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Shifts with two generators on the hyperfinite II_1-factor

Keiichi Watanabe

1. Introduction

R. T. Powers([6]) introduced a concept of a shift on the hyperfinite II_1-factor $\mathfrak{R}$, which is an identity preserving $*$-endmorphism σ such that $\cap_{k=1}^{\infty}\sigma^k(\mathfrak{R}) = \mathbb{C}1$. He defined the index of σ as the Jones index $[\mathfrak{R}:\sigma(\mathfrak{R})]$. He discussed on conjugacy or on outer conjugacy of binary shifts which is a class of shifts of index two on $\mathfrak{R}$. A shift σ on $\mathfrak{R}$ is said to be a binary shift if there is a unitary element $u \in \mathfrak{R}$ with $u^2 = 1$ which satisfies $\mathfrak{R} = \{\sigma^k(u) \ ; \ k \geq 0\}''$ and $u\sigma^k(u) = \pm\sigma^k(u)u$ for $k \in \mathbb{N}$. There are uncountably many non conjugate, at least countably many non outer conjugate binary shifts on $\mathfrak{R}$. Enomoto, Choda and Watatani considered a general shift σ on a group von Neumann algebra $R_m(G)$ on a group G twisted by a multiplier m such that the shift σ is induced from a shift on G, and they generalized results of Powers' binary shifts. Bures and Yin also independently studied the shifts as mentioned above.

In this paper we consider a class of shifts which have two generators in a sense. At first, we shall show that a shift with two generators assumed some conditions is a shift induced from the ones on the restricted direct product $G = \Pi_{i=0}^{\infty}\mathbb{Z}_2 \oplus \mathbb{Z}_2$. Secondly, we give a sufficient condition for a multiplier m which makes $R_m(G)$ a factor. Then, under some condition, we shall express the relative commutant algebras $\sigma^n(R_m(G))' \cap R_m(G)$ in terms of the three sequences which determine the multiplier m.

2. Shifts with two generators

Let G be a countable discrete group. A multiplier m on G is a map from $G \times G$ into $\mathbb{T} = \{z \in \mathbb{C} ; |z| = 1\}$ such that $m(1_G, x) = m(x, 1_G) = 1$ and $m(x,y)m(xy,z) = m(x,yz)m(y,z)$ for $x,y,z \in G$. We denote by λ_m the left regular projective representation of G associated with m on the Hilbert space $l^2(G)$. That is, $(\lambda_m(x)\xi)(y) = m(x, x^{-1}y)\xi(x^{-1}y)$ for $\xi \in l^2(G)$.

Powers defined binary shifts using a generator(one unitary operator) and commutation relations between two images of the generator under the shift.

Definition 2.1. Let σ be a shift on the hyperfinite II_1-factor $\mathfrak{R}$. Then we say that σ has two generators if there exist two unitary operators u and v in $\mathfrak{R}$ which satisfy the following conditions (1),(2) and (3);

(1) $u^2 = 1$ and $v^2 = 1$

(2) $\{\sigma^i(u), \sigma^j(v); i,j \geq 0\}'' = \mathfrak{R}$

(3) $\sigma^i(u)\sigma^j(v) = \sigma^j(v)\sigma^i(u)$ or $-\sigma^j(v)\sigma^i(u)$,

$u\sigma^i(u) = \sigma^i(u)u$ or $-\sigma^i(u)u$ and

$v\sigma^i(v) = \sigma^i(v)v$ or $-\sigma^i(v)v$.

A function $a:\mathbb{Z} \longrightarrow \{0,1\}$ is called a signature sequence if $a(n) = a(-n)$ for any $n \in \mathbb{Z}$. For any shift σ which has two generators, there exist three signature sequences b, a_u and a_v such that

$$\sigma^i(u)\sigma^j(v) = (-1)^{b(i-j)}\sigma^j(v)\sigma^i(u),$$

$$u\sigma^i(u) = (-1)^{a_u(i)}\sigma^i(u)u \quad \text{and}$$

$$v\sigma^i(v) = (-1)^{a_v(i)}\sigma^i(v)v.$$

Here we should note that it need not be $b(0) = 0$.

