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On the maximal toral action on aspherical fibered 4-manifolds over S^1

Dedicated to Professor Shoôrô Araki on his 60th birthday

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Introduction.

In this note,we shall consider piecewise linear closed aspherical 4-manifolds M fibered over S^1. In particular, we consider the following problem ;

(1) Is the center $z(\pi_1(M))$ of the fundamental group of M finitely generated ?

(2) If (1) is affirmative,say,$z(\pi_1(M)) \cong Z^k$ $(k \geq 1)$, then does M admit a topological action of k- dimensional toral group ?

We say that M admits a maximal toral action when (1) and (2) are true.

In this note,we shall prove the following

THEOREM A. *Let M be as above. If rank of the center of the fundamental group $\pi_1(M)$ is greater than 1,then M admits a maximal torus action.*

Concerning the case of rank$(z(\pi_1(M))) = 1$, we shall prove the followings,where F is the the typical fiber F, p the projection and h the attaching map.

THEOREM B. *Assume F is irreducible.If h is a homeomorphism of a finite order, then M admits a T^1 action.*

THEOREM C. *Assume F is irreducible. If F admits a maximal toral action and $p_*(z(\pi_1(M))) = 1$, then M admits a maximal toral action.*

In this note, we shall use the following notations;

1. $\mathbf{Z}$,$\mathbf{R}$ denote the groups of integers or reals respectively.
2. $z(G)$ denotes the center of a group G.
3. $\mathbf{Z}^k$ denotes the direct sum $\mathbf{Z} \oplus \cdots \oplus \mathbf{Z}$ (m times).
4. T^k denotes the group $SO(2) \times \cdots \times SO(2)$ (k times).
5. Let N be a manifold and $h : N \to N$ a homeomorphism. Then N_h denotes the manifold

$$\mathbf{R} \times_{\mathbf{Z}} N,$$

where $\mathbf{Z}$ acts on $\mathbf{R} \times N$ by $n(x,a) = (x - n, h^n(a))$.

6. The sequenc of groups and homomorphisms

$$1 \longrightarrow G_1 \longrightarrow G_2 \longrightarrow G_3 \longrightarrow 1$$

is exact,unless the contrary is stated.

1.Preliminaries

In this section,we shall list some basic facts and some results on 3-dimensional manifolds. Let N be a closed 3-dimensional manifold. We refer basic definitions to [H1] and [W].

PROPOSITION 1.1. *If N is aspherical and its universal covering* $\tilde{N}$ *is homeomorphic to* $\mathbf{R}^3$*,then N has no fake 3-cell.*

PROOF. Let $p : \tilde{N} \to N$ be the projection and D a 3-cell in N. Then D lifts to $\mathbf{R}^3$ and hence D is not fake. ∎

COROLLARY 1. *Let N be as in Proposition 1.1. Then* $N = \wp(N)$ *(*$\wp(N)$ *denotes the Poincare associate of N).*

See [H1]([H1],Chap.10).

COROLLARY 2. *If N is aspherical and fibered over* S^1*, then* $N = \wp(N)$*.*

PROPOSITION 1.2 ([S]). *If N is irreducible, then N is prime.*

PROPOSITION 1.3. *If N is aspherical and fibered over* S^1*,then N is irreducible.*

PROOF. Since N is aspherical,N is prime.It follows from 3.13 in [H1] that N is irreducible. ∎

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We can define a homomorphism ([Z]);

$$\Omega : Homeo(N)/Isot(N) \to Aut(\pi_1(N))/Inn(\pi_1(N))$$

We have the following

THEOREM 1.1([W]). *If N is orientable,irreducible and sufficiently large,then* Ω *is injective.*

COROLLARY. *Let N be as above and* $h : N \to N$ *a homeomorphism such that* $h_*^n : \pi_1(N) \to \pi_1(N)$ *is an inner automorphism. Then there exists a homeomorphism* $h' : N \to N$ *such that* h' *is isotopic to h and* h'^n *is isotopic to the identity.*

THEOREM 1.2([H1],6.6). *If* $H_1(N)$ *is infinite,then N is sufficiently large.*

PROPOSITION 1.4. *If M is aspherical and fiberd over* S^1*,then M is sufficiently large.*

PROOF. Let $M = I \times_{\phi} F$. We have the following exact sequence;

$$\to H_i(M) \to H_{i-1}(F) \xrightarrow{id-\phi_*} H_{i-1}(F) \to H_{i-1}(M) \to H_{i-2}(F) \xrightarrow{id-\phi_*} H_{i-2}(F) \to$$

Since $id - \phi_* : H_0(F) \to H_0(F)$ is zero map,we have the following exact sequence;

$$\to H_2(M) \to H_1(F) \to H_1(F) \to H_1(M) \to H_0(F) \to 0.$$

It follows that the order of $H_1(M) = \infty$. Since M is irreducible, it follows from Theorem 4 that M is sufficiently large. ∎

PROPOSITION 1.5. *If $z(\pi_1(N))$ contains $\mathbf{Z}^2$,then $z(\pi_1(N))$ is finitely generated.*

This follows from Theorem 9.14 in [H1].

