On special infinitesimal holomorphically projective transformations

By

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K. Yano [1][1] has proved that in a Riemannian space any conformal transformation which transforms every Einstein space into an Einstein space is a concircular one, and a space of constant curvature is transformed into a space of constant curvature by a concircular transformation. In this paper we shall study the formal analogue of these results to an analytic infinitesimal holomorphically projective transformation in a Kählerian space. In § 2 we shall define an analytic special infinitesimal holomorphically projective transformation which transforms every Kähler Einstein space into an Kähler Einstein space. Next after introducing the special holomorphically projective curvature tensor which is invariant under an analytic special holomorphically projective transformation, we shall prove that this transformation preserves a space of constant holomorphic sectional curvature. In § 3, we shall prove some theorems concerning with an analytic SHP-transformation in a Kählerian space, these are valid for analytic HP-transformation in a Kähler Einstien space.

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§ 1. Preliminaries

A vector field v^i is called an infinitesimal holomorphically projective transformation or briefly an HP-transformation if it satisfies

$$(1.\ 1)\qquad \underset{v}{\pounds}\left\{ {h \atop ji} \right\} = \delta_j^h \rho_i + \delta_i^h \rho_j - \varphi_j{}^h \tilde{\rho}_i - \varphi_i{}^h \tilde{\rho}_j$$

where ρ_i is a vector $\tilde{\rho}_i = \varphi_i{}^r \rho_r$ and $\varphi_j{}^i$ is the complex structure. We shall call ρ_i the associated vector of the HP-transformation. $\underset{v}{\pounds}$ denotes the Lie differentiation with respect to v^i. Contracting (1. 1) with respect to h and i, we get $\nabla_j \nabla_r v^r = 2(n+1)\rho_j$, which shows that ρ_j is gradient, where ∇_j denotes the operator of

1) The number in brackets [ ] refers to Bibliography at the end of the paper.

covariant differentiation with respect to $\{^{h}_{i\,j}\}$ and $2n$ is the dimension of the space.

A vector field v^i is called analytic on a Kählerian space if it satisfies

$$\underset{v}{\pounds}\varphi_j{}^h=-\varphi_j{}^r\nabla_r v^h+\varphi_r{}^h\nabla_j v^r=0. \tag{1. 2}$$

We shall give here preliminary formulas on Kählerian space X_{2n},

$$\varphi_j{}^r\varphi_r{}^i=-\delta_j^i, \qquad g_{ji}=\varphi_j{}^r\varphi_i{}^s g_{rs}, \tag{1. 3}$$
$$\nabla_k\varphi_j{}^h=0, \qquad \nabla_k g_{ji}=0.$$

Then the following identity holds good,

$$R_{ji}=\varphi_j{}^r\varphi_i{}^s R_{rs} \tag{1. 4}$$

where R_{ji} is the Ricci tensor.

A space whose curvature tensor takes the form

$$R_{kji}{}^h=\frac{1}{k}(g_{ki}\delta_j^h-g_{ji}\delta_k^h+\varphi_{ki}\varphi_j{}^h-\varphi_{ji}\varphi_k{}^h+2\varphi_{kj}\varphi_i{}^h) \tag{1. 5}$$

is called a space of constant holomorphic sectional curvature[2], where $\frac{1}{k}=-\frac{R}{4n(n+1)}$ and $R=g^{ji}R_{ji}$.

§ 2. Analytic SHP-transformations

A vector field v^i is called an analytic special infinitesimal holomorphically projective transformation or briefly an analytic SHP-transformation if it satisfies

$$\underset{v}{\pounds}\{^{h}_{ji}\}=\delta_j^h\rho_i+\delta_i^h\rho_j-\varphi_j{}^h\rho_i-\varphi_i{}^h\rho_j, \tag{2. 1}$$

$$\underset{v}{\pounds}\varphi_j{}^h=0, \tag{2. 2}$$

$$\underset{v}{\pounds}g_{ji}=k\nabla_j\rho_i, \qquad k=-\frac{4n(n+1)}{R}=\text{const.} \tag{2. 3}$$

From these equations we have

THEOREM 2. 1. *In order that an analytic HP-transformation of a Kählerian space transforms every Kähler Einstein space into a Kähler Einstein space, it is necessary and sufficient that the transformation is an analytic SHP-transformation.*

Proof. When an analytic HP-transformation preserves an Einstein space, it holds that

2) Tashiro, Y. [2].

