Nihonkai Math. J.
Vol. 11(2000), 97-101

CHARACTERIZATION OF POSINORMAL OPERATORS

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Abstract

In this paper, we give a characterization of posinormal operators. And we introduce a new class of operators and show M-paranormality of such operators.

1. Introduction.

In [1], H.C. Rhaly Jr. introduced and studied posinormal operators. He showed a characterization of posinormality and spectral properties of posinormal operators. Moreover, he gave many fruitful examples of posinormal operators for the Casáro operator. In this paper, first we give another characterization of posinormality. Next we introduce p-posinormal operators and give a characterization of it. Finally, we show that p-posinormal operators are M-paranormal.

Let $\mathcal{H}$ be a complex separable Hilbert space and $\mathrm{B}(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. In what follows, an operator means a bounded linear operator on $\mathcal{H}$. An operator T is said to be a posinormal operator if there exists a positive operator $P \in B(\mathcal{H})$ such that $TT^* = T^*PT$. Here, an operator P just mentioned above is called an interrupter of T. The set of all posinormal operators in $\mathrm{B}(\mathcal{H})$ is denoted by $P(\mathcal{H})$ (see [1]). Let p be $0 < p \leq 1$. An operator T is said to be a p-hyponormal operator if $(T^*T)^p \geq (TT^*)^p$. An operator $T \in B(\mathcal{H})$ is said to be p-posinormal if $(TT^*)^p \leq \lambda^2(T^*T)^p$ for some positive number λ. We denote the set of all p-posinormal operators by p-$P(\mathcal{H})$. By Rhaly's characterization of posinormality, we can see that 1-posinormal operators are posinormal (cf. Theorem B). According to [3], an operator $T \in B(\mathcal{H})$ is said to be M-paranormal if there exists $\lambda > 0$ such that $\|Tx\|^2 \leq \lambda\|T^2x\|$ for all $x \in \mathcal{H}$ with $\|x\| = 1$. Let call P an interrupter of T with degree p if $|T^*|^{2p} = |T|^pP|T|^p$.

2. Result.

First, we need the following theorems.

Theorem A (Fujii, Nakamoto and Watanabe[2]). *Let* $A \geq 0$ *and* $B \geq 0$. *If* T *satisfies* $T^*T \leq A^2$ *and* $TT^* \leq B^2$, *then inequality*

$$|(T|T|^{p+q-1}x, y)| \leq \|A^p x\| \|B^q y\|$$

holds for all $x, y \in \mathcal{H}$, $0 \leq p, q \leq 1$ *with* $p + q \geq 1$.

Theorem B (H.C. Rhaly[1], Th.2.1). *For* $T \in B(\mathcal{H})$, *the following statements are equivalent :*
(1) T *is posinormal* ;
(2) Ran $T \subset$ Ran T^*;
(3) $TT^* \leq \lambda^2 T^*T$ *for some* $\lambda > 0$; *and*
(4) *there exists* $S \in B(\mathcal{H})$ *such that* $T = T^*S$.
Moreover, if (1),(2),(3) *and* (4) *hold, then there is a unique operator* S *such that*
(a) $\|S\| = \inf\{\mu |\ TT^* \leq \mu T^*T\}$;
(b) $\ker T = \ker S$; *and*
(c) Ran$S \subset \overline{\text{Ran}}T$.

By Theorems A and B, we have the following lemma.

Lemma 1. *If* $T \in P(\mathcal{H})$, *then there exists* $\lambda > 0$ *such that* $|(T|T|^{p+q-1}x, y)| \leq \||T|^p x\| \|\lambda^q |T|^q y\|$ *for all* $x, y \in \mathcal{H}$, $0 \leq p, q \leq 1$ *with* $p + q \geq 1$.

Proof. In Theorem A, put $A = |T|$ and $B = \lambda|T|$. Since $T^*T = |T|^2$ and $T^*T \leq \lambda^2 |T|^2 = B^2$ by Theorem B, we have

$$|(T|T|^{p+q-1}x, y)| \leq \||T|^p x\| \|\lambda^q |T|^q y\|$$

for all $x, y \in \mathcal{H}$, $0 \leq p, q \leq 1$ with $p + q \geq 1$. It completes the proof.

The following theorem is another characterization of posinormality, which is different from Rhaley's one.

Theorem 2. *T is posinormal if, and only if, there exists $\lambda > 0$ such that $|(T|T|x, y)| \leq \lambda \||T|x\| \||T|y\|$ for all $x, y \in \mathcal{H}$.*

Proof. Assume that T is posinormal. By Lemma 1 we have

$$|(T|T|^{p+q-1}x, y)| \leq \lambda^p \||T|^p x\| \||T|^q y\|$$

for all $x, y \in \mathcal{H}$, $0 \leq p, q \leq 1$ with $p + q \geq 1$. Letting $p = q = 1$, we have

$$|(T|T|x, y)| \leq \lambda \||T|x\| \||T|y\|. \tag{1}$$

Conversely, assume that (1) holds. Let $T = U|T|$ be the polar decomposition of T. For any $y \in \mathcal{H}$, we put $x = U^*y$. Then by (1) it holds

$$|(U|T|^2U^*y, y)| \leq \lambda \||T|U^*y\| \||T|y\|.$$

Since $(U|T|^2U^*y, y) = (T^*y, T^*y) = \|T^*y\|^2$ and $\||T|y\| = \|Ty\|$, we have $\|T^*y\|^2 \leq \lambda \|T^*y\| \|Ty\|$. Hence $\|T^*y\|^2 \leq \lambda^2 \|Ty\|^2$, that is, $TT^* \leq \lambda^2 T^*T$. By Theorem B, we have $T \in P(\mathcal{H})$. This completes the proof.