EXAMPLE 1. *The following gives periodic homeomorphisms ϕ_i of $S^1 \times S^1$, which are needed in the sequel, and describes the corresponding manifold $N_{\phi_i} = N_i$.*

(1) $\phi_1 = 1. N_1 = S^1 \times S^1 \times S^1$.

$$\pi_1(N_1) =< \alpha, \beta, t : [\alpha, \beta] = [\alpha, t] = [\beta, t] = 1 > .$$

(2) $\phi_2(x, y) = (-x, -y)$. $\phi_2^2 = 1$.

$$\pi_1(N_2) =< \alpha, \beta, t : [\alpha, \beta] = 1, t\alpha t^{-1} = \alpha^{-1}, t\beta t^{-1} = \beta^{-1} > .$$

(3) $\phi_3(x, y) = (x, -y)$. $\phi_3^2 = 1$.

$$\pi_1(N_3) =< \alpha, \beta, t : [\alpha, \beta] = 1,\ t\alpha t^{-1} = \alpha,\ t\beta t^{-1} = \beta^{-1} > .$$

(4) $\phi_4(x, y) = (x + y, -y)$. $\phi_4^2 = 1$.

$$\pi_1(N_4) =< \alpha, \beta, t : [\alpha, \beta] = 1,\ t\alpha t^{-1} = \alpha,\ t\beta t^{-1} = \alpha\beta^{-1} > .$$

EXAMPLE 2. *The following gives all periodic homeomorphisms $(S^1)^3 = T^3$ and describes the corresponding manifold $M_{\phi_i} = M_i$.*

In the following,α, β, γ are generators of $\pi_1(T^3)$ and

$$(*) \quad [\alpha, \beta] = [\beta, \gamma] = [\gamma, \alpha] = 1.$$

(1) $\phi_1 = 1$. $M_1 = T^4$.
(2) $\phi_2(x, y, z) = x + y, -y, -z)$. $\phi_2^2 = 1$.

$$\pi_1(M_2) =< \alpha, \beta, \gamma, t : (*), t\alpha t^{-1} = \alpha, t\beta t^{-1} = \alpha\beta^{-1}, t\gamma t^{-1} = \gamma^{-1} > .$$

(3) $\phi_3(x, y, z) = (x, -y, -z)$. $\phi_3^2 = 1$.

$$\pi_1(M_3) =< \alpha, \beta, \gamma, t : (*), t\alpha t^{-1} = \alpha, t\beta t^{-1} = \beta^{-1}, t\gamma t^{-1} = \gamma^{-1} > .$$

(4) $\phi_4(x, y, z) = (x, y, -z)$. $\phi_4^2 = 1$.

$$\pi_1(M_4) =< \alpha,\beta,\gamma,t : (*), t\alpha t^{-1} = \alpha, t\beta t^{-1} = \beta, t\gamma t^{-1} = \gamma^{-1} > .$$

(5) $\phi_4(x,y,z) = (x+z, y, -z)$. $\phi_5^2 = 1$.

$$\pi_1(M_5) =< \alpha,\beta,\gamma,t : (*), t\alpha t^{-1} = \alpha, t\beta t^{-1} = \beta, t\gamma t^{-1} = \alpha\gamma^{-1} > .$$

(6) $\phi_6(x,y,z) = (x+z, -z, y-z)$. $\phi_6^3 = 1$.

$$\pi_1(M_6) =< \alpha,\beta,\gamma,t : (*), t\alpha t^{-1} = \alpha, t\beta t^{-1} = \gamma, t\gamma t^{-1} = \alpha\beta^{-1}\gamma^{-1} > .$$

(7) $\phi_7(x,y,z) = (x, -z, y-z)$. $\phi_7^3 = 1$.

$$\pi_1(M_7) =< \alpha,\beta,\gamma,t : (*), t\alpha t^{-1} = \alpha, t\beta t^{-1} = \gamma, t\gamma t^{-1} = \beta^{-1}\gamma^{-1} > .$$