(2. 4) $$\underset{v}{\pounds}\left(R_{ji}-\frac{R}{2n}g_{ji}\right)=\underset{v}{\pounds}R_{ji}-\frac{R}{2n}\underset{v}{\pounds}g_{ji}=0.$$

On the other hand, the following identity is known

(2. 5) $$\nabla_k\underset{v}{\pounds}\left\{{h \atop ji}\right\}-\nabla_j\underset{v}{\pounds}\left\{{h \atop ki}\right\}=\underset{v}{\pounds}R_{kji}{}^h.$$

From (2. 1), (2. 2) and (2. 5) we have

(2. 6) $$\underset{v}{\pounds}R_{kji}{}^h=\delta_j^h\nabla_k\rho_i-\delta_k^h\nabla_j\rho_i-\varphi_j{}^h\nabla_k\tilde{\rho}_i+\varphi_k{}^h\nabla_j\tilde{\rho}_i-(\nabla_k\tilde{\rho}_j-\nabla_j\tilde{\rho}_k)\varphi_i{}^h.$$

Contracting (2. 6) with respect to h and k, we get

$$\underset{v}{\pounds}R_{ji}=-2n\nabla_j\rho_i-2\varphi_j{}^r\varphi_i{}^s\nabla_r\rho_s.$$

Taking account of (1. 4) and (2. 2) we have

(2. 7) $$\nabla_j\rho_i=\varphi_j{}^r\varphi_i{}^s\nabla_r\rho_s,$$

$$\underset{v}{\pounds}R_{ji}=-2(n+1)\nabla_j\rho_i.$$

Substituting the last equation into (2. 4) we obtain (2. 3), i. e:

$$\underset{v}{\pounds}g_{ji}=-\frac{4n(n+1)}{R}\nabla_j\rho_i=k\nabla_j\rho_i.$$

The converse is evident.

Now we shall define a tensor $Z_{kji}{}^h$ as following

(2. 8) $$Z_{kji}{}^h\equiv R_{kji}{}^h-\frac{1}{k}(g_{ki}\delta_j^h-g_{ji}\delta_k^h+\varphi_{ki}\varphi_j{}^h-\varphi_{ji}\varphi_k{}^h+2\varphi_{kj}\varphi_i{}^h),$$

$$\frac{1}{k}=-\frac{R}{4n(n+1)}=\text{const.}.$$

We shall show that an analytic SHP-transformation leaves this tensor invariant. Transvecting (2. 3) with $\varphi_j{}^k$ and taking account of (1. 3) and (2. 2), we obtain

(2. 9) $$\underset{v}{\pounds}\varphi_{ji}=-k\nabla_j\tilde{\rho}_i..$$

If we operate $\underset{v}{\pounds}$ to (2. 8), then by virtue of (2. 3), (2. 6) and (2. 9) we have

(2. 10) $$\underset{v}{\pounds}Z_{kji}{}^h=\underset{v}{\pounds}R_{kji}{}^h-\frac{1}{k}(\delta_j^h\underset{v}{\pounds}g_{ki}-\delta_k^h\underset{v}{\pounds}g_{ji}+\varphi_j{}^h\underset{v}{\pounds}\varphi_{ki}-\varphi_k{}^h\underset{v}{\pounds}\varphi_{ji}+2\varphi_i{}^h\underset{v}{\pounds}\varphi_{kj})$$
$$=0.$$

Conversely if an analytic HP-transformation v^i leaves $Z_{kji}{}^h$ invariant, then (2. 10) holds. Contracting (2. 10) with respect to h and k, we get

$$0=\underset{v}{\pounds} Z_{ji}=\underset{v}{\pounds}\left(R_{ji}-\frac{R}{2n}g_{ji}\right),$$

where $Z_{ji}=Z_{kji}{}^{k}$. Thus by virtue of Theorm 2. 1, v^{i} is an analytic SHP-transformation. We shall call $Z_{kji}{}^{h}$ a special holomorphically projective curvature tensor or briefly a SHP curvature tensor. Then we have the following.

THEOREM 2. 2. *In a Kählerian space, an analytic HP-transformation preserves a SHP-curvature tensor, if and only if it is an analytic SHP-transformation.*

COROLLARY. *A necessary and sufficient condition for $Z_{kji}{}^{h}=0$ is that the space is a space of constant holomorphic sectional curvature, i, e. a space whose curvature tensor $R_{kji}{}^{h}$ takes the form* (1. 5).