Due to a characterization of group shifts by Bures and Yin([1; Proposition 2.1]), a shift which has two generators is a group shift. Enomoto, Choda and Watatani introduced in [4] a notion of a commutation relator in order to generalize results of Powers' binary shifts to the shifts induced by the shifts on a countable discrete group. They showed that there exists a one to one correspondence between the set of all commutation relators and a class of bicharacter on the restricted direct product of the group. Here we need a slightly modified their results as follows.

Definition 2.2. Let G be a countable discrete group. Let $G_i = G$ for $i = 0,1,2,\cdots$ and $X = \amalg_{i=0}^{\infty} G_i$, the restricted direct product. Let $\tilde{G}_i$ be the set of elements $(x_j)_{j\geq 0}$ in X such that $x_j = 1_G$ for $j \neq i$. We denote by ρ_i the canonical isomorphism from G to

$\tilde{G}_i$ in X. A function $a:\mathbb{Z} \times G \times G \longrightarrow \mathbb{T}$ is called a commutation relator if

(1) $a(n;gh,k) = a(n;g,k)a(n;h,k)$

(2) $a(n;g,hk) = a(n;g,h)a(n;g,k)$

(3) $a(n;g,h) = \overline{a(-n;h,g)}$

for any $n \in \mathbb{Z}$, $g,h,k \in G$.

Let Comm(G) be the set of all commutation relators. Let σ be the canonical shift on X. Let $\mathrm{Bich}(\dot{X},\mathbb{T})$ be the set of all functions $m:X \times X \longrightarrow \mathbb{T}$ such that

(a) m is a bicharacter

(b) $m(\sigma(x),\sigma(y)) = m(x,y)$

(c) $m(\rho_i(g),\rho_j(h)) = 1$ if $i < j$.

Let u be a mapping from $X_0 = \cup_{i=0}^{\infty}\rho_i(G)$ to the unitary group U(B(H)) of B(H) on a separable Hilbert space H. Then u is called a generator representation with respect to an element a in Comm(G) if u satisfies the following conditions;

(i) $u(\rho_i(g))u(\rho_j(h)) = a(i-j;g,h)u(\rho_j(h))u(\rho_i(g))$ for $g,h \in G$

(ii) $u(\rho_i(g))u(\rho_i(h)) = a(0;g,h)u(\rho_i(gh))$ for $g,h \in G$.

Remark 2.3. Under the situation in [4], we should consider that $a(0;g,h) = 1$, $g,h \in G$ provided that G is abelian. Indeed, the restriction of u to $\rho_i(G)$ is defined to be a unitary representation in [4; Definition 3.2]. It is the only difference of the above definition of Comm(G) from the one in [4] that a commutation relator is defined on $\{0\} \times G \times G$ and may not equal 1. Also the only difference between two definitions of $\mathrm{Bich}(X,\mathbb{T})$ is the conditions "if $i<j$" or "if $i\leq j$" in (c).

Then we have the following lemmas. Proofs of those are similar to the proofs of the corresponding lemmas in [4], and it should be omitted.

Lemma 2.4(cf.[4;Lemma 3.1]). There is a one to one correspondence between Comm(G) and Bich(X,$\mathbb{T}$) such that

$$m(x,y) = \prod_{\substack{(i,j)\in(\mathbb{N}\cup\{0\})\times(\mathbb{N}\cup\{0\})\\ i\geq j}} a(i-j;x(i),y(j))$$

$$a(n;g,h) = \begin{cases} m(\rho_i(g),\rho_j(h))/m(\rho_j(h),\rho_i(g)) & \text{if } n = i-j > 0 \\ m(\rho_i(g),\rho_i(h)) & \text{if } n = 0. \end{cases}$$

Lemma 2.5(cf.[4;Lemma 3.3]). There exists a one to one correspondence between the set of all projective representations u from X into U(B(H)) with respect to elements in Bich(X,$\mathbb{T}$) and the set of all generator representations from X_0 into U(B(H)).