(8) $\phi_8(x,y,z) = (x, z, -y)$. $\phi_8^4 = 1$.

$$\pi_1(M_8) =< \alpha,\beta,\gamma,t : (*), t\alpha t^{-1} = \alpha, t\beta t^{-1} = \gamma^{-1}, t\gamma t^{-1} = \beta > .$$

(9) $\phi_9(x,y,z) = (x+z, z, -y)$. $\phi_9^4 = 1$.

$$\pi_1(M_9) =< \alpha,\beta,\gamma,t : (*), t\alpha t^{-1} = \alpha, t\beta t^{-1} = \gamma^{-1}, t\gamma t^{-1} = \alpha\beta > .$$

(10) $\phi_{10}(x,y,z) = (x, -z, y+z)$. $\phi_{10}^6 = 1$.

$$\pi_1(M_{10}) =< \alpha,\beta,\gamma,t : (*), t\alpha t^{-1} = \alpha, t\beta t^{-1} = \gamma, t\gamma t^{-1} = \beta^{-1}\gamma > .$$

(11) $\phi_{11}(x,y,z) = (-x, -y, -z)$. $\phi_{11}^2 = 1$.

$$\pi_1(M_{11}) =< \alpha,\beta,\gamma,t : (*), t\alpha t^{-1} = \alpha^{-1}, t\beta t^{-1} = \beta^{-1}, t\gamma t^{-1} = \gamma^{-1} > .$$

(12) $\phi_{12}(x,y,z) = (-x+y, y, -z)$. $\phi_{12}^2 = 1$.

$$\pi_1(M_{12}) =< \alpha,\beta,\gamma,t : (*), t\alpha t^{-1} = \alpha^{-1}, t\beta t^{-1} = \alpha\beta, t\gamma t^{-1} = \gamma^{-1} > .$$

(13) $\phi_{13}(x,y,z) = (-x+z, -z, y-z)$. $\phi_{13}^3 = 1$.

$$\pi_1(M_{13}) =< \alpha,\beta,\gamma,t : (*), t\alpha t^{-1} = \alpha^{-1}, t\beta t^{-1} = \gamma, t\gamma t^{-1} = \alpha\beta^{-1}\gamma^{-1} > .$$

(14) $\phi_{14}(x,y,z) = (-x, -z, y-z)$. $\phi_{14}^3 = 1$.

$$\pi_1(M_{14}) =< \alpha,\beta,\gamma,t : (*), t\alpha t^{-1} = \alpha^{-1}, t\beta t^{-1} = \gamma, t\gamma t^{-1} = \beta^{-1}\gamma^{-1} > .$$

(15) $\phi_{15}(x,y,z) = (-x, z, -y)$. $\phi_{15}^4 = 1$.

$$\pi_1(M_{15}) =< \alpha,\beta,\gamma,t : (*), t\alpha t^{-1} = \alpha^{-1}, t\beta t^{-1} = \gamma^{-1}, t\gamma t^{-1} = \beta > .$$

(16) $\phi_{16}(x,y,z) = (-x+z, z, -y)$. $\phi_{16}^4 = 1$.

$$\pi_1(M_{16}) =< \alpha,\beta,\gamma,t : (*), t\alpha t^{-1} = \alpha^{-1}, t\beta t^{-1} = \gamma^{-1}, t\gamma t^{-1} = \alpha\beta > .$$

(17) $\phi_{17}(x,y,z) = (-x,-z,y+z)$. $\phi_{17}^6 = 1$.

$$\pi_1(M_{17}) =< \alpha,\beta,\gamma,t : (*), t\alpha t^{-1} = \alpha^{-1}, t\beta t^{-1} = \gamma, t\gamma t^{-1} = \beta^{-1}\gamma > .$$

These follow from results in [H1] and [H2].

3. 4-manifolds

In this section,we shall consider the problem stated in Introduction. Let M be a 4-dimensional closed aspherical manifold fibered over S^1 with F as a fiber and a projection p. We assume F is irreducible. Then we have the following

PROPOSITION. *There exists a homeomorphism $h : F \to F$ such that M is homeomorphic to $\mathbf{R} \times_{\mathbf{Z}} F$,where the group $\mathbf{Z}$ acts on $\mathbf{R} \times F$ by the formula;*

$$n(t,x) = (t-n, h^n(x)).$$

This follows from the standard arguments.

We have the following exact sequence ;

$$(1) \qquad 1 \longrightarrow \pi_1(F) \longrightarrow \pi_1(M) \xrightarrow{p_*} \mathbf{Z} \longrightarrow 1$$

Let $t \in \pi_1(M)$ be an element such that $p_*(t) = 1 \in \mathbf{Z}$.

