that for each $A \in \mathcal{F}(C)$ there is a compact convex $B_A \subset X$ containing A such that, for every $y \in B_A \setminus B$, there exists $x \in B_A \cap C$ with*

$$y \in \text{int}_X \{v \in X : \psi(x, v) \subseteq L(v)\}.$$

Then there exists $\bar{y} \in B$ such that $\varphi(x, \bar{y}) \not\subseteq K(\bar{y})$ for all $x \in C$.

Proof. Define two set-valued maps $F, G : C \rightrightarrows X$ as follows:

$$F(x) = \{y \in X : \varphi(x, y) \not\subseteq K(y)\}$$

and

$$G(x) = \{y \in X : \psi(x, y) \not\subseteq L(y)\}.$$

The existence of a generalized vector equilibrium for φ in B with respect to K is now equivalent to

$$\bigcap_{x \in C} B \cap F(x) \neq \emptyset$$

Hence, we need only to check the assumptions of Lemma 2.1 for F and G .

(0i) Let $x \in C$ and let $y \in G(x)$; then $\psi(x, y) \not\subseteq L(y)$; it follows, by assumption (A0), that $\varphi(x, y) \not\subseteq K(y)$; thus $y \in F(x)$.

(i) Let $A \in \mathcal{F}(C)$ and let $y \in coA$. By assumption (A1), we have $\psi(A, y) \not\subseteq L(y)$. Hence there exists $x \in A$ such that $\psi(x, y) \not\subseteq L(y)$; that is $y \in G(x)$. Hence $co(A) \subset \bigcup_{x \in A} G(x)$. We conclude that G is a KKM map.

- (ii) Let us consider $(x, y) \in C \times X$ with $y \notin F(x)$. Suppose, contrary to our claim, that $y \in cl_X F(x')$ for all $x' \in C$. Therefore, for every $x' \in C$, there is a net (y_α) in X converging to y and satisfying $\varphi(x', y_\alpha) \notin K(y_\alpha)$. It follows that $\varphi(x', y) \notin K(y)$ for all $x' \in C$ since φ is K -transfer semicontinuous in y , and hence that $y \in F(x')$ for all $x' \in C$, a contradiction. We conclude that F is transfer closed-valued.
- (iii) It suffices to see that assumption (A3) leads to

$$B_A \setminus B \subset \bigcup_{x \in B_A \cap C} int_X G(x);$$

hence $B_A \cap \bigcap_{x \in B_A \cap C} cl_X G(x) \neq \emptyset$.

The proof is complete. ■

Theorem 2.1 generalizes [3, Theorem 2.1], which is proved by means of a Fan-Browder fixed point theorem - an immediate consequence of the Fan-KKM theorem. As we will mention in the 'Assumptions analysis' subsection, our hypotheses are more general than those used in [3]. Besides, the scalar version of this result extends [17, Theorem 4] (we take $C_A = co(A \cup R) \cap X$ where R is the convex compact which contains C in [17, Theorem 4, (4iii)]). Other particular cases are [1, Theorem 2], [19, Theorem 2.1], [20, Theorem 2.11], [18, Theorem 1], [15, Corollary 2.4], [16, Lemma 2.1] and [4, Theorem 2]. The origin of this kind of results goes back to Ky Fan [7]. His classical minimax inequality can be deduced from our result by setting $E = \mathbb{R}$, $K(x) = \mathbb{R}_+$ and $\varphi(x, y) = \psi(x, y) = f(x, y) - \sup_{x \in C} f(x, x)$ for all $x, y \in C$.

3 Assumptions analysis

In this section, we analyze the requirements of Theorem 2.1 by presenting different situations where assumptions (A0)-(A3) hold true. Let $(P(y))_{y \in C}$ a family of proper convex closed cones on E with $int P(y) \neq \emptyset$ for all $y \in C$.

- Pseudomonotonicity

Remark 3.1. (A0) holds provided one of the following statements is satisfied.

(a) $\varphi = \psi$ and $K = L$.