From now on, we consider the case of $G = \mathbb{Z}_2\oplus\mathbb{Z}_2$. Let a_1, a_2, b be signature sequences such that $a_1(0) = a_2(0) = 0$ and b(0) may not equal 0.

Let a be an element in Comm($\mathbb{Z}_2\oplus\mathbb{Z}_2$) determined by the following conditions;

$$a(n;\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}1\\0\end{pmatrix}) = (-1)^{a_1(n)}$$

$$a(n;\begin{pmatrix}0\\1\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix}) = (-1)^{a_2(n)}$$

$$a(n;\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix}) = a(n;\begin{pmatrix}0\\1\end{pmatrix},\begin{pmatrix}1\\0\end{pmatrix}) = (-1)^{b(n)}.$$

Then there exists an element m in Bich(X,$\mathbb{T}$) by Lemma 2.4, and the canonical $\mathbb{Z}_2\oplus\mathbb{Z}_2$-shift σ associated with m has two generators

$\lambda_m(e_0^1)$ and $\lambda_m(e_0^2)$, where we put $e_j^1 = \rho_j(\binom{1}{0})$ and $e_k^2 = \rho_k(\binom{0}{1})$.

Proposition 2.6. Let σ is a shift which has two generators u and v. Put $\mathfrak{P} = \{\sigma^i(u); i\geq 0\}''$ and $\mathfrak{Q} = \{\sigma^j(v); j\geq 0\}''$. Assume the following conditions (1) and (2);

(1) $\mathfrak{P} \cap \mathfrak{Q} = \mathbb{C}1$

(2) $\dim(\mathfrak{P}) = \infty$ and $\dim(\mathfrak{Q}) = \infty$.

Then σ is conjugate to a $\mathbb{Z}_2\oplus\mathbb{Z}_2$-shift.

Proof. Put $S = \{u,v\}$ and denote by $G_\sigma(S)$ the group of unitaries generated by $\{\sigma^k(S); k\geq 0\}$. Let G be the quotient group $G_\sigma(S)/G_\sigma(S)\cap\mathbb{C}$. Due to [1;Proposition 2.1], it is enough to prove that G is isomorphic to $\amalg_{i=0}^\infty \mathbb{Z}_2\oplus\mathbb{Z}_2$. However we can define a group isomorphism Φ from G to $\amalg_{i=0}^\infty \mathbb{Z}_2\oplus\mathbb{Z}_2$ as follows;

$$\Phi([u^{k(0)}\sigma(u)^{k(1)}\cdots\sigma^n(u)^{k(n)}v^{l(0)}\sigma(v)^{l(1)}\cdots\sigma^n(v)^{l(n)}])$$
$$= k(0)e_0^1 + \cdots + k(n)e_n^1 + l(0)e_0^2 + \cdots + l(n)e_n^2.$$

Indeed, if $[u^{k(0)}\sigma(u)^{k(1)}\cdots\sigma^n(u)^{k(n)}v^{l(0)}\sigma(v)^{l(1)}\cdots\sigma^n(v)^{l(n)}] = 1_G$, then there is a scalar $\alpha \in \mathbb{C}$ such that $u^{k(0)}\sigma(u)^{k(1)}\cdots\sigma^n(u)^{k(n)} = \alpha v^{l(0)}\sigma(v)^{l(1)}\cdots\sigma^n(v)^{l(n)}$. By the assumption (1), we have $u^{k(0)}\sigma(u)^{k(1)}\cdots\sigma^n(u)^{k(n)} \in \mathbb{C}1$. Suppose that there exists a number $k(i) \neq 0$. Putting $i_0 = \max\{i; k(i) \neq 0, 0<i\leq n\}$, we can easily see $\mathfrak{P} = \{\sigma^i(u); 0\leq i<i_0\}''$. This contradicts to $\dim(\mathfrak{P}) = \infty$. Thus we have $k(0) = \cdots = k(n) = l(0) = \cdots = l(n) = 0$. Thus G is isomorphic to $\amalg_{i=0}^\infty \mathbb{Z}_2\oplus\mathbb{Z}_2$. □