The following Propositions are easily proved.

PROPOSITION 3.1. *If αt^n $(n \geq 1)$ is contained in $z(\pi_1(M))$, $h_*^n(\beta) = \alpha^{-1}\beta\alpha$ for β any element of $\pi_1(F)$.*

PROPOSITION 3.2. *If $z(\pi_1(F))$ is finitely generated, then $z(\pi_1(M)) \cap \pi_1(F) = z(\pi_1(F))^{h_*}$.*

We consider the oriented double $\tilde{F}$ of $F = N_3, N_4$. Recall F is written as follows. Define an action of $\mathbf{Z}$ on $\mathbf{R} \times T^2$ by the formula; $n(x,(z_1,z_2)) = (x-n, h_0^n(z_1,z_2))$. Then F is homeomorphic to the orbit space $(\mathbf{R} \times T^2)/\mathbf{Z}$.Then $\tilde{F}$ is represented as the orbit space $(\mathbf{R} \times T^2)/2\mathbf{Z}$.We write element of F and $\tilde{F}$ as $[x,(z_1,z_2)]_1$ and $[x,(z_1,z_2)]_0$,respectively. Then the natural covering map $\pi : \tilde{F} \to F$ is given by $\pi[x,(z_1,z_2)]_0 = [x,(z_1,z_2)]_1$ and the non-trivial covering transformation $w : \tilde{F} \to \tilde{F}$ given by $w[x,(z_1,z_2)]_0 = [x-1, h_0(z_1,z_2)]_0$.

We have the following

THEOREM 3.3. *If* $F = N_3, N_4$ *and* $z(\pi_1(F))^{h_*} = \mathbf{Z}^2$,then h lifts to a homeomrphism $\tilde{h}$ of $\tilde{F}$.

PROOF. We shall consider only $F = N_4$. Recall that

$$\pi(F) = <\alpha, \beta, t : [\alpha, \beta] = 1, t\alpha t^{-1} = \alpha t \beta t^{-1} = \alpha\beta^{-1}>$$

$$z(\pi(F)) = <\alpha, t^2>$$

Since $h_*(t^2) = t^2$,$z(\pi(F))$ is h_*-invariant. This implies that there exists a homeomorphism $\tilde{h} : \tilde{F} \to \tilde{F}$ such that the following diagram is commutative.

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \pi_1(\tilde{F}) & \longrightarrow & \pi_1(F) & \longrightarrow & \mathbf{Z}_2 & \longrightarrow & 1 \\ & & \downarrow \tilde{h}_* & & \downarrow h_* & & \downarrow id & & \\ 1 & \longrightarrow & \pi_1(\tilde{F}) & \longrightarrow & \pi_1(F) & \longrightarrow & \mathbf{Z}_2 & \longrightarrow & 1 \end{array} \quad (2)$$

Note that we can speak of rank of $z(\pi_1(M))$ since $\pi_1(M)$ is torsion free.

3.1. The case of rank $z(\pi_1(M))$=4.

We have the following

PROPOSITION 3.3([H1],11.6). *Let* G *be a group. If the index of* $z(G)$ *in* G *is finite,then* $[G, G]$*(=commutator group) is finite.*

It follows from this that $\pi_1(M) \cong \mathbf{Z}^4$. Then M is homeomorphic to T^4.

3.2. The case of rank $z(\pi_1(M))$ =3.

It follows from (1) that rank of $z(\pi_1(F))$ is at least 2 . Proposition 1.5 shows that $z(\pi_1(F))$ and hence $z(\pi_1(M))$ is also finitely generated.

PROPOSITION 3.5. $p_* z(\pi_1(M)) \neq 1$.

PROOF. Assume the contrary. We have

$$z(\pi_1(M)) \cap \pi_1(F) \subset z(\pi_1(F)) \subset \pi_1(F).$$

It follows from Proposition 3.4 that $\pi_1(F) \cong \mathbf{Z}^3$ and hence $F \cong T^3$. From the assumption,we have

$$z(\pi_1(M)) \subset \{(\alpha, 0) : \alpha \in \pi_1(F)\},$$

and hence we have

$$z(\pi_1(M)) \subset \pi_1(F).$$

Since $\pi_1(F)$ is abelian,we have $\pi_1(F)/z(\pi_1(M))$ is finite.Consider the commutative diagram;

$$\begin{array}{ccccccccc}
1 & \longrightarrow & z(\pi_1(M)) & \longrightarrow & \pi_1(F) & \longrightarrow & \pi_1(F)/z(\pi_1(M)) & \longrightarrow & 1 \\
 & & \downarrow \tilde{h}_* & & \downarrow h_* & & \downarrow \bar{h}_* & & \\
1 & \longrightarrow & z(\pi_1(M)) & \longrightarrow & \pi_1(F) & \longrightarrow & \pi_1(F)/z(\pi_1(M)) & \longrightarrow & 1
\end{array}$$

It is clear that $\bar{h}_*$ is of finite order and $\tilde{h}_*$ is identity,and hence h_* is of finite order,which contradicts the assumption.