(b) $X = C$, $K(y) = -L(y) = -\text{int } P(y)$, $\psi(x, y) = \varphi(y, x)$ for all $x, y \in C$, and φ is P_x -pseudomonotone, that is,

$$\varphi(x, y) \not\subseteq \text{int } P(x) \implies \varphi(y, x) \not\subseteq -\text{int } P(x) \quad \forall x, y \in C.$$

- **Convexity.** Suppose that $X = C$.

Remark 3.2. (A1) holds provided that, for every $y \in C$, one has either

(a) $\psi(y, y) \not\subseteq L(y)$, and

(b) the set $\{x \in C : \psi(x, y) \subseteq L(y)\}$ is convex,

or

(i) $L(y) = -\text{int } P(y)$ and $P(y)$ is w -pointed²,

(ii) $\psi(y, y) \subseteq P(y)$, and

(iii) ψ is left P_y -quasiconvex.

Indeed, if we suppose in contrary that there exist $x_1, \dots, x_n \in C$, $\lambda_1, \dots, \lambda_n \in [0, 1]$, and $y = \sum_{j=1}^n \lambda_j x_j$ such that $\sum_{i=1}^n \lambda_i = 1$ and

$$\psi(x_i, y) \subseteq L(y) \tag{1}$$

for each $i = 1, \dots, n$. For the first assertion, assumption (b) shows that $\psi(y, y) \subseteq L(y)$, which contradicts (a).

For the second one, (iii) implies

$$\psi(x_{i_0}, y) \subseteq \psi(y, y) + P(y) \subseteq P(y) + P(y) \subseteq P(y)$$

for some $i_0 \in \{1, \dots, n\}$; this contradicts (1) since $P(y)$ is w -pointed. The assertion is proved.

Moreover condition (b) in Remark 3.2 is satisfied provided that $L(y)$ is convex, and for all $x_1, x_2 \in C$ and all $\lambda \in [0, 1]$,

$$\psi(\lambda x_1 + (1 - \lambda)x_2, y) \subseteq \lambda \psi(x_1, y) + (1 - \lambda) \psi(x_2, y),$$

In case of $L(y) = -\text{int } P(y)$, (b) holds if either

²A cone P is w -pointed if $P \cap -\text{int } P = \emptyset$.

- ψ is right P -convex, or
- ψ is right P -quasiconvex.

- Continuity

Remark 3.3. (A2) holds provided that one of the following statement is satisfied.

- (a) φ is (transfer) u.s.c. in y with compact values and if K has an open graph.
- (b) φ is (transfer) u.s.c. in y and $K(x) = O$ for all $x \in C$, where O is an open subset of E .
- (c) For each $x \in C$, the set $\{y \in X : \varphi(x, y) \not\subseteq K(y)\}$ is closed in C .

While the assertion “(c) \Rightarrow (A2)” is obvious, the other assertions follow from Proposition 1.1.

- Coercivity. Recall first the following definition.

Definition 3.1. We will say that a set-valued map $\phi : D \rightarrow 2^E$ is K -compact if the set $cl\{y \in D : \phi(y) \not\subseteq K(y)\}$ is compact in Y .

It has to be observed that this definition is equivalent to the following fact: there exists a nonempty compact subset B in D such that $\phi(y) \subseteq K(y)$ for all $y \in D \setminus B$.

Remark 3.4. (A3) holds if one of the following statements is satisfied.

- (a) X is compact.
- (b) There is $x_0 \in C$ such that $\psi(x_0, \cdot)$ is K -compact.
- (c) There is a nonempty compact subset B in X such that for each $y \in X \setminus B$ there exists $x \in B \cap C$ such that $\psi(x, y) \subseteq L(y)$.
- (d) There is a nonempty compact subset B of X and a compact convex subset $B' \in X$ such that for each $y \in X \setminus B$ there exists $x \in B' \cap C$ with

$$y \in \text{int}_X \{v \in X : \psi(x, v) \subseteq L(v)\}.$$

Besides, when the classical assumption (c) of Remark 3.3 is satisfied, (A3) holds provided that

- (e) there is a nonempty compact subset B in X such that for each $A \in \mathcal{F}(C)$ there is a compact convex $B_A \subset X$ containing A such that, for every $y \in C \setminus B$, there exists $x \in B_A \cap C$ with $\varphi(x, y) \subseteq K(y)$.

