On some F-connections in almost Hermitian manifolds

By

Sumio SAWAKI and Satoshi KOTO

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§1. Introduction

In almost Hermitian manifolds, many affine connections have been introduced independently and discussed systematically from the various viewpoints by many authors such as J. A. Schouten and K. Yano [2], K. Yano [1] and T. Suguri [5]. In the present paper, we shall try to discuss somewhat systematically, from another angle, the remarkable connections already obtained, to introduce some possibly new connections and to give the geometrical interpretation to some of them. As an appendix, we shall make a remark about the connection introduced by M. Obata [4]. In the last section we shall consider a space with the special Nijenhuis tensor which is anti-symmetric in all its indices and describe a few properties of the space, but this is a question to be further investigated in the future.

§2. Linear operators

Let X_{2n} be a $2n$-dimensional differential manifold of class C^2 admitting an almost complex structure defined by the tensor field $F_i^{\cdot h}$ of class C^1:

$$F_j^{\cdot h}F_i^{\cdot j}=-A_i^h \tag{2.1}$$

where A_i^h denotes the unit tensor. As is well known, it is always possible to give an almost Hermitian structure g_{ih} to an almost complex manifold:

$$F_i^{\cdot l}F_h^{\cdot k}g_{lk}=g_{ih} \tag{2.2}$$

If we put $F_{ih}=F_i^{\cdot k}g_{kh}$, then it is easily seen that F_{ih} is anti-symmetric in its lower indices. Let $P_{ji}^{\cdot\cdot h}$ or $P_{jih}=P_{ji}^{\cdot\cdot k}g_{kh}$ a tensor in the almost Hermitian manifold and we define the following linear operators operating on the tensor P_{jih} by K. Yano [3] or by M. Obata [4]:

$$\Phi_{ih}P_{jih}=\frac{1}{2}(P_{jih}-F_i^{\cdot a}F_h^{\cdot b}P_{jab}), \tag{2.3}$$

$$^*\Phi_{ih}P_{jih}=\frac{1}{2}(P_{jih}+F_i^{\cdot a}F_h^{\cdot b}P_{jab}).$$

Similarly, $\Phi_{ji}P_{jih}$, $*\Phi_{ji}P_{jih}$, $\Phi_{jh}P_{jih}$ and $*\Phi_{jh}P_{jih}$ will be defined. We have then for the two operators with the same indices:

$$\Phi\cdot\Phi=\Phi\,,\quad *\Phi\cdot *\Phi=*\Phi\,,\quad \Phi\cdot *\Phi=*\Phi\cdot\Phi=0\,,\quad \Phi+*\Phi=E \tag{2.4}$$

where E denotes identity operator.

If $\Phi_{ji}P_{jih}=0$ ($*\Phi_{ji}P_{jih}=0$), we say that P_{jih} is hybrid (pure) in ji and similarly define hybrid (pure) in ih and jh.

And also we define the following operators for later purpose:

$$\Psi_{ih}P_{jih}=\frac{1}{2}(P_{jih}-F_i^{\cdot a}F_h^{\cdot b}P_{jba})\,, \tag{2.5}$$

$$*\Psi_{ih}P_{jih}=\frac{1}{2}(P_{jih}+F_i^{\cdot a}F_h^{\cdot b}P_{jba})\,.$$

Similarly, $\Psi_{ji}P_{jih}$, $*\Psi_{ji}P_{jih}$, $\Psi_{jh}P_{jih}$ and $*\Psi_{jh}P_{jih}$ will be defined. Also for these operators, the two operators with the same indices have the following properties:

$$\Psi\cdot\Psi=\Psi\,,\quad *\Psi\cdot *\Psi=*\Psi\,,\quad \Psi\cdot *\Psi=*\Psi\cdot\Psi=0\,,\quad \Psi+*\Psi=E\,. \tag{2.6}$$

From these definitions, for the same indices, we obtain

$$\Phi\cdot\Psi=\Psi\cdot\Phi\,,\qquad *\Phi\cdot *\Psi=*\Psi\cdot *\Phi\,. \tag{2.7}$$

Moreover, if $P_{j[ih]}=0$, then we have

$$\Phi_{ih}P_{jih}=\Psi_{ih}P_{jih}\,. \tag{2.8}$$

LEMMA 2.1. *Given a tensor P_{jih}, in order that there exist a tensor Q_{jih} such that*

$$\Psi_{ih}Q_{jih}=P_{jih}\quad (*\Psi_{ih}Q_{jih}=P_{jih})\,,$$

it is necessary and sufficient that $*\Psi_{ih}P_{jih}=0$ ($\Psi_{ih}P_{jih}=0$).

PROOF. If $\Psi_{ih}Q_{jih}=P_{jih}$, then by (2.6) we have $*\Psi_{ih}\cdot\Psi_{ih}Q_{jih}=*\Psi_{ih}P_{jih}=0$. Conversely if $*\Psi_{ih}P_{jih}=0$, then by (2.6) $\Psi_{ih}P_{jih}+*\Psi_{ih}P_{jih}=P_{jih}$ and therefore we have $\Psi_{ih}P_{jih}=P_{jih}$ which shows that we may choose P_{jih} itself as a solution of $\Psi_{ih}Q_{jih}=P_{jih}$.