Remark 2.7. In the assumption of the previous proposition, if $u \in \sigma(\mathfrak{R})$ then we have $\dim(\mathfrak{P}) = \infty$. Indeed, one can immediately see that the set $\{u,\sigma(u),\sigma^2(u),\cdots\}$ is linearly independent.

3. Factor condition and relative commutant algebras

Let $G = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ and $X = \amalg_{i=0}^{\infty} G_i$, where $G_i = G$. Let s be the canonical shift on X which is defined by $s(e_i^1) = e_{i+1}^1$ and $s(e_i^2) = e_{i+1}^2$. Let a_1, a_2 and b be signature sequences which need not be $b(0) = 0$. Then there is an element m in $\mathrm{Bich}(X,\mathbb{T})$ corresponding to a_1, a_2 and b by Lemma 2.4. We denote by ρ the character of the second exterior product $X \wedge X$ which is induced by m via

$$\rho(g \wedge h) := m(g,h)\overline{m(h,g)}, \quad g,h \in X.$$

In this section we study some suffitient conditions in terms of a_1, a_2 and b for m to be non-degenerate, i.e, $\lambda_m(X)''$ is a factor. Also we shall give some examples to show that some analogue of the results of n-shifts are false.

Lemma 3.1. Suppose $g = \sum_{j=0}^{\infty} g_j^1 e_j^1 + g_j^2 e_j^2 \in X$, where $g_j^1, g_j^2 \in \mathbb{Z}_2$ Then $\rho(g \wedge s^m(X)) = 1$ if and only if

$$\left.\begin{aligned} &\textstyle\sum_{j=0}^{\infty} g_j^1 a_1(k-j) + g_j^2 b(k-j) = 0 \\ &\textstyle\sum_{j=0}^{\infty} g_j^1 b(k-j) + g_j^2 a_2(k-j) = 0 \end{aligned}\right. \qquad \text{for all } k \geq m.$$

Proof. Since $s^m(X)$ is generated by $\{e_k^1, e_l^2; k,l \geq m\}$, $\rho(g \wedge s^m(X)) = 1$ if and only if $\rho(g \wedge e_k^1) = \rho(g \wedge e_l^2) = 1$ for all k, $l \geq m$. On the other hand, we have $\rho(e_j^1 \wedge e_k^1) = m(e_j^1, e_k^1)\overline{m(e_k^1, e_j^1)} = a_1(k-j; \binom{1}{0}, \binom{1}{0}) = (-1)^{a_1(k-j)}$. Similarly, $\rho(e_j^2 \wedge e_k^2) = (-1)^{a_2(k-j)}$ and $\rho(e_j^1 \wedge e_k^2) =$

$\rho(e_j^2 \wedge e_k^1) = (-1)^{b(k-j)}$. Then it follows that

$$\rho(g \wedge e_k^1) = \rho(\Sigma_{j=0}^{\infty} g_j^1 e_j^1 \wedge e_k^1 + g_j^2 e_j^2 \wedge e_k^1) = \Pi_{j=0}^{\infty} \rho(e_j^1 \wedge e_k^1)^{g_j^1} \rho(e_j^2 \wedge e_k^1)^{g_j^2}$$
$$= (-1)^{\Sigma_{j=0}^{\infty} g_j^1 a_1(k-j) + g_j^2 b(k-j)}.$$

The second equality follows similarly. This completes the proof. □

Price showed that a multiplier m on $\amalg_{i=0}^{\infty} \mathbb{Z}_2$ associated with a signature sequence a is nondegenerate if and only if a is not periodic.