4 Applications

Throughout this section, and otherwise stated, C is supposed to be a nonempty closed convex subset in a t.v.s X , and E to be a real topological vector space. Assume also we are given a family $\{P(x) : x \in C\}$ of convex cones in E with $\text{int } P(x) \neq \emptyset$ for all $x \in C$. Also, we denote by $L(X, E)$ the space of all linear bounded applications from X to E .

4.1 Generalized vector variational like-inequalities

Let us consider a set-valued operator T from C into $L(X, E)$, and a bifunction η from C to itself. We write for $\Pi \subset L(X, E)$ and $x \in C$, $\langle \Pi, x \rangle = \{\langle \pi, x \rangle : \pi \in \Pi\}$, where $\langle \pi, x \rangle$ denotes the evaluation of the linear mapping π at x which is supposed to be continuous on $L(X, E) \times X^3$.

The generalized vector variational inequality problem (GVVLIP) takes the following form:

$$\text{Find } \bar{x} \in C \text{ such that } \langle T\bar{x}, \eta(y, \bar{x}) \rangle \not\subseteq -\text{int } P(\bar{x}) \forall y \in C.$$

Thus (GVVLIP) is a particular case of (GVEP) if we take

$$\varphi(x, y) = \{\langle t, \eta(y, x) \rangle : t \in Tx\}.$$

For the reader's convenience, we recall the following definitions.

Definition 4.1. (see [2])

1) T is said to be η -pseudomonotone if, for all $x, y \in C$,

$$\langle Tx, \eta(y, x) \rangle \not\subseteq -\text{int } P(x) \Rightarrow \langle Ty, \eta(y, x) \rangle \not\subseteq -\text{int } P(x).$$

³A typical situation when X is a reflexive Banach and E is a Banach

2) T is said to be V -hemicontinuous if for any $x, y \in C$ and $t \in]0, 1[$ $T(tx + (1-t)y) \rightarrow T(y)$ as $t \rightarrow 0_+$ (that is, for any $z_t \in T(tx + (1-t)y)$ there exists $z \in Ty$ such that for any $a \in C$, $\langle z_t, a \rangle \rightarrow \langle z, a \rangle$ as $t \rightarrow 0_+$).

It has to be observed that when T is single-valued, we recover the hemicontinuity used in [5]. If $\eta(x, y) = x - y$ for all $x, y \in C$, η is dropped from the definition of pseudomonotonicity.

The linearization lemma plays a significant role in variational inequalities. Chen [5] extended this lemma to the single-valued vector case. For our need in this subsection, we state it in the set-valued case by using standard Minty's argument. Consider the following problem, which may be seen as a dual problem of (GVVLIP),

$$\text{Find } \bar{x} \in C \text{ such that } \langle Ty, \eta(y, \bar{x}) \rangle \not\subseteq -\text{int } P(\bar{x}) \quad \forall y \in C. \quad (\text{GVVLIP}^*)$$

Lemma 4.1. *Suppose that $\eta(\cdot, x)$ is affine and $\eta(x, x) = 0$ for each $x \in C$. If T is η -pseudomonotone and V -hemicontinuous then (GVVLIP) and (GVVLIP*) are equivalent.*

Proof. We only need to show that every solution of (GVVLIP*) is a solution of (GVVLIP); the reverse assertion follows clearly from the η -pseudomonotonicity of T . To this end, let $\bar{x} \in C$ such that

$$\langle Ty, \eta(y, \bar{x}) \rangle \not\subseteq -\text{int } P(\bar{x}) \quad \forall y \in C.$$

Let $y \in C$ be fixed and set $y_t = ty + (1-t)\bar{x}$ for $t \in [0, 1]$, which is in the convex subset C . Then

$$\langle Ty_t, \eta(y_t, \bar{x}) \rangle \not\subseteq -\text{int } P(\bar{x}).$$

Hence

$$\langle Ty_t, \eta(y, \bar{x}) \rangle \not\subseteq -\text{int } P(\bar{x}).$$

By the V -hemicontinuity of T and the closedness of $E \setminus (-\text{int } P(\bar{x}))$, it follows that

$$\langle T\bar{x}, \eta(y, \bar{x}) \rangle \not\subseteq -\text{int } P(\bar{x}).$$

The last assertion holds for an arbitrary $y \in C$. The lemma is proved. ■

As an application of Theorem 2.1, we are now in position to formulate the following existence result for (GVVLLIP).