§ 3. Properties of the tensor $A_{ji}^{\cdot\cdot h}$

If we put

$$A_{ji}^{\cdot\cdot h}=\frac{1}{2}F^{lh}(\overset{\circ}{\nabla}_l F_{ji}+\overset{\circ}{\nabla}_i F_{jl})\,,\qquad A_{jih}=A_{ji}^{\cdot\cdot l}g_{lh} \tag{3.1}$$

where $\overset{\circ}{\nabla}_j$ denotes the covariant differentiation with respect to the Christoffel symbols $\left\{ {h \atop ji} \right\}$, then we have

(3.2) $\Psi_{ih}(F_j^{\cdot l}\overset{\circ}{\nabla}_i F_{hl})=A_{jih}\,,\qquad \Psi_{ih}(F_j^{\cdot l}\overset{\circ}{\nabla}_h F_{il})=A_{jhi}\,,$

(3.3) $\Psi_{ih}(\overset{\circ}{\nabla}_i F_{jh})=F_j^{\cdot l}A_{lih}\,,\qquad \Psi_{ih}(\overset{\circ}{\nabla}_h F_{ji})=F_j^{\cdot l}A_{lhi}\,,$

(3.4) $\Psi_{ih}A_{jih}=A_{jih}\,,\qquad \Psi_{ih}A_{jhi}=A_{jhi}\,,$

$$\Psi_{ih}A_{hij}=-\frac{1}{2}A_{jih}\,,\qquad \Psi_{ih}A_{ijh}=-\frac{1}{2}A_{jih}\,,$$

$$\Psi_{ih}A_{hji}=-\frac{1}{2}A_{jhi}\,,\qquad \Psi_{ih}A_{ihj}=-\frac{1}{2}A_{jhi}\,,$$

(3.5) $\Psi_{ih}(F_j^{\cdot s}F_{sih})=A_{jih}-A_{jhi}\,,$

where $F_{jih}=3\overset{\circ}{\nabla}_{[j}F_{ih]}\,,$

(3.6) $F_{(h}^{\cdot l}\overset{\circ}{\nabla}_{j)}F_{il}=A_{hji}+A_{jhi}\,,$

(3.7) $A_{(jih)}=0$ and $A_{ji}^{\cdot\cdot l}=0\,.$

For the Nijenhuis tensor $N_{ji}^{\cdot\cdot h}=\frac{1}{2}(F_{[j}^{\cdot a}\overset{\circ}{\nabla}_{|a|}F_{i]}^{\cdot h}-F_{[j}^{\cdot a}\overset{\circ}{\nabla}_{i]}F_a^{\cdot h})$, we have

(3.8) $$\Psi_{ih}N_{jih}=\frac{1}{2}(N_{jih}+N_{jhi})\,,$$

(3.9) $$N_{jih}+N_{jhi}=2(A_{jih}+A_{jhi})\,.$$

From (3.5), (3.7) and (3.9), we obtain the following two lemmas.

LEMMA 3.1. *If* $F_{jih}=0$, *then we have* $A_{jih}=A_{jhi}$ *and*

$$A_{jih}+A_{ihj}+A_{hji}=0\,.$$

LEMMA 3.2. $N_{j(ih)}=0$ *if and only if* $A_{j(ih)}=0\,.$

Finally, the following relation can be easily verified:

(3.10) $$^*\Phi_{ji}\cdot\Phi_{ih}A_{jih}=0\,.$$

This implies that $A_{ji}^{\cdot\cdot h}$ is pure in all indices [9, p. 79].

§ 4. *F*-connection

In an almost Hermitian manifold an affine connection Γ_{ji}^h is called an F-connection if $\nabla_j F_{ih}=0$ where ∇_j denotes the covariant differentiation with respect to Γ_{ji}^h.

THEOREM 4.1. *In order that an affine connection* Γ_{ji}^h *in an almost Hermitian manifold may be an F-connection it is necessary and sufficient that* Γ_{ji}^h *be written in the form*

(4.1) $$\Gamma_{ji}^h=\left\{{h \atop ji}\right\}-\frac{1}{2}F^{lh}(\overset{\circ}{\nabla}_j F_{il})+(\Psi_{il}Q_{jil})g^{lh}\,,$$

where Q_{jih} *is an arbitrary tensor.*

PROOF. If we put $\Gamma_{ji}^h=\left\{{h \atop ji}\right\}+Q_{ji}^{\cdot\cdot h}$ for a tensor $Q_{ij}^{\cdot\cdot h}$, Γ_{ji}^h is an affine connection. We have then

$$\nabla_j F_{ih} = \overset{\circ}{\nabla}_j F_{ih} - (Q_{ji}^{\cdot\cdot a} F_{ah} + Q_{jh}^{\cdot\cdot a} F_{ia}) .$$