Example 3.2. Let $a_1 = a_2 = (0,0,1,0,1,0,1,\cdots)$ and $b = (1,0,0,\cdots)$. Namely, $a_1(0) = a_2(0) = 0$, $a_1(2j-1) = a_2(2j-1) = 0$, $a_1(2j) = a_2(2j) = 1$, $b(0) = 1$ and $b(j) = 0$ for all $j \in \mathbb{N}$. Then a_1, a_2 and b are not periodic at all, however the corresponding element $m \in \mathrm{Bich}(X,\mathbb{T})$ is degenerate. Indeed, putting $g = e_0^1 + e_0^2 + e_2^1 + e_2^2$, it is easy to see that $\rho(g \wedge X) = 1$.

A signature sequence a is called to be essentially periodic if there exist integers $p > 0$ and $N \geq 0$ such that $a(i+p) = a(i)$ for all $i \geq N$(cf.[5;Definition 3.1]).

Lemma 3.3. Let a_1, a_2 and b be nonzero signature sequences which may not be $b(0) = 0$. Let m be an element in $\mathrm{Bich}(X,\mathbb{T})$ corresponding to a_1, a_2 and b. If $\lambda_m(X)''$ is not a factor, then a_1, a_2 and b are essentially periodic.

This proposition can be proved by a similar way to some proof of the corresponding result for n-shifts(see [1] for instance). However we give a proof for completeness.

Proof. Since $\lambda_m(X)''$ is not a factor, there exists a nonzero element g in X such that $\rho(g\wedge X) = 1$. Put $i_1 = \min\{j; g_j^1 \neq 0\}$, $j_1 = \max\{j; g_j^1 \neq 0\}$, $i_2 = \min\{j; g_j^2 \neq 0\}$ and $j_2 = \max\{j; g_j^2 \neq 0\}$. Also we put $d_1 = \max\{i\in\mathbb{N}; a_1(i) \neq 0\}$, $d_2 = \max\{i\in\mathbb{N}; a_2(i) \neq 0\}$ and $d_b = \max\{i\in \mathbb{N}\cup\{0\}; b(i) \neq 0\}$. Then by Lemma 3.1, g satisfies the following formulae;

$$(*) \quad \sum_{j=i_1}^{j_1} g_j^1 a_1(k-j) + \sum_{j=i_2}^{j_2} g_j^2 b(k-j) = 0$$

$$(**) \quad \sum_{j=i_2}^{j_2} g_j^2 a_2(k-j) + \sum_{j=i_1}^{j_1} g_j^1 b(k-j) = 0$$

for all integer $k \geq 0$. We may assume that $j_1 \leq j_2$. Especially, when $k > d_b + j_2$, we have $g_j^1 b(k-j) = 0$ for any $j \geq 0$. It follows from the formula (**) that

$$a_2(k-i_2) = \Phi(a_2(k-i_2-1), a_2(k-i_2-2), \cdots, a_2(k-j_2))$$

for $k > d_b + j_2$, where Φ is a fixed linear function. Let $r = j_2-i_2$ and assume first that $r > 0$. Then we have

$$a_2(k) = \Phi(a_2(k-1), a_2(k-2), \cdots, a_2(k-r)) \quad \text{for } k > d_b + r.$$

Since there are only finitely many distinct values for an n-tuple from $\mathbb{Z}_2$, there is a number $l > 0$ such that $(a_2(d_b+r), \cdots, a_2(d_b+1)) = (a_2(d_b+r+l), \cdots, a_2(d_b+1+l))$. Then we have $a_2(d_b+r+1) = \Phi(a_2(d_b+r), \cdots, a_2(d_b+1)) = \Phi(a_2(d_b+r+l), \cdots, a_2(d_b+1+l)) = a_2(d_b+r+1+l)$. Thus $a_2(k) = a_2(k+l)$ for $k > d_b+r$.