PROPOSITION 3.6. *F is one of T^3, N_3, N_4.*

PROOF. It follows from Theorem 3.2 that there exists a subgroup Z^2 of $\pi_1(F)$ with finite index. Then $\pi_1(F)$ is finitely generated. It follows from [H1]([H1] Theorem 12.10) that F is one of T^3, N_3, N_4. ∎

PROPOSITION 3.7. *h is of finite order.*

PROOF. Assume the contrary. If F is orientable,it follows from Theorem 1.1 that h_* is also of infinite order modulo $\mathrm{Inn}(\pi_1(F))$,which contradicts Proposition 3.5. If F is not orientable,consider the orientable double $\tilde{F}$ It follows from Proposition 3.3 that h lifts to homeomorphism $\tilde{h} : \tilde{F} \to \tilde{F}$. We put $\tilde{M} = \tilde{F}_{\tilde{h}}$. Then it is clear that h_* is of infinite order mod $\mathrm{Inn}(\pi_1(F))$ if and only if $\bar{h}_*$ is of infinite order mod $\mathrm{Inn}(\pi_1(\bar{F}))$. ∎

3.2.1. The case of $F = T^3$.

We have the following

THEOREM 3.1. *If $F = T^3$, then M admits a maximal toral action.*

PROOF. Note that M is one of manifolds in Example 2. Since $z(\pi_1(M)) = \mathbf{Z}^3$,$M = M_4$ or M_5. We can define a T^3-action on M_5 as follows ;
First define of $\mathbf{R}^3$ on $\mathbf{R} \times T^3$ by the formula ;

$$(x, y, z)(t, (z_1, z_2, z_3)) = (t + x, ((exp(2\pi i y)z_1, (exp(2\pi i z)z_2, z_3)).$$

This is compatible with the action of ϕ_5. It is easy to show that this action induces an action of T^3. ∎

3.2.2. The case of $F = N_3$ or N_4.

We shall consider only the case of $F = N_4$. Let $\tilde{F} = T^3$ be the oriented double of F and $\tilde{h} : \tilde{F} \to \tilde{F}$ be the lifting of h. Put $\tilde{M} = \tilde{F}_{\tilde{h}}$.
Now we define a new bundle structure over S^1 on N_4. The followings are easily shown ;

$$\pi_1(N_4) =< \alpha, \beta, t : [\alpha, \beta] = 1, t\alpha t^{-1} = \alpha, t\beta t^{-1} = \alpha\beta^{-1} >$$

$$[\pi_1(N_4), \pi_1(N_4)] = < \alpha\beta^{-2} >$$
$$H_1(N_4) = < \bar{\beta} > \oplus < \bar{t} >,$$

where $\bar{x}$ denotes the image of x by the projection $\pi_1(N_4) \to H_1(N_4)$.

Define a homomorphism $\rho_1 : H_1(N_4) \to Z \quad \bar{\beta} \oplus \bar{t} \mapsto 1$ and ρ composition of the natural projection $\pi_1(N_4) \to H_1(N_4)$ and ρ_1. Clearly $\ker \rho = < \alpha\beta^{-2}, t >$.

Now we define a fiber bundle $N \to N_4 \to S^1$ associated to the exact sequence ;

$$1 \longrightarrow \ker\rho \longrightarrow \pi_1(N_4) \longrightarrow Z = < \bar{\beta}\bar{t} > \longrightarrow 1.$$

Note that $\pi_1(N) = \ker\rho$ and $N \cong T^2$. We write $N_4 = N_{h_0}$. We have the following

Lemma. *h_* preserves $\ker \rho$.*

Proof. Let $\pi : \tilde{N}_4 \to N_4$ be the projection. We have the following commutative diagram ;

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \pi_1(T^2) & \longrightarrow & \pi_1(\tilde{N}_4) & \longrightarrow & Z & \longrightarrow & 1 \\
 & & \downarrow \tilde{\pi}_* & & \downarrow \pi_* & & \downarrow 2 & & \\
1 & \longrightarrow & \pi_1(T^2) & \longrightarrow & \pi_1(N_4) & \longrightarrow & Z & \longrightarrow & 1.
\end{array}$$