We can obtain the following identities,

(2. 12) $$Z_{(kj)i}{}^{h}=0, \qquad Z_{(kji)}{}^{h}=0, \qquad Z_{[kji]}{}^{h}=0,$$

(1. 13) $$Z_{kjih}=Z_{ihkj}, \quad \text{where } Z_{kjih}=g_{rh}Z_{kji}{}^{r},$$

(2. 14) $$\nabla_{l}Z_{kji}{}^{h}+\nabla_{k}Z_{jli}{}^{h}+\nabla_{j}Z_{lki}{}^{h}=0,$$

(2. 15) $$Z_{ji}\equiv Z_{rji}{}^{r}=R_{ji}-\frac{R}{2n}g_{ji}, \qquad Z\equiv g^{ji}Z_{ji}=0,$$

(2. 16) $$Z_{kji}{}^{h}=\varphi_{k}{}^{r}\varphi_{j}{}^{s}Z_{rsi}{}^{h}, \qquad Z_{kji}{}^{h}=-\varphi_{i}{}^{r}\varphi_{s}{}^{h}Z_{kjr}{}^{s}.$$

§ 3. Some theorems on analytic SHP-transformations

The following identies[3)] are well known

(3. 1) $$\underset{v}{\pounds}\left\{{h \atop ji}\right\}=\nabla_{j}\nabla_{i}v^{h}+R_{rji}{}^{h}v^{r},$$

(3. 2) $$\underset{v}{\pounds}\left\{{h \atop ji}\right\}=\frac{1}{2}g^{hl}(\nabla_{j}\underset{v}{\pounds}g_{il}+\nabla_{i}\underset{v}{\pounds}g_{jl}-\nabla_{l}\underset{v}{\pounds}g_{ji}).$$

If we substitute (2. 3) into (3. 2), we obtain

$$\begin{aligned}\underset{v}{\pounds}\left\{{h \atop ji}\right\}&=\frac{k}{2}g^{hl}(\nabla_{j}\nabla_{i}\rho_{l}+\nabla_{i}\nabla_{j}\rho_{l}-\nabla_{l}\nabla_{j}\rho_{i})\\&=\frac{k}{2}g^{hl}[\nabla_{j}\nabla_{i}\rho_{l}-(\nabla_{l}\nabla_{j}\rho_{i}-\nabla_{j}\nabla_{l}\rho_{i})]\\&=\frac{k}{2}(\nabla_{i}\nabla_{j}\rho^{h}+R_{rij}{}^{h}\rho^{r}).\end{aligned}$$

From (3. 1) we have

3) Yano, K. [3].

$$\underset{v}{\pounds}\left\{ {h \atop ji} \right\} = \frac{k}{2} \underset{\rho}{\pounds}\left\{ {h \atop ji} \right\},$$

$$\underset{\rho}{\pounds}\left\{ {h \atop ji} \right\} = \delta_j^h \sigma_i + \delta_i^h \sigma_j - \varphi_j{}^h \sigma_i - \varphi_i{}^h \sigma_j$$

where we have put $\sigma_i = \frac{2}{k}\rho_i$.

In the next place

$$\underset{\rho}{\pounds} g_{ji} = \nabla_j \rho_i + \nabla_i \rho_j = 2\nabla_j \rho_i = k \nabla_j \sigma_i.$$

Hence ρ^i is a SHP-transformation. Moreover by virtue of (2. 7) we get

$$\underset{\rho}{\pounds} \varphi_j{}^i = -\varphi_j{}^r \nabla_r \rho^i + \varphi_r{}^i \nabla_j \rho^r = 0.$$

This shows that ρ^i is analytic. We have the following

THEOREM 3. 1. *The associated vector of an analytic SHP-transformation is also an analytic SHP-transformation.*

Since φ_{ji} is anti-symmetirc we have

$$\nabla_j \tilde{\rho}_i + \nabla_i \tilde{\rho}_j = 0,$$

by virtue of (2. 9). Therefore the vector $\tilde{\rho}_i$ is a Killing one.

From (2. 3) it follows

$$\underset{v}{\pounds} g_{ji} = \nabla_j v_i + \nabla_i v_j = k \nabla_j \rho_i = \frac{k}{2}(\nabla_j \rho_i + \nabla_i \rho_j),$$

$$\nabla_j\left(v_i - \frac{k}{2}\rho_i\right) + \nabla_i\left(v_j - \frac{k}{2}\rho_j\right) = 0$$

which means that

$$(3.\ 3) \qquad p_i \equiv v_i - \frac{k}{2}\rho_i$$

is a Killing vector. If we put $q_i \equiv \frac{k}{2}\tilde{\rho}_i$, then q_i is also a Killing vector, and we get $\frac{k}{2}\rho^i = \varphi_r{}^i q^r$. Thus we obtain

$$v^i = p^i + \varphi_r{}^i q^r.$$

THEOREM 3. 2. *In a Kählerian space, an analytic SHP-transformation v^i is uniquely decomposed in the form*

$$(3.\ 4) \qquad v^i = p^i + \varphi_r{}^i q^r$$

where p^i and q^i are both Killing vectors, and $\varphi_r{}^i q^r$ is qradient.

We shall prove the uniqueness of this decomposition. In the first place we have the following lemma.