Now assume that $r = 0$. It follows from the formula (**) that $g_{i_2}^2 a_2(k-i_2) = 0$ for $k > d_b+j_2$. Hence we have $a_2(k) = 0$ for all k

$> d_b$.

Similarly we obtain the essentially periodicity of a_1 and b. This completes the proof. □

When the signature sequences have only finite supports, we can realize the relative commutant algebras $\sigma^n(\mathfrak{R})' \cap \mathfrak{R}$ concretely as well as n-shifts. The result contains a suffitient condition for a_1, a_2 and b such that m makes $\lambda_m(X)''$ a factor.

Theorem 3.4. Let a_1, a_2 and b be nonzero signature sequences whose supports are finite. Put $d_1 = \max\{i \in \mathbb{N}; a_1(i) \neq 0\}$, $d_2 = \max\{i \in \mathbb{N}; a_2(i) \neq 0\}$ and $d_b = \max\{i \in \mathbb{N} \cup \{0\}; b(i) \neq 0\}$. Let m be the multiplier associated with a_1, a_2 and b. Consider the following conditions (i) and (ii);

(i) $d_b \leq d_1 \leq d_2$ and $d_b < d_2$

(ii) $d_1 \leq d_2 \leq d_b$ and $d_1 < d_b$.

If either (i) or (ii) is satisfied, then $\lambda_m(X)''$ is a hyperfinite II_1-factor. Moreover, for the case of (i), we have

$\sigma^n(\mathfrak{R})' \cap \mathfrak{R} = \mathbb{C}1$ if $0 \leq n \leq d_1$,

$\sigma^n(\mathfrak{R})' \cap \mathfrak{R} = \{\lambda_m(e_i^1) ; 0 \leq i \leq n-d_1-1\}''$ if $d_1+1 \leq n \leq d_2$ and

$\sigma^n(\mathfrak{R})' \cap \mathfrak{R} = \{\lambda_m(e_i^1), \lambda_m(e_j^2) ; 0 \leq i \leq n-d_1-1, 0 \leq j \leq n-d_2-1\}''$ if $d_2+1 \leq n$.

For the case of (ii), we have

$\sigma^n(\mathfrak{R})' \cap \mathfrak{R} = \mathbb{C}1$ if $0 \leq n \leq d_b$ and

$\sigma^n(\mathfrak{R})' \cap \mathfrak{R} = \{\lambda_m(e_i^1), \lambda_m(e_j^2) ; 0 \leq i,j \leq n-d_b-1\}''$ if $d_b+1 \leq n$.

Proof. For a subgroup Y of X, we put $D_Y = \{g \in X ; \rho(g \wedge h) = 1$

for all $h \in Y\}$. We shall prove the theorem only for the case of (i) (a similar proof works for the case of (ii)). Due to [1; Corollary 1.3], it is sufficient to show the following;

$$D_{s^n(X)} = \{0\} \quad \text{if} \quad 0 \le n \le d_1 ,$$

$$D_{s^n(X)} = [e_i^1 ; 0 \le i \le n-d_1-1] \quad \text{if} \quad d_1+1 \le n \le d_2$$

$$D_{s^n(X)} = [e_i^1, e_j^2 ; 0 \le i \le n-d_1-1, 0 \le j \le n-d_2-1] \quad \text{if} \quad d_2+1 \le n,$$

where we denote by $[e_i^1 ; 0 \le i \le n-d_1-1]$ the subgroup of X generated by e_i^1, $0 \le i \le n-d_1-1$. It is clear that $D_{s^n(X)}$ containes the right side of the above formula in each case. We shall show the reverse inclusion. At first, we assume that the condition (i) is satisfied. Suppose that there exists a nonzero element g in $D_{s^n(X)}$. Put $j_1 = \max\{j; g_j^1 \neq 0\}$ and $j_2 = \max\{j; g_j^2 \neq 0\}$. It follows from Lemma 3.1 that