It follows that we can take generators α_1, β_1, and t_1 of $\pi_1(\tilde{N}_4)$ such that $\pi_*(\alpha_1) = \alpha, \pi_*(\beta_1) = \beta$ and $\pi_*(t_1) = t^2$. According to diagram (2), we have $\tilde{h}(\alpha_1) = \alpha, \tilde{h}(\beta_1) = \beta$ and hence $h_*(\alpha) = \alpha$ and $h_*(\beta) = \beta$. Now Lemma is proved by the direct computation using the fact that $h_*(t^2) = t^2$. ∎

Let $G = < h >$ be the subgroup of homeomorphisms of N_4. According to results in [CR](section 2 in [CR]),we have an exact sequence ;

$$1 \longrightarrow \pi_1(N_4) \longrightarrow \Gamma \longrightarrow G \longrightarrow 1$$

It follows from Theorem 61.1 in [Z] that h preserves the above bundle structure over S^1. Thus we get a homeomorphism $h_1 : N \to N$ such that $h_0 \circ h_1 = h_1 \circ h_0$. Then M is homeomorphic to the manifold $R^2 \times_{Z^2} N$, where Z^2 acts on $R^2 \times N$ by $(n, m)(x, y, z) = (x - n, y - m, h_1^n h_0^m(z))$.

Moreover N admits an action of T^1 with respect to which h_1 and h_0 are equivariant.

In fact,we have the following commutative diagram ;

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \ker\rho = < \alpha\beta^{-2} > & \longrightarrow & \pi_1(N_4) & \longrightarrow & Z = < \tilde{\beta}\tilde{t} > & \longrightarrow & 1 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
1 & \longrightarrow & Z = < \alpha > & \longrightarrow & z(\pi_1(N_4)) & \longrightarrow & Z = < \tilde{\alpha}\tilde{t^2} > & \longrightarrow & 1.
\end{array}$$

Decompose $N = T^1 \times T^1$ such that the first factor T^1 corresponds to $< \alpha >$. Define an action of T^1 on N by $z(z_1, z_2) = (zz_1, z_2)$. It is clear that h_0 and h_1 are equivariant with this action. Now define an action of T^3 on M. First define an action of $\mathbf{R}^3$ on $\mathbf{R}^2 \times T^2$ by the formula ;

$$(t_1, t_2, t_3)(x, y, (z_1, z_2)) = (x + t_1, y + t_2, ((exp 2\pi i t_3) z_1, z_2)).$$

It is easily to see that this action is commutative with the above action of $\mathbf{Z}^2$ and hence we get an action of $\mathbf{R}^3$ on M. It is also easy to see the restriction to the subgroup $\{(2n, 2m, l) \in \mathbf{Z}^3\}$ of $\mathbf{R}^3$ is trivial and hence we get an action of T^3 on M. Thus we have the following

THEOREM 3.2. *If $F = N_3$ or N_4, then M admits a maximal toral action.*

3.3. The case of rank $z(\pi_1(M))$=2.

In this case,we assume F is irreducible and sufficiently large. It follows from results in [H1](Corollary 12.8 in [H1]) that F is a Seifert fibered space and hence $z(\pi_1(F))$ and $z(\pi_1(M))$ are finitely generated.

3.3.1. The case of $F = T^3$.

3.3.1.1. The case when h is of finite order.

In this case,M is one of manifolds $M_2, M_3, M_6, M_7, M_8, M_9$ or M_{10} in Example 2. We shall show that M_{10},for example,admits a maximal toral action.

Recall that

$$M_{10} = T^3_{\phi_{10}},$$

$$\phi_{10} : T^3 \to T^3 \quad (z_1, z_2, z_3) \mapsto (z_1, z_3^{-1}, z_2 z_3)$$

Define an action of $\mathbf{R}^2$ on $\mathbf{R} \times T^3$ by

$$(t_1, t_2)(x, (z_1, z_2, z_3)) = (x + t, ((exp(2\pi i t_2) z_1, z_2, z_3)).$$

It is easy to see that this action is compatible with ϕ_{10} and induces an action of $T^2 = \mathbf{R}^2/\mathbf{Z}^2$ on M_{10}.