In order that $\nabla_j F_{ih}=0$, it is necessary and sufficient that we have

$$\overset{\circ}{\nabla}_j F_{ih} = Q_{ji}^{\cdot\cdot a} F_{ah} + Q_{jh}^{\cdot\cdot a} F_{ia} , \tag{4.2}$$

which is equivalent to

$$F_h^{\cdot b}(\overset{\circ}{\nabla}_j F_{ib}) = Q_{jih} + F_i^{\cdot a} F_h^{\cdot b} Q_{jba} . \tag{4.3}$$

This is written in the form

$$*\Psi_{ih} Q_{jih} = \frac{1}{2} F_h^{\cdot l}(\overset{\circ}{\nabla}_j F_{il}) . \tag{4.4}$$

From the lemma 2.1, there exists a tensor Q_{jih} satisfying (4.4) if and only if $\Psi_{ih}\left(\frac{1}{2} F_h^{\cdot l} \overset{\circ}{\nabla}_j F_{il}\right)=0$ but it follows easily from $F_{ji}=-F_{ij}$. Then, in order to have $\nabla_j F_{ih}=0$ it is necessary and sufficient that Q_{jih} be of the form

$$Q_{jih} = \frac{1}{2} F_h^{\cdot l} \overset{\circ}{\nabla}_j F_{il} + \Psi_{ih} Q_{jih} .$$

Thus, Γ_{ji}^h has the following form

$$\Gamma_{ji}^h = \left\{ {h \atop ji} \right\} - \frac{1}{2} F^{lh} \overset{\circ}{\nabla}_j F_{il} + (\Psi_{il} Q_{jil}) g^{lh} .$$

And Γ_{ji}^h is called a metric connection if $\nabla_j g_{ih}=0$. We have defined F-connection corresponding to the Obata's ϕ-connection in an almost complex manifold, i. e. $\nabla_j \phi_i^{\cdot h}=0$ for the almost complex structure $\phi_i^{\cdot h}$ [4], but if a ϕ-connection is metric, it is also a metric F-connection in an almost Hermitian manifold.

Next, we shall prove two lemmas needed to establish the unity of the tensor satisfying some conditions.

LEMMA 4.1. *If a tensor P_{jih} satisfies $P_{jih}+P_{ijh}=0$ and $*\Psi_{ih} P_{jih}=0$, then P_{jih} vanishes.*

PROOF. From $*\Psi_{ih} P_{jih}=0$, we have

$$P_{jih} + F_i^{\cdot a} F_h^{\cdot b} P_{jba} = 0 \tag{4.5}$$

and by $P_{(ji)h}=0$, (4.5) becomes

$$P_{jih} + F_j^{\cdot a} F_h^{\cdot b} P_{bia} = 0 . \tag{4.6}$$

From (4.5) and (4.6), we obtain

$$F_j^{\cdot a} F_h^{\cdot b} P_{bia} - F_i^{\cdot a} F_h^{\cdot b} P_{jba} = 0 \tag{4.7}$$

or

$$P_{jih} - F_i^{\cdot a} F_h^{\cdot b} P_{jba} = 0 . \tag{4.8}$$

Thus by (4.5) and (4.8), we have

$$P_{jih}=0 .$$

The following lemma is also proved similarly.

LEMMA 4.2. *If a tensor P_{jih} satisfies $P_{jih}+P_{hij}=0$ and $*\Psi_{ih}P_{jih}=0$, then P_{jih} vanishes.*

Now if we put

$$\overset{1}{\Gamma}{}^{h}_{ji}=\left\{ {h \atop ji} \right\}-\frac{1}{2}F^{lh}(\overset{\circ}{\nabla}_j F_{il}) , \tag{4.9}$$

this connection is a metric F-connection by virtue of theorem 4.1.

THEOREM 4.2. *In an almost Hermitian manifold, $\Gamma^{h}_{ji}=\overset{1}{\Gamma}{}^{h}_{ji}+P_{ji}{}^{h}$ is a metric connection if and only if P_{jih} is anti-symmetric with respect to the indices i and h.*

PROOF.
$$\begin{aligned}\nabla_j g_{ih}&=\partial_j g_{ih}-\Gamma^{a}_{ji}g_{ah}-\Gamma^{a}_{jh}g_{ia}\\ &=\overset{\circ}{\nabla}_j g_{ih}-P_{ji}{}^{a}g_{ah}-P_{jh}{}^{a}g_{ia}\\ &=-(P_{jih}+P_{jhi}) .\end{aligned}$$