$$(*) \quad \textstyle\sum_{j=0}^{j_1} g_j^1 a_1(k-j) + \sum_{j=0}^{j_2} g_j^2 b(k-j) = 0$$

$$(**) \quad \textstyle\sum_{j=0}^{j_2} g_j^2 a_2(k-j) + \sum_{j=0}^{j_1} g_j^1 b(k-j) = 0$$

for each integer $k \ge n$.

Step(1). Let $0 \le n \le d_1$. If $j_1 \le j_2$, then we can apply the formula (**) for $k = d_2 + j_2 \ge d_1 \ge n$. It follows from $d_b < d_2$ that all terms in the formula (**) except of $g_{j_2}^2 a_2(d_2)$ equal to 0. Hence $g_{j_2}^2 a_2(d_2) = 0$. Since $a_2(d_2) = 1$, we have $g_{j_2}^2 = 0$. This is a contradiction. If $j_1 > j_2$, then we can apply the formula (*) for $k = d_1 + j_1 \ge n$. Then all terms in (*) except of $g_{j_1}^1 a_1(d_1)$ equal to 0. Hence $g_{j_1}^1 a_1(d_1) = 0$. Since $a_1(d_1) = 1$, we have $g_{j_1}^1 = 0$. This is a contradiction. Therefore we conclude that $g_j^1 = 0$ for all integers $j \geq 0$.

Step(2). Let $d_1+1 \leq n$. Assume that $g_s^2 = 1$ for some $s \geq 0$. Then a similar argument in step(1) immediately yields a contradiction. Therefore $g_j^2 = 0$ for all integers $j \geq 0$. Assume that $g_s^1 = 1$ for some $s \geq n-d_1$. If $j_1 \leq j_2$, we have by the assumption that $j_1 \geq s \geq n-d_1$. Thus we can apply the formula (**) for $k = d_2 + j_2 \geq d_1 + j_1 \geq n$, and we have a contradiction as step(1). If $j_1 > j_2$, considering the formula (*) for $k = d_1 + j_1 \geq n$, we reach a contradiction similarly. Therefore $g_j^1 = 0$ for all integers $j \geq n-d_1$.

Step(3). Let $d_2+1 \leq n$. Assume that $g_s^2 = 1$ for some $s \geq n-d_2$. Then we have $j_2 \geq s \geq n-d_2$. Considering the formula (**) for $k = d_2+ j_2 \geq n$, we have a contradiction as above. Therefore $g_j^2 = 0$ for all integers $j \geq n-d_2$. Clearly these arguments complete the proof. □

Remark 3.5. In the above proposition, we can not drop the condition (i) or (ii). Let $a_1 = a_2 = b$. Then $\rho(g \wedge X) = 1$ for any g in X such that $g_j^1 = g_j^2$ for all $j \geq 0$.

Corollary 3.6. There are at least a countable infinity of outer conjugacy classes among the $\mathbb{Z}_2 \oplus \mathbb{Z}_2$-shifts.

Proof. Fix any integer $d \geq 1$. Let $a_1^d = a_2^d = (0,\cdots,0,\overset{(d)}{1},0,\cdots)$ and $b = (1,0,0,\cdots)$. Namely, $a_1^d(d) = a_2^d(d) = 1$ and $a_1^d(i) = a_2^d(i) = 0$ for $i \neq d$, $b(0) = 1$ and $b(i) = 0$ for $i \neq 0$. Then $d_b < d_1 = d_2$. Denote by σ_d the $\mathbb{Z}_2 \oplus \mathbb{Z}_2$-shift associated with a_1^d, a_2^d and b.