3.3.1.2. The case when h is of infite order.

Since $z(\pi_1(M)) \cong \mathbf{Z}^2$ and $p_* z(\pi_1(M)) = 1$, we have

$$z(\pi_1(F))^{h_*} = \{\alpha \in z(\pi_1(F)); h_*(\alpha) = \alpha\} \cong \mathbf{Z}^2.$$

Hence we have

$$h_* = \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix},$$

in other words,

$$h : T^3 \to T^3 \qquad (z_1, z_2, z_3) \mapsto (z_1 z_3^a, z_2 z_3^b, z_3).$$

Define an action of T^2 on M by

$$(z_1, z_2)[t, (z_3, z_4, z_5)] = [t, (z_1 z_3, z_2 z_4, z_5)].$$

This action is the one we want.

3.3.2. The case of $F = N_4$ or N_3.

We shall consider only N_4.
We have the following commutative diagram;

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbf{Z}^2 & \longrightarrow & \pi_1(\tilde{F}) & \longrightarrow & \mathbf{Z} & \longrightarrow & 1 \\ & & \downarrow \tilde{\pi}_* & & \downarrow \pi_* & & \downarrow 2 & & \\ 1 & \longrightarrow & \mathbf{Z}^2 & \longrightarrow & \pi_1(F) & \longrightarrow & \mathbf{Z} & \longrightarrow & 1 \end{array} \tag{3}$$

We can take generators α_1, β_1 and t_1 of $\pi_1(\tilde{F})$ such that $\pi_*(\alpha_1) = \alpha, \pi_*(\beta_1) = \beta$ and $\pi_*(t_1) = t^2$.

3.3.2.1. The case when h is of infinite order.

Since $\mathrm{rank} z(\pi_1(F))^{h_*} = \mathrm{rank} z(\pi_1(F))$, we have $h_*(\alpha) = \alpha$ and $h_*(t^2) = t^2$ and hence $\tilde{h}_*(\alpha_1) = \alpha_1$ and $\tilde{h}_*(t_1) = t_1$. This implies that $\tilde{h} : \tilde{F} \to \tilde{F}$ is assumed to be $\tilde{h}[x, (z_1, z_2)]_0 = [x, (z_1 z_2^a, z_2)]_0$. Define an action of $\mathbf{R}^2$ on $\tilde{F}$ by the formula;

$$(t_1, t_2)[x, (z_1, z_2)]_0 = [x + t_1, (\exp(2\pi i t_2) z_1, z_2)]_0$$

By direct computations, we can show that $\tilde{h}$ and w are equivariant with respect to this action. It is easy to see the above action of $\mathbf{R}^2$ defines of T^2 on $\tilde{F}$ with respect to which $\tilde{h}$ and w are equivariant. This defines an action of T^2 on F with respect to which h is equivariant and hence M admits an action of T^2.

3.3.2.2. The case when h is of finite order.

In this case, we have $z(\pi_1(F))^{h_*} = \mathbf{Z}$. We can construct an action of T^2 on F which satisfies the following;

Let $ev^x : T^2 \to F$ be the map defined by $ev^x(t) = tx$. Then $\mathrm{Im}\{ev^x_* : \pi_1(T^2) \to \pi_1(F)\} = z(\pi_1(F))$.

Consider the action of T^1 which is obtained by the restriction of the above action to $z(\pi_1(F))^{h_*}$.

We have an exact sequence;

$$1 \longrightarrow \mathbf{Z} \longrightarrow \pi_1(F) \longrightarrow N \longrightarrow 1,$$

where $h_*|\mathbf{Z}$ =id.

Since h_* is of finite order,there exists a normal subgroup Γ of $\pi_1(F)$ with the properties;

(i) the index $[\pi_1(F):\Gamma]$:finite

(ii) $h_*(\Gamma)=\Gamma$

(iii) Γ is an extension of Z by a group N,where N is the fundamental group of an orientable surface.

Let F_1 be the covering of F associated to Γ. Then F_1 is an orientable 3-manifold fibered over S having T^1 as fiber. It follows from Theorem 11 in [CR2] that the restriction of Ω to the subset

$$G(T^1,F_1)=\{(g,H):g\in GL(1,Z),H:F_1\to F_1,H(tx)=g(t)H(x)\}$$

is surjective. Let h_1 be the lifting of h to $F1$. Note that h_1 exists ,because $\pi_1(F_1)$ is invariant under h_*. Then $h_{1*}=\Omega(g,H)$. It follows from results in [W] that h_1 is isotopic to a fiber preserving homeomorphism of F_1;$h_1(tx)=th_1(x)$. This implies h is also T^1-invariant.Thus we have the following

THEOREM 3.3. *If $F=N_3,N_4$,then M admits a maximal toral action.*

3.3.3. The case of F=other Seifert fibered space.

Assume h is of infinite order.Then $z(\pi_1(F))$ contains $\mathbf{Z}^2$. Thus we may assume that h is of finite order.