By using the tensor A_{jih}, we can write the four remarkable F-connections in an almost Hermitian manifold in the following forms:

$$\overset{1}{\Gamma}{}^{h}_{ji}=\left\{ {h \atop ji} \right\}-\frac{1}{2}F^{lh}\overset{\circ}{\nabla}_j F_{il} , \tag{4.10}$$

$$\overset{2}{\Gamma}{}^{h}_{ji}=\overset{1}{\Gamma}{}^{h}_{ji}-A_{ji}{}^{h} , \tag{4.11}$$

$$\overset{3}{\Gamma}{}^{h}_{ji}=\overset{1}{\Gamma}{}^{h}_{ji}+A_{ji}{}^{h} , \tag{4.12}$$

$$\overset{4}{\Gamma}{}^{h}_{ji}=\overset{1}{\Gamma}{}^{h}_{ji}-\frac{1}{2}(A_{ji}{}^{h}-A_{j}{}^{h}{}_{i}) . \tag{4.13}$$

In addition to these connections, we put

$$'\overset{2}{\Gamma}{}^{h}_{ji}=\overset{1}{\Gamma}{}^{h}_{ji}-A_{j}{}^{h}{}_{i} , \tag{4.14}$$

$$'\overset{3}{\Gamma}{}^{h}_{ji}=\overset{1}{\Gamma}{}^{h}_{ji}+A_{j}{}^{h}{}_{i} , \tag{4.15}$$

$$\overset{5}{\Gamma}{}^{h}_{ji}=\overset{1}{\Gamma}{}^{h}_{ji}+\gamma(A_{ji}{}^{h}+A_{j}{}^{h}{}_{i}) , \tag{4.16}$$

$$\overset{6}{\Gamma}{}^{h}_{ji}=\overset{1}{\Gamma}{}^{h}_{ji}+\gamma F_{j}{}^{l}(A_{li}{}^{h}-A_{l}{}^{h}{}_{i}) , \tag{4.17}$$

$$\overset{7}{\Gamma}{}^{h}_{ji}=\overset{1}{\Gamma}{}^{h}_{ji}+\gamma F_{j}{}^{l}(A_{li}{}^{h}+A_{l}{}^{h}{}_{i}) , \tag{4.18}$$

where γ is an arbitrary constant.

$\overset{1}{\Gamma}$ has been defined by A. Lichnerowicz [7], $\overset{2}{\Gamma}$ and $\overset{3}{\Gamma}$ by K. Yano [1] and $\overset{4}{\Gamma}$ by J. A. Schouten and K. Yano [2] and P. Liebermann [6], and a geometrical characterization for each of them has been given by J. A. Schouten and K. Yano [2]. These connections can be obtained, from the theorem 4.1 and properties of A_{jih} in

§3, by putting a suitable tensor in the place of Q_{jih} in (4.1).

For instace, if we put $Q_{jih}=-F_j^{\cdot l}\overset{\circ}{\nabla}_i F_{hl}$, from (3.2), we obtain $\overset{2}{\Gamma}$, if we put $Q_{jih}=\frac{1}{2}F_j^{\cdot s}\underset{3}{F}_{sih}$, from (3.5), we obtain $\overset{4}{\Gamma}$ and if we put $Q_{jih}=F_j^{\cdot l}\overset{\circ}{\nabla}_h F_{il}$, from (3.2), we obtain $'\Gamma$.

Consequently, of course, these connections are F-connections and, from theorem 4.2, $\overset{1}{\Gamma}$, $\overset{4}{\Gamma}$ and $\overset{6}{\Gamma}$ are metric. In this way, we can construct many F-connections, but it is not easy to give a geometrical interpretation to every new connection.

Now, when Γ_{ji}^h is an F-connection, $'\Gamma_{ji}^h\equiv\Gamma_{ij}^h$ is not necessarily an F-connection, but from

$$'\Gamma_{ji}^h=\Gamma_{ji}^h-2S_{ji}^{\cdot\cdot h}, \tag{4.19}$$

where $S_{ji}^{\cdot\cdot h}$ is the torsion tensor of Γ_{ji}^h, and theorem 4.1, we can conclude that $'\Gamma_{ji}^h$ is also an F-connection if and only if $^*\Psi_{ih}S_{jih}=0$. By using this fact, we have the following theorem.

THEOREM 4.3. *In order that there may exist in an almost Hermitian manifold a metric F-connection* Γ_{ji}^h *where* $'\Gamma_{ji}^h\equiv\Gamma_{ij}^h$ *is also an F-connection, it is necessary and sufficient that the manifold be Kählerian.*

PROOF. In fact, if $'\Gamma_{ji}^h\equiv\Gamma_{ji}^h$ is a an F-connection, because of $^*\Psi_{ih}S_{jih}=0$ and $S_{(ji)h}=0$, by lemma 4.1, we get $S_{jih}=0$. But, since Γ_{ji}^h is metric, by using this Γ_{ji}^h, we can write N_{jih} in the form

$$N_{jih}=\frac{1}{2}(S_{jih}-F_j^{\cdot a}F_i^{\cdot b}S_{abh}-F_j^{\cdot a}F_h^{\cdot b}S_{aib}-F_i^{\cdot a}F_h^{\cdot b}S_{jab})\,. \tag{4.20}$$

And therefore $N_{jih}=0$.