Theorem 4.1. *Suppose that*

- (i) *the mapping $\text{int } P(\cdot)$ has an open graph in $C \times L(X, E)$;*
- (ii) *for each $x \in C$, $\eta(\cdot, x)$ is affine, $\eta(x, \cdot)$ is continuous and $\eta(x, x) = 0$;*
- (iii) *T is compact valued, η -pseudomonotone and V -hemicontinuous;*
- (iv) *there is a nonempty compact subset B in C such that for each $A \in \mathcal{F}(C)$ there is a compact convex $B_A \subset C$ containing A such that, for every $y \in B_A \setminus B$, there exists $x \in B_A \cap C$ with*

$$y \in \text{int}_C \{v \in C : \langle Tv, \eta(x, v) \rangle \subseteq -\text{int } P(v)\}.$$

Then (GVVLLIP) has at least one solution, which is in B .

Proof. Set $\varphi(x, y) = \langle Tx, \eta(x, y) \rangle$, $\psi(x, y) = \langle Ty, \eta(x, y) \rangle$ and $K(x) = -\text{int } P(x)$ for all $x, y \in C$. Let us show that the assumptions of Theorem 2.1 are satisfied:

(A0) follows clearly from the η -pseudomonotonicity of T .

(A1) First we have,

- $\psi(x, x) = \langle Tx, \eta(x, x) \rangle = 0 \notin -\text{int } P(x)$ for all $x \in C$, and
- for y being fixed in C , the set

$$\{x \in C : \langle Ty, \eta(x, y) \rangle \subseteq -\text{int } P(y)\}$$

is convex since $\eta(\cdot, y)$ is affine.

Thus conditions (a) and (b) of Remark 3.2 hold, which lead to (A1).

(A2) Fix x in C and let (y_α) be a net on C converging to $y \in C$ such that

$$\langle Tx, \eta(x, y_\alpha) \rangle \not\subseteq -\text{int } P(y_\alpha).$$

Therefore there exists $z_\alpha \in Tx$ such that

$$\langle z_\alpha, \eta(x, y_\alpha) \rangle \not\subseteq -\text{int } P(y_\alpha).$$

Since Tx is compact then, passing to a subnet, if necessarily, we may assume that z_α converges to $z \in Tx$. By the continuity of the map $\eta(x, \cdot)$ and $\langle \cdot, \cdot \rangle$, we get $\langle z_\alpha, \eta(x, y_\alpha) \rangle \rightarrow \langle z, \eta(x, y) \rangle$. Hence, according to (i), we obtain

$$\langle z, \eta(x, y) \rangle \not\subseteq -\text{int } P(y);$$

thus

$$\langle Tx, \eta(x, y) \rangle \not\subseteq -\text{int } P(y).$$

(A3) follows easily from (iv). Therefore, from Theorem 2.1, there exists $\bar{x} \in B$ such that

$$\langle Ty, \eta(y, \bar{x}) \rangle \not\subseteq -\text{int } P(\bar{x}) \quad \forall y \in C.$$

Hence (GVLLIP*) has a solution in B , which completes the proof of the theorem according to Lemma 4.1. \blacksquare

The coercivity condition (iv) is better than that formulated in [12], namely, there is a compact B of X and $x_o \in C \cap B$ such that

$$\langle Tx_o, \eta(x_o, y) \rangle \subset -\text{int } P(y) \quad \forall y \in C \setminus B.$$

4.2 Vector complementarity problems

A natural extension of the classical nonlinear complementarity problem, (CP) for short, is considered as follows. Let T be a single-valued operator from C , which is supposed to be a convex closed cone, to $L(X, E)$. The vector complementarity problem considered in this subsequent, (VCP) for short, is to find $\bar{x} \in C$ such that

$$\langle T(\bar{x}), \bar{x} \rangle \notin \text{int } P(\bar{x}), \text{ and } \langle T(\bar{x}), y \rangle \notin -\text{int } P(\bar{x}) \text{ for all } y \in C.$$

This problem collapses to (CP) when $E = \mathbb{R}$ and $P(x) = \mathbb{R}_+$ for all $x \in C$.