It follows from the proceeding theorem that $\sigma_d^n(\mathfrak{R})' \cap \mathfrak{R} = \mathbb{C}1$ if $n \le d$ and $\sigma_d^n(\mathfrak{R})' \cap \mathfrak{R} \ne \mathbb{C}1$ if $n \ge d+1$. Since the relative commutant algebras are invariant under outer conjugacy, σ_{d_1} is not outer conjugate to σ_{d_2} if $d_1 \ne d_2$. Thus we have countable infinity of outer conjugacy classes among the $\mathbb{Z}_2 \oplus \mathbb{Z}_2$-shifts. This completes the proof. □

Finally, we shall consider a converse of Proposition 3.3 and Theorem 3.4. Let a_1, a_2 and b are signature sequences and let m be the element in $\mathrm{Bich}(X,\mathbb{T})$ associated with a_1, a_2 and b. Assume that $\lambda_m(X)''$ is a factor. Is it necessary that one of a_1, a_2 and b is not essentially periodic? Is it necessary that all of a_1, a_2 and b have finite supports? The following example gives an answer to the above question.

Example 3.7. Let $a_1 = (0,1,0,0,\cdots)$, $a_2 = (0,1,1,1,\cdots)$ and $b = (1,0,0,\cdots)$. Namely, $a_1(1) = 1$ and $a_1(i) = 0$ if $i \ne 1$, $a_2(i) = 1$ if $i \ne 0$ and $a_2(0) = 0$, $b(0) = 1$ and $b(i) = 0$ if $i \ne 0$. Then all of these signature sequences are essentially periodic, and a_2 does not have finite support. However $\lambda_m(X)''$ is a factor.

Proof. It is sufficient to show that $\{g \in X \ ; \ \rho(g \wedge X) = 1\} = \{0\}$. Suppose that there exists a nonzero element $g = \sum_{j=0}^{\infty} g_j^1 e_j^1 + g_j^2 e_j^2$ in X such that $\rho(g \wedge X) = 1$.
Then g satisfies the following conditions;

(1) $g_1^1 + g_0^2 = 0$

(2) $g^1_{k-1} + g^1_{k+1} + g^2_k = 0$ for all $k \geq 1$

(3) $\sum_{\substack{j\geq 0 \\ j\neq k}} g^2_j + g^1_k = 0$ for all $k \geq 0$.

Suppose that $\sum_{j\geq 0} g^2_j = 1$. Considering the formula (3) for sufficiently large k, we have $g^1_k = 1$ for infinitely many k. This is a contradiction, and we have $\sum_{j\geq 0} g^2_j = 0$. Adding this and the equation (3), we have $g^2_k = g^1_k$ for all $k \geq 0$. So we put $g_k = g^1_k = g^2_k$. From the equation (1) and (3) for $k = 1$, one can see that $\sum_{j\geq 2} g_j = 0$. Therefore $g_0 + g_1 = 0$. Also we see from the equation (2) for $k = 1$ that $g_0 + g_2 + g_1 = 0$. Hence $g_2 = 0$.

Suppose that $g_{3(m-1)} = g_{3(m-1)+1}$ and $g_{3(m-1)+2} = 0$ for some positive integer m. It follows from the equation (2) that $g_{k-1} + g_{k+1} + g_k = 0$. Since $g_{3m-1} = 0$, we have $g_{3m-2} = g_{3m}$ from the equation for $k = 3m-1$. Similarly, the equation for $k = 3m$ says that $g_{3m+1} = g_{3m}$. When $k = 3m+1$, we have $g_{3m} + g_{3m+2} + g_{3m+1} = 0$, and so $g_{3m+2} = 0$. By an induction, we conclude that $g_{3m} = g_{3m+1} = g_0$ and $g_{3m+2} = 0$ for all integers $m \geq 0$. Since g belongs to the restricted direct product, we have $g = 0$. This completes the proof.□

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Keiichi Watanabe
Department of Mathematics
Faculty of Science
Niigata University
Niigata, 950-21 Japan

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