On the other hand, when $\nabla_j F_{ih}=0$, we have

$$F_{jih}=-\frac{2}{3}(F_h^{\cdot a}S_{jia}+F_i^{\cdot a}S_{hja}+F_j^{\cdot a}S_{iha}) \tag{4.21}$$

and consequently it follows $F_{jih}=0$. $N_{jih}=0$ and $F_{jih}=0$ imply the manifold is Kählerian.

The converse is evident.

REMARK 1. Let $\overset{1}{S}_{ji}^{\cdot\cdot h}$ be the torsion tensor of $\overset{1}{\Gamma}_{ji}^h$. Then we have

$$\overset{1}{S}_{jih}=\frac{1}{4}F_h^{\cdot l}(\nabla_j F_{il}-\nabla_i F_{jl}) \tag{4.22}$$

$$=\frac{1}{2}N_{jih}+\frac{1}{4}{}^*\Phi_{ji}(F_h^{\cdot l}F_{jil})$$

and hence

$$\Phi_{ji}\overset{1}{S}_{jih}=\frac{1}{2}N_{jih}\,. \tag{4.23}$$

Since for a Hermitian manifold we have $N_{jih}=0$, from (4.23), it follows that an almost Hermitian manifold is Hermitian if and only if $\Phi_{ji}\overset{1}{S}_{jih}=0$.

REMARK 2. Let $\overset{4}{S}_{jih}$ be the torsion tensor of metric F-connection $\overset{4}{\Gamma}{}^h_{ji}$. From (3.5), we have

$$\overset{4}{S}_{jih}=\overset{1}{S}_{jih}-\frac{1}{4}[\Psi_{ih}(F_j{}^lF_{lih})-\Psi_{jh}(F_i{}^lF_{ljh})]\,. \tag{4.24}$$

Substitutiry (4.22) into (4.24), we have

$$\overset{4}{S}_{jih}=\frac{1}{2}N_{jih}+\frac{1}{8}(F_h{}^aF_{jia}-F_i{}^bF_{hjb}-F_j{}^cF_{ihc}-F_j{}^cF_i{}^bF_h{}^aF_{cba}) \tag{4.25}$$

$$=\frac{1}{2}N_{jih}-\frac{1}{2}\Phi_{ji}\cdot{}^*\Phi_{ih}(F_j{}^cF_{cih})\,. \tag{4.26}$$

And therefore, from the fact that two metric F-connections coincide with each other if and only if their torsion tensors coincide with each other, it follows that the Obata's connection [4, p. 73] coincides with $\overset{4}{\Gamma}{}^h_{ji}$.

Or multiplying (4.26) by ${}^*\Phi_{ji}$, since ${}^*\Phi_{ji}N_{jih}=0$ and (2.4), we have

$${}^*\Phi_{ji}\overset{4}{S}_{jih}=0\,. \tag{4.27}$$

Now (4.27) shows that $\overset{4}{S}_{jih}$ is pure in ji but by J. A. Schouten and K. Yano [2] or T. Suguri [5] we know that metric F-connection of which torsion tensor is pure in ji is unique, and therefore also through this fact we reach the same result.

§5. On the geometrical interperetation

We can give the geometric interpretation completely analogous to [2] to the connections $'\overset{2}{\Gamma}$ and $'\overset{3}{\Gamma}$, and some geometrical interpretation to the connection $\overset{5}{\Gamma}$ and $\overset{7}{\Gamma}$ also.

1. Consider two contravariant vectors u^h and $v^h\equiv F_s{}^h u^s$, and a covariant vector $F_{ih}u^h$ which contains the vector u^h. We transport the covariant vector $F_{ih}u^h$ parallelly with respect to the Riemannian connection $\left\{ {h \atop ji} \right\}$ and with respect to the affine connection $'\overset{2}{\Gamma}{}^h_{ji}$ from the point ξ^h to the point $\xi^h+\varepsilon v^h$. Then we get at ξ^h and $\xi^h+\varepsilon v^h$ respectively

$$F_{ih}u^h+\varepsilon v^j\left\{ {l \atop ji} \right\}F_{lh}u^h$$

or

$$F_{ih}u^h+\varepsilon F_s{}^j u^s\left\{ {l \atop ji} \right\}F_{lh}u^h\,, \tag{5.1}$$

and

$$F_{ih}u^h+\varepsilon v^j\,'\overset{2}{\Gamma}{}^l_{ji}F_{lh}u^h$$

or

(5.2) $$F_{ih}u^h+\varepsilon F_s^{\cdot j}u^s\left[\left\{{l \atop ji}\right\}-\frac{1}{2}(\overset{\circ}{\nabla}_j F_{it})F^{tl}-A_{j\cdot i}^{\cdot l}\right]F_{lh}u^h\,.$$

In this case, we can verify that (5.1) coincides with (5.2). To prove this fact, it is sufficient to show the following relation:

(5.3) $$F_s^{\cdot j}\left[\frac{1}{2}(\overset{\circ}{\nabla}_j F_{it})F^{tl}+A_{j\cdot i}^{\cdot l}\right]F_{lh}u^s u^h=0$$

holds for any vector u^h, that is,

(5.4) $$[(\overset{\circ}{\nabla}_j F_{it})F^{tl}+2A_{j\cdot i}^{\cdot l}]F_s^{\cdot j}F_{lh}+[(\overset{\circ}{\nabla}_j F_{it})F^{tl}+2A_{j\cdot i}^{\cdot l}]F_h^{\cdot j}F_{ls}=0$$

holds.