Proposition 3. *If T is posinormal, then T is* M*-paranormal.*

Proof. By the hypothesis, there exists a positive number λ such that $\|T^*x\| \leq \lambda \|Tx\|$ for all $x \in \mathcal{H}$ with $\|x\| = 1$. Hence $\|Tx\|^2 = (T^*Tx, x) \leq \|T^*Tx\| \|x\| = \|T^*Tx\| \leq \lambda \|T^2x\|$, that is, T is M-paranormal.

By the definitions of p-hyponormal and p-posinormal operators, we can easily have the following results.

Proposition 4.
(1) *If T is posinormal, then $T \in p\text{-}P(\mathcal{H})$ for every p $(0 < p \leq 1)$.*
(2) *If T is p-hyponormal, then $T \in p\text{-}P(\mathcal{H})$.*
(3) *$T \in p\text{-}P(\mathcal{H})$ if and only if there exists $\lambda > 0$ such that $\||T^*|^p x\| \leq \lambda \||T|^p x\|$ for all $x \in \mathcal{H}$.*

Proposition 5. *For $T \in B(\mathcal{H})$, the following statements are equivalent:*
(1) $T \in p\text{-}P(\mathcal{H})$;
(2) $\mathrm{Ran}(|T^*|^p) \subset \mathrm{Ran}(|T|^p)$;
(3) *there exists* $S \in B(\mathcal{H})$ *such that* $|T^*|^p = |T|^p S$; *and*
(4) *there exists a positive operator such that* $|T^*|^{2p} = |T|^p P |T|^p$.

Theorem 6. *Let $T = U|T| \in B(\mathcal{H})$ be the polar decomposition of T. Then $T \in p\text{-}P(\mathcal{H})$ if, and only if, there exists a positive number λ such that*

$$|(U|T|^{2p}x, y)| \leq \lambda \| |T|^p x \| \| |T|^p y \| \tag{2}$$

for all $x, y \in \mathcal{H}$.

Proof. Suppose that (2) holds. For any $y \in \mathcal{H}$, put $x = U^* y$. Then

$$|(U|T|^{2p}U^* y, y)| \leq \lambda \| |T|^p U^* y \| \| |T|^p y \|.$$

Since $\| |T|^p U^* y \|^2 = (U|T|^{2p}U^* y, y) = (|T^*|^{2p} y, y) = \| |T^*|^p y \|^2$, we have $\| |T^*| y \| \leq \lambda \| |T|^p y \|$, that is, $T \in p\text{-}P(\mathcal{H})$.
Next, suppose $T \in p\text{-}P(\mathcal{H})$. Then, by (3) of Proposition 4,

$$\begin{aligned} |(U|T|^{2p}x, y)| &= |(|T|^{2p}x, U^* y)| \leq \| |T|^p x \| \| |T|^p U^* y \| \\ &= \| |T|^p x \| \| |T^*|^p y \| \leq \lambda \| |T|^p x \| \| |T|^p y \|. \end{aligned}$$

Hence (2) holds. This completes the proof.

Finally, for the proof of Theorem 7, we need the following theorem.

Theorem C (McCarthy[4]). *Let $A \geq 0$. Then, for all $x \in \mathcal{H}$,*
(1) $(Ax, x)^r \leq \|x\|^{2(r-1)} (A^r x, x)$ *if* $1 \leq r$; *and*
(2) $(Ax, x)^r \geq \|x\|^{2(r-1)} (A^r x, x)$ *if* $0 \leq r \leq 1$.

Theorem 7. *If $T \in p\text{-}P(\mathcal{H})$, then T is* M*-paranormal.*

Proof. Let $T = U|T|$ be the polar decomposition of T. Since $T \in p\text{-}P(\mathcal{H})$, it is clear that

$$U|T|^{2p}U^* \leq \lambda^2 |T|^{2p} \quad \text{and} \quad |T|^{2p} \leq \lambda^2 U^* |T|^{2p} U.$$

Hence, by Theorem C, for all $x \in \mathcal{H}$ with $\|x\| = 1$,

$$\lambda^2\|T^2x\|^2 = \lambda^2(T^*TTx, Tx) = \lambda^2\left((|T|^{2p})^{\frac{1}{p}}\frac{Tx}{\|Tx\|}, \frac{Tx}{\|Tx\|}\right)$$

$$\geq \lambda^2\frac{(|T|^{2p}Tx, Tx)^{\frac{1}{p}}\|Tx\|^2}{\|Tx\|^{\frac{2}{p}}} = \lambda^2\frac{(U^*|T|^{2p}U|T|x, |T|x)^{\frac{1}{p}}\|Tx\|^2}{\|Tx\|^{\frac{2}{p}}}$$

$$\geq \frac{(|T|^{2p+2}x, x)^{\frac{1}{p}}\|Tx\|^2}{\|Tx\|^{\frac{2}{p}}} = \frac{((|T|^2)^{p+1}x, x)^{\frac{1}{p}}\|Tx\|^2}{\|Tx\|^{\frac{2}{p}}}$$

$$\geq \frac{(|T|^2x, x)^{\frac{1}{p}+1}\|Tx\|^2}{\|Tx\|^{\frac{2}{p}}} = \|Tx\|^4.$$

Therefore, $\|Tx\|^2 \leq \lambda\|T^2x\|$, that is, T is M-paranormal.

Acknowledgement. The author would like to express his thanks to the referee for his helpful and many valuable suggestions.

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Received September 15, 1999 Revised March 3, 2000