In this case,by the same arguments as above,we can prove that M admits a T^2-action. Thus we have

THEOREM 3.4. *If F is the Seifert fibered space other that T^3,N_4,N_4,then M admits a maximal toral action.*

3.4.The case of rank $z(\pi_1(M))$=1.

3.4.1 The case when h is of finite order.

We have the following

THEOREM. *If h is finite order,say of order n,then M admits an S^1-action.*

PROOF. Define an action of R on M by the formula;

$$s[t,x]=[t+s,x].$$

It is easy to see that this action is well defined and nZ acts trivially.

$$[t',x']=[t,x]\Rightarrow t'=t-m,x'=h^m(x)$$
$$s[t',x']=[t'+s,x']=[t-m+s,h^n(x)]=[t+s,x]=s[t,x]$$

$$mn[t,x]=[t+nm,x]=[t+nm,(h^n)^m(x)]=[t,x]$$

Then R/nZ acts on M. This action is effective. In fact, we have

$$\begin{aligned} s[t,x]=[t,x] &\Leftrightarrow [t+s,x]=[t,x] \\ &\Leftrightarrow t+s=t-m, x=h^m(x) \\ &\Leftrightarrow s=-m, m\equiv 0 \pmod n \\ &\Leftrightarrow s\in nZ \qquad \blacksquare \end{aligned}$$

3.4.2. The case when h is of infinite order.

In this case we have the following

THEOREM. *Let M be with $p_*(z(\pi_1(M)))=1$. If the fiber F admits a maximal toral action, then M does also.*

PROOF. Let $\tilde{X}$ denote the universal covering space of X. Then $\tilde{M}$ is homeomorphic to $\tilde{F}\times\mathbf{R}$. Since the action on F is injective , so $\tilde{F}$ splits as $\mathbf{R}^k\times W$ where k is the rank of $z(\pi_1(F))$. There exists the following central exact sequence

$$1\to z(\pi_1(F))\to\pi_1(F)\to\Gamma\to 1.$$

Associated to this sequence, we have

$$F=(\mathbf{R}^k\times W)/\pi_1(F)=(\mathbf{R}^k/z(\pi_1(F)\times W)/\Gamma=(T^k\times W)/\Gamma.$$

Then $T^k\times W$ admits a natural T^k action compatible with Γ action.

Let r be the rank of $z(\pi_1(M))$. Since $z(\pi_1(M))$ is a subgroup of $z(\pi_1(F))$,$r\le k$ and h_* is an identity on $z(\pi(_1(M))$. On the covering space corresponding to the factor $z(\pi_1(M))$, we have the T^r action by the same argument of 3.3.2.2, which is commutative with the lifting of h.So we can construct T^r action on $M=F_h$.

REFERENCES

[CR1] P.E.Conner and F.Raymond, *Manifolds with few periodic homeomorphisms*, Proc.Second Conference on Compact Transformation Groups,Part II,Lecture Notes in Math. vol. 299,Springer-Verlag,New York, (1972), 1–75.

[CR2] ———, *Deforming homotopy equivalence to homeomorphisms in aspherical manifolds*, Bull.AMS **83** (1977), 36–85.

[H1] J.Hempel, "3-Manifolds," Princeton Univ.Press, New Jersey, 1976.

[H2] J.Hempel, *Free cyclic actions on $S^1\times S^1\times S^1$*, Proc.AMS **48** (1975), 221–227.

[J] W.Jaco, "Lectures on Three-Manifolds topology," Amer.Math.Soc..

[M] A.Morgan, *The classification of flat solvmanifolds*, Trans.Amer.Math.Soc. **239** (1978), 321–365.

[O] P.Orlik, "Seifert Manifolds," Lectures Notes in Math.291, Springer, Berlin, 1970.
[S] P.Scott, *The classification of compact 3-manifolds*, London Math.Soc.Lecture note 48.
[W] F.Waldhausen, *On irreducible 3-manifolds which are sufficiently large*, Ann.of Math. **87** (1968), 56–88.
[Wa] T.Watabe, *Maximal toral action on aspherical manifolds* $\Gamma\backslash G/K$ *and* G/H., J.Math.Soc.Japan **40** (1988), 629–645.
[Z] H.Zieschang, "Finite groups of Mapping Classes of Surfaces," Lectures Notes in Math.875, Springer, Berlin, 1981.

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