Now, substituting (3.1) into the first term of (5.4), this equality can be easily verified by a straightforward calculation.

Next, we shall show that the affine connection with such properties is unique. Namely, let Γ_{ji}^h an arbitrary F-connection and u^h an arbitrary vector. We transport the covariant vector $F_{ih}u^h$ parallelly with respect to the connection $\left\{{h \atop ji}\right\}$ and with respect to Γ_{ji}^h from the point ξ^h to the point $\xi^h+\varepsilon F_s^{\cdot h}u^s$ respectively. In this case, if the two transported vectors coincide with each other, Γ_{ji}^h coincides with $'\overset{2}{\Gamma}{}_{ji}^h$.

In fact, an arbitrary F-connection Γ_{ji}^h, by virtue of theorem 4.1, can be written as

(5.5) $$\Gamma_{ji}^h={}'\Gamma_{ji}^h+P_{ji}^{\cdot\cdot h}\,,\qquad {}^*\Psi_{ih}P_{jih}=0\,.$$

And therefore, when

(5.6) $$F_{ih}u^h+\varepsilon F_s^{\cdot j}u^s[{}'\overset{2}{\Gamma}{}_{ji}^l+P_{ji}^{\cdot\cdot l}]F_{lh}u^h$$

coincides with (5.1), that is,

(5.7) $$F_s^{\cdot j}P_{ji}^{\cdot\cdot l}F_{lh}+F_h^{\cdot j}P_{ji}^{\cdot\cdot l}F_{ls}=0\,,$$

it is sufficient to show

(5.8) $$P_{ji}^{\cdot\cdot h}=0\,.$$

From (5.7), we get

(5.9) $$P_{jih}+P_{hij}=0\,.$$

Thus together with ${}^*\Psi_{ih}P_{jih}=0$, from lemma 4.2, we have $P_{jih}=0$.

2. We consider the geodesics of the Riemannian metric $ds^2=g_{ih}d\xi^i d\xi^h$ and the autoparallel curve with respect to the connection $'\overset{3}{\Gamma}{}_{ji}^h$.

When the contravariant vector $F_k^{\cdot h}\dfrac{dx^k}{ds}$ is displaced parallelly along the two curves respectively, it is verified that they coincide with each other. And moreover the unity of the connection with such properties will be verified. For, from the relations

$$\frac{d}{ds}\left(F_{k}^{\cdot h}\frac{dx^{k}}{ds}\right)+\left\{ {h \atop ji} \right\}\frac{dx^{j}}{ds}\left(\frac{dx^{k}}{ds}F_{k}^{\cdot i}\right)=0\,, \tag{5.10}$$

$$\frac{d}{ds}\left(F_{k}^{\cdot h}\frac{dx^{k}}{ds}\right)+'\overset{3}{\Gamma}{}_{ji}^{h}\frac{dx^{j}}{ds}\left(\frac{dx^{k}}{ds}F_{k}^{\cdot i}\right)=0\,, \tag{5.11}$$

it is sufficient to show that the following relation is verified:

$$\left\{ {h \atop ji} \right\}F_{k}^{\cdot i}-'\overset{3}{\Gamma}{}_{ji}^{h}F_{k}^{\cdot i}+\left\{ {h \atop ki} \right\}F_{j}^{\cdot i}-'\overset{3}{\Gamma}{}_{ki}^{h}F_{j}^{\cdot i}=0 \tag{5.12}$$

or

$$\frac{1}{2}F^{lh}F_{k}^{\cdot i}(\overset{\circ}{\nabla}_{j}F_{il})+\frac{1}{2}F^{lh}F_{j}^{\cdot i}(\overset{\circ}{\nabla}_{k}F_{il})-F_{k}^{\cdot i}A_{j\cdot i}^{\cdot h}-F_{j}^{\cdot i}A_{k\cdot i}^{\cdot h}=0\,. \tag{5.13}$$

This is easily verified from (3.1). Finally, the unity can be proved in the same way as in the case of $'\overset{2}{\Gamma}{}_{ji}^{h}$.

3. According to J. A. Schouten and K. Yano [2], the affine connection satisfying the three conditions:

a) $\nabla_{j}g_{ih}=0$ b) $\nabla_{j}F_{ih}=0$

c) $^{*}\Phi_{ji}{}^{*}S_{ji}{}^{h}=0$ (i.e. in each invariant R_2 there exist parallelograms)

does not exist except $\overset{4}{\Gamma}{}_{ji}^{h}$.