By means of vector variational inequalities, we can formulate the following existence theorem for (VCP).

Theorem 4.2. *Suppose that*

- (i) *the set-valued map $\text{int } P(\cdot)$ has an open graph in $C \times L(X, E)$;*
- (ii) *T is pseudomonotone and hemicontinuous;*
- (iv) *there is a nonempty compact subset B in C such that for each $A \in \mathcal{F}(C)$ there is a compact convex $B_A \subset C$ containing A such that, for every $y \in B_A \setminus B$, there exists $x \in B_A \cap C$ with*

$$y \in \text{int}_C \{v \in C : \langle Tv, x - v \rangle \in -\text{int } P(v)\}.$$

Then (VCP) has at least one solution, which is in B .

Proof. It is clear that all the assumptions of Theorem 4.1 are satisfied with $\eta(x, y) = x - y$ for all $x, y \in C$. Therefore there exists $\bar{x} \in B$ such that

$$\langle T\bar{x}, z - \bar{x} \rangle \notin -\text{int } P(\bar{x}) \quad \forall z \in C. \quad (2)$$

Since C is a convex cone, then setting in (2), $z = 0$ and $z = y + \bar{x}$ for an arbitrary $y \in C$, we get respectively

$$\langle T\bar{x}, \bar{x} \rangle \notin \text{int } P(\bar{x}) \text{ and } \langle T\bar{x}, y \rangle \notin -\text{int } P(\bar{x}).$$

Hence we conclude that \bar{x} is also a solution to (VCP). ■

4.3 Vector optimization

Here, to convey an idea about the use of vector variational-like inequalities in vector optimization which involves smooth vector mappings, we prove the existence of solutions to weak vector optimization problems, (WVOP) for short, by considering the concept of invexity. Let us state the problem as follows.

$$\text{Find } \bar{x} \in C \text{ such that } \phi(y) - \phi(\bar{x}) \notin -\text{int } P \text{ for all } y \in C, \quad (\text{WVOP})$$

where $\phi : C \rightarrow E$ be a given vector-valued function and P is a given convex cone in E .

Let $\eta : C \times C \rightarrow X$ be a given function, and denote by $\nabla\phi$ the Fréchet derivative of ϕ once the latter is assumed to be Fréchet differentiable.

Definition 4.2. Suppose that ϕ is Fréchet differentiable. ϕ is said to be P -invex with respect to η if

$$\phi(y) - \phi(x) - \langle \nabla\phi(x), \eta(y, x) \rangle \in P \quad \forall x, y \in C.$$

If $E = \mathbb{R}$ and $P = \mathbb{R}^+$ then we recover the real invexity introduced by Hanson [10] and later labeled so by Craven [6] due to its "invariance" under "convex" transformations.

Theorem 4.3. Suppose that P is a convex cone in E with $\text{int } P \neq \emptyset$, and let $\phi : C \rightarrow E$ be a Fréchet differentiable mapping. Assume that

- (i) $\langle \nabla\phi(x), \eta(y, x) \rangle \notin -\text{int } P$ implies $\langle \nabla\phi(y), \eta(y, x) \rangle \notin -\text{int } P$ for all $x, y \in C$;
- (ii) ϕ is P -invex with respect to η ;
- (iii) $\nabla\phi$ is hemicontinuous;
- (iv) for each $x \in C$, $\eta(\cdot, x)$ is affine, $\eta(x, \cdot)$ is continuous and $\eta(x, x) = 0$;
- (v) there is a compact subset B in X such that for every finite subset A in C there is a compact convex $C_A \subset X$ containing A such as to satisfy, for every $y \in C \setminus B$, there exists $x \in C_A \cap C$ with $\langle \nabla\phi(x), \eta(x, y) \rangle \in -\text{int } P$.

Then (WVOP) has at least one solution.

Proof. First, by virtue of Theorem 4.1 with $T := \nabla\phi$, we get

$$\langle \nabla\phi(\bar{x}), \eta(y, \bar{x}) \rangle \notin -\text{int } P \quad \forall y \in C.$$

Then the P -invexity of ϕ allows us to conclude. ■

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