LEMMA. *In a Kählerian space, the associated vector of an analytic SHP-transformation is a Ricci direction, i. e.* ρ^i *satisfies* $R_{kr}\rho^r=\frac{R}{2n}\rho_k$.

Proof. Substituting (2. 3) $\nabla_j\rho_i=\frac{1}{k}\underset{v}{\pounds}g_{ji}$ into the Ricci's formula

$$\nabla_k\nabla_j\rho_i-\nabla_j\nabla_k\rho_i=-R_{kji}{}^r\rho_r,$$

we obtain

$$R_{kji}{}^r\rho_r=\frac{1}{k}(\nabla_j\underset{v}{\pounds}g_{ki}-\nabla_k\underset{v}{\pounds}g_{ji}). \tag{3. 5}$$

From the formula[4]

$$\underset{v}{\pounds}\nabla_j g_{ki}-\nabla_j\underset{v}{\pounds}g_{ki}=-g_{ri}\underset{v}{\pounds}\left\{{r \atop jk}\right\}-g_{kr}\underset{v}{\pounds}\left\{{r \atop ji}\right\},$$

it follows

$$\nabla_j\underset{v}{\pounds}g_{ki}=g_{ri}\underset{v}{\pounds}\left\{{r \atop jk}\right\}-g_{kr}\underset{v}{\pounds}\left\{{r \atop ji}\right\}.$$

Substituting the last equation into (3. 5), we have

$$R_{kji}{}^r\rho_r=\frac{1}{k}(g_{kr}\underset{v}{\pounds}\left\{{r \atop ji}\right\}-g_{jr}\underset{v}{\pounds}\left\{{r \atop ki}\right\}).$$

If we substitute (2. 1) into the above equation, we obtain

$$R_{kji}{}^r\rho_r=\frac{1}{k}(g_{ki}\rho_j-g_{ji}\rho_k+\varphi_{ki}\tilde{\rho}_j-\varphi_{ji}\tilde{\rho}_k+2\varphi_{kj}\tilde{\rho}_i). \tag{3. 6}$$

Transvecting this with g^{ij}, we have

$$R_k{}^r\rho_r=\frac{R}{2n}\rho_k. \tag{3. 7}$$

Now, if we have two decomposition of (3. 4)

$$v^i=p^i+\varphi_r{}^iq^r={}'p^i+\varphi_r{}^i{}'q^r,$$

then

$$p^i-{}'p^i=\varphi_r{}^i({}'q^r-q^r).$$

If we put $\xi^i\equiv p^i-{}'p^i$, then ξ^i is a Killing vector and at the same time gradient, and therefore it holds that

$$\nabla_j\xi^h=0.$$

4) Yano, K. [3].

By the Ricci's formula, we have

$$\nabla_j \nabla_i \xi^h - \nabla_i \nabla_j \xi^h = R_{jir}{}^h \xi^r = 0$$

from which we obtain

$$R_{ir}\xi^r = 0. \tag{3. 8}$$

From (3. 3) we have $\rho_i = \frac{2}{k}(v_i - p_i)$, and substituting into (3. 7), it follows

$$R_k{}^r(v_r - p_r) = -(n+1)(v_k - p_k).$$

Similarly

$$R_k{}^r(v_r - {}'p_r) = -(n+1)(v_k - {}'p_k)$$

and from the last two equations we have

$$R_k{}^r(p_r - {}'p_r) = -(n+1)(p_k - {}'p_k)$$

$$R_k{}^r \xi_r = -(n+1)\xi_k.$$

Hence from (3.8) we have $\xi_k = p_k - {}'p_k = 0$ and $q_k - {}'q_k = 0$, therefore the decomposition is unique.

Now, from (3. 6), we get

$$\left[R_{kji}{}^r - \frac{1}{k}(g_{ki}\delta_j{}^r - g_{ji}\delta_k{}^r + \varphi_{ki}\varphi_j{}^r - \varphi_{ji}\varphi_k{}^r + 2\varphi_{kj}\varphi_i)\right]\rho_r = 0 \tag{3. 9}$$

which is equivalent to

$$Z_{kji}{}^r \rho_r = 0. \tag{3. 10}$$

Since the associated vector ρ_i of an analytic SHP-transformation is also an analytic SHP-transformation and is gradient, we have from (3. 10) and theorem (2. 1) the following.

THEOREM 3. 2. *In a Kählerian space, if the vector space of all analytic gradient SHP-transformation is transitive at each point, then the space is of constant holomorphic curvature.*

Moreover applying Lemma in [[4], Appendix II] we obtain from (3. 9)

THEOREM 3. 3. *If a Kählerian space admits an analytic SHP-transformation, then its local homogeneous holonomy group at any point is the full unitary group $U(n)$.*

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