But, for instance, the following connection

$$\Gamma_{ji}^{h}=\overset{4}{\Gamma}{}_{ji}^{h}+\gamma(A_{ji}{}^{h}+A_{j\cdot i}^{\cdot h})+\lambda F_{j}^{\cdot a}(A_{ai}{}^{h}+A_{a\cdot i}^{\cdot h})\,, \tag{5.14}$$

where γ and λ are arbitrary constants, satisfies the two conditions b) and c).

In fact, by (3.9) and the fact that N_{jih} is pure in ji and ih, we have

$$^{*}\Phi_{ji}(A_{jih}+A_{jhi}-A_{ijh}-A_{ihj})=\frac{1}{2}{}^{*}\Phi_{ji}(2N_{jih}-N_{hji}+N_{hij})=0\,. \tag{5.15}$$

On the other hand

$$^{*}\Phi_{ji}\{F_{j}^{\cdot a}(A_{aih}+A_{ahi})-F_{i}^{\cdot a}(A_{ajh}+A_{ahj})\}=0 \tag{5.16}$$

is verified, from (3.1), by a straightforward calculation. Thus, for its torsion tensor $S_{ji}{}^{h}$, we have

$$^{*}\Phi_{ji}S_{jih}=0\,.$$

THEOREM 5.1. *The general connection Γ_{ji}^{h} satisfying the two conditions* b) *and* c) *can be written as*

$$\Gamma_{ji}^{h}=\overset{4}{\Gamma}{}_{ji}^{h}+2T_{ji}{}^{h}+T_{ij}{}^{h}-T^{h}_{\cdot ij}+F_{i}^{\cdot a}F_{j}^{\cdot b}T_{ab}{}^{h}-F^{ha}F_{j}^{\cdot b}T_{abi} \tag{5.17}$$

where T_{jih} is an arbitrary tensor in which $T_{jih}=T_{jhi}$ and $*\Psi_{ji}T_{jih}=0$.

PROOF. Putting $\Gamma^h_{ji}=\overset{4}{\Gamma}{}^h_{ji}+P_{ji}{}^h$, its torsion tensor S_{jih} becomes

(5.18) $$S_{jih}=\overset{4}{S}_{jih}+\frac{1}{2}(P_{jih}-P_{ijh}).$$

Here Γ^h_{ji} satisfies the conditions b) and c) if and only if $*\Psi_{ih}P_{jih}=0$ and $*\Phi_{ji}(P_{jih}-P_{ijh})=0$, that is,

(5.19) $$P_{jih}+F_i{}^aF_h{}^bP_{jba}=0$$

and

(5.20) $$P_{jih}-P_{ijh}+F_i{}^aF_j{}^b(P_{bah}-P_{abh})=0.$$

From (5.19) and (5.20), we can derive

(5.21) $$P_{jhi}+P_{ihj}+F_i{}^aF_j{}^b(P_{bha}+P_{ahb})=0.$$

And then putting

(5.22) $$T_{jih}=P_{jih}+P_{jhi},$$

if we substitute (5.22) into (5.21), we have

(5.23) $$P_{jih}+P_{ijh}-(T_{jih}+T_{ijh})+F_i{}^aF_j{}^b(P_{bah}+P_{abh})-F_i{}^aF_j{}^b(T_{bha}+T_{ahb})=0.$$

From (5.20)+(5.23), we obtain

(5.24) $$P_{jih}+F_i{}^aF_j{}^bP_{bah}-\frac{1}{2}(T_{jih}+T_{ijh})-\frac{1}{2}F_i{}^aF_j{}^b(T_{bah}+T_{ahb})=0$$

or

(5.25) $$-F_i{}^aF_h{}^bP_{jba}+F_i{}^aF_j{}^bP_{bah}-\frac{1}{2}(T_{jih}+T_{ijh})-\frac{1}{2}F_i{}^aF_j{}^b(T_{bah}+T_{ahb})=0$$

and on multiplying (5.25) by $F^j{}_kF^i{}_l$ and using $P_{jhi}=T_{jih}-P_{jih}$, we get

(5.26) $$P_{jih}-F_j{}^bF_i{}^aP_{bah}+\frac{1}{2}F_h{}^aF_j{}^b(T_{bai}+T_{abi})-\frac{1}{2}(T_{jih}-T_{hij})=0.$$

By (5.24)+(5.26), we have

(5.27) $$4P_{jih}=2T_{jih}+T_{ijh}-T_{hij}+F_i{}^aF_j{}^bT_{abh}-F_h{}^aF_j{}^bT_{abi}+F_i{}^aF_j{}^bT_{bah}-F_h{}^aF_j{}^bT_{bai},$$

but by virtue of $*\Psi_{ih}T_{jih}=0$, $F_i{}^aF_j{}^bT_{bah}-F_h{}^aF_j{}^bT_{bai}=0$ and thus we have (5.17).

§6. Half-Hermitian manifold

If an almost Hermitian manifold has the Nijenhuis tensor which is anti-symmetric in all indices, then the manifold will be called a half-Hermitian manifold. In a half-Hermitian manifold, from lemma 3.2, $\overset{5}{\Gamma}$ and $\overset{7}{\Gamma}$ coincide with $\overset{1}{\Gamma}$. Moreover $\overset{2}{\Gamma}$, $\overset{3}{'\Gamma}$ and $\overset{4}{\Gamma}$ coincide with each other and they become metric. Next, an F-connection Γ_{ji}^h will be said to be half-symmetric if its torsion tensor $S_{ji}{}^h$ satisfies

$$\Psi_{ih}\Psi_{ji}S_{jih}=0\,. \tag{6.1}$$

Now we have

THEOREM 6.1. *In order that in an almost Hermitian manifold there exist a half-symmetric metric F-connection, it is necessary and sufficient that the manifold be a half-Hermitian manifold.*

PROOF. If Γ_{ji}^h is a half-symmetric metric F-connection,

$$4\Psi_{ih}\Psi_{ji}S_{jih}=S_{jih}-F_j{}^aF_i{}^bS_{abh}-F_i{}^lF_h{}^m(S_{jml}-F_j{}^aF_m{}^bS_{abl})=0\,. \tag{6.2}$$

Hence, we have

$$\begin{aligned}&4\Psi_{ih}\Psi_{ji}S_{jih}+4\Psi_{ih}\Psi_{jh}S_{jhi}\\&=S_{jih}-F_j{}^aF_i{}^bS_{abh}-F_i{}^bF_h{}^cS_{jcb}-F_i{}^bF_j{}^aS_{ahb}\\&\quad+S_{jhi}-F_j{}^aF_h{}^bS_{abi}-F_h{}^bF_i{}^cS_{jcb}-F_h{}^bF_j{}^aS_{aib}=0\,.\end{aligned} \tag{6.3}$$

But, since Γ_{ji}^h is a metric F-connection, the Nijenhuis tensor N_{jih} has the form (4.20).

By using (4.20), from (6.3), we have

$$N_{jih}+N_{jhi}=0\,.$$

Conversely if $N_{jih}+N_{jhi}=0$, we may choose $\overset{1}{\Gamma}{}_{ji}^h$ as the desired connection, because

$$\begin{aligned}4\Psi_{ih}\Psi_{ji}\overset{1}{S}_{jih}&=2\Psi_{ih}N_{jih}\\&=N_{jih}+N_{jhi}=0\,.\end{aligned} \tag{6.4}$$

THEOREM 6.2. *In order that in an almost Hermitian manifold there exist a metric F-connection Γ_{ji}^h with the torsion tensor satisfying $S_{j(ih)}=0$ it is necessary and sufficient that the manifold be a half-Hermitian manifold.*

PROOF. If $S_{j(ih)}=0$, from (4.20) we have

$$N_{jih}+N_{jhi}=0\,.$$

Conversely if $N_{jih}+N_{jhi}=0$, we can prove that the connection $\overset{3}{\Gamma}{}_{ji}^h$ is a desired one.

In fact, if $N_{jih}+N_{jhi}=0$, from (3.9) and theorem 4.2, $\overset{3}{\Gamma}{}_{ji}^h$ is metric. And from,

$$2\overset{3}{S}_{jih}=\frac{1}{2}(F_h{}^l\nabla_jF_{il}-F_h{}^l\nabla_iF_{jl})+A_{jih}-A_{ijh} \tag{6.5}$$

and

$$\frac{1}{2}(N_{jih}+N_{jhi})=A_{jih}+A_{jhi}=0\,, \tag{6.6}$$

we have

$$2(\overset{3}{S}_{jih}+\overset{3}{S}_{jhi})=\frac{1}{6}A_{(jih)}\,. \tag{6.7}$$

But, by virtue of (3.7), the second term vanishes.

Remark. In a half-Hermitian manifold, there does not exist a metric F-connection of which torsin tenson satisfies $S_{j(ih)}=0$ except $\overset{3}{\Gamma}{}^h_{ji}$.

For, if we put a general metric F-connection

$$\Gamma^h_{ji}=\overset{3}{\Gamma}{}^h_{ji}+P_{ji}{}^h \tag{6.8}$$

where $P_{j(ih)}=0$ and $*\Psi_{ih}P_{jih}=0$, and let S_{jih} its torsion tensor, by the assumption, we have

$$S_{j(ih)}=\overset{3}{S}_{j(ih)}+\frac{1}{4}(P_{jih}-P_{ijh}+P_{jhi}-P_{hji})=\frac{1}{4}(P_{ihj}+P_{hij})=0\,. \tag{6.9}$$

Thus from (6.9) and $*\Psi_{ih}P_{jih}=0$, by virtue of lemma 4.1 we have $P_{jih}=0$ as claimed.

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