Nihonkai Math. J.
Vol.17(2006), 125-133

REVERSES OF OPERATOR INEQUALITIES ON OPERATOR MEANS

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ABSTRACT. In this note, we improve the non-commutative Kantorovich inequality as follows: If A, B satisfy $0 < m \leq A, B \leq M$, then for each $\mu \in [0,1]$

$$A \nabla_\mu B \leq \frac{M \nabla_\mu m}{M \,!_\mu\, m} A \,!_\mu\, B,$$

where $A \,!_\mu\, B$ is the μ-harmonic mean and $A\nabla_\mu B$ is the μ-arithmetic mean. Next we discuss the optimality of the constant $(\sqrt{M}-\sqrt{m})^2$ in the difference reverse inequality

$$A\nabla B - A \,!\, B \leq (\sqrt{M}-\sqrt{m})^2$$

for all positive invertible A, B with $0 < m \leq A, B \leq M$.

In addition, we compare the μ-geometric mean $A \,\sharp_\mu\, B$ with $A\nabla_\mu B$, $A \,!_\mu\, B$ and $\frac{1}{2}(A\nabla_\mu B + A \,!_\mu\, B)$ for positive operators A and B.

1. Noncommutative Kantorovich inequality. Let Φ be a unital positive linear map on $B(H)$, the C^*-algebra of all bounded linear operators on a Hilbert space H. Then Kadison's Schwarz inequality asserts

$$\Phi(A^{-1})^{-1} \leq \Phi(A) \tag{1}$$

for all positive invertible $A \in B(H)$.

If Φ is defined on $B(H) \oplus B(H)$ by

$$\Phi(A \oplus B) = \frac{1}{2}(A+B) \quad \text{for } A, B \in B(H), \tag{2}$$

then Φ satisfies

$$\Phi((A \oplus B)^{-1})^{-1} = A \,!\, B, \quad \Phi(A \oplus B) = A\nabla B \tag{3}$$

for all positive invertible $A, B \in B(H)$, where $A \,!\, B$ is the harmonic operator mean and $A\nabla B$ is the arithmetic operator mean in the sense of Kubo-Ando [5]. Consequently, Kadison's Schwarz inequality implies the arithmetic-harmonic mean inequality, i.e., $A \,!\, B \leq A\nabla B$, cf. [1] and [3].

By the same discussion as in above, the weighted arithmetic-harmonic mean inequality, i.e., $A \,!_\mu\, B \leq A\nabla_\mu B$ for $\mu \in [0,1]$, is proved.

1991 *Mathematics Subject Classification.* 47A63, 47A64 and 47B15.
Key words and phrases. Kantorovich inequality, reverse inequality, operator mean.

Moreover a reverse of Kadison's Schwarz inequality is known as follows:

$$\Phi(A) \le \frac{(M+m)^2}{4Mm}\Phi(A^{-1})^{-1} \tag{4}$$

if A satisfies $0 < m \le A \le M$ for some constants $m < M$, cf. [4, Theorem 1.32] and [3]. Thus it follows that

$$A\nabla_\mu B \le \frac{(M+m)^2}{4Mm} A!_\mu B \tag{5}$$

for A, B with $0 < m \le A, B \le M$. It is nothing but the noncommutative Kantorovich inequality introduced in [1] (for the case $\mu = \frac{1}{2}$), cf. [3]. We here remark the following facts:

(1) The Kantorovich constant $\frac{(M+m)^2}{4Mm}$ is understood as the ratio of $M\nabla m$ by $M!m$, that is,

$$\frac{(M+m)^2}{4Mm} = \frac{M\nabla m}{M!m}.$$

(2) The Kantorovich constant is the maximum among $\{\frac{M\nabla_\mu m}{M!_\mu m}; \mu \in [0,1]\}$. That is,

$$\frac{(M+m)^2}{4Mm} = \frac{M\nabla m}{M!m} \ge \frac{M\nabla_\mu m}{M!_\mu m}$$

for all $\mu \in [0,1]$.

Based on these facts, we prove the following improvement:

Theorem 1. *If A, B satisfy $0 < m \le A, B \le M$, then for each $\mu \in [0,1]$*

$$A\nabla_\mu B \le \frac{M\nabla_\mu m}{M!_\mu m} A!_\mu B. \tag{6}$$

Proof. We put $K_\mu = \frac{M\nabla_\mu m}{M!_\mu m}$, $C = A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$ and $h = \frac{M}{m}$. Then it suffices to show that

$$1\nabla_\mu C \le K_\mu \; 1!_\mu C,$$

by the transformer inequality, or equivalently,

$$1\nabla_\mu t \le K_\mu \; 1!_\mu t \text{ for } t \in [h^{-1}, h].$$

This follows from $K_\mu = \max\{\frac{1\nabla_\mu t}{1 \, !_\mu \, t}; t \in [h^{-1}, h]\}$.

2. Reverse inequalities of difference type. A difference version of the noncommutative Kantorovich inequality is also introduced by

$$A\nabla B - A \,!\, B \le (\sqrt{M} - \sqrt{m})^2 \tag{7}$$

for all positive invertible $A, B \in B(H)$ with $0 < m \leq A, B \leq M$, cf. [1, Theorem 6]. More generally, it has already known in [4, Theorem 1.32] that

$$\Phi(A) - \Phi(A^{-1})^{-1} \leq (\sqrt{M} - \sqrt{m})^2 \tag{8}$$

for all positive invertible $A \in B(H)$ with $0 < m \leq A \leq M$.

On the other hand, the optimality of the constant $(\sqrt{M} - \sqrt{m})^2$ has been discussed. It is shown by the following example in [2, Example 2.4]:

Example 2. *Let A and B be 2×2 matrices defined by*

$$A = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \frac{1}{3}\begin{pmatrix} 4 & 2\sqrt{2} \\ 2\sqrt{2} & 11 \end{pmatrix}.$$

Then we can take $m = 1$ and $M = 4$ because spectra of both A and B are $\{1, 4\}$. Furthermore we have

$$A \nabla B = \frac{1}{3}\begin{pmatrix} 8 & \sqrt{2} \\ \sqrt{2} & 7 \end{pmatrix}$$

and

$$A \,!\, B = \frac{2}{9}\begin{pmatrix} 8 & \sqrt{2} \\ \sqrt{2} & 7 \end{pmatrix}.$$

We pay our attention to the fact that

$$A \,!\, B = \frac{2}{3} A \nabla B$$

in this example, and show that it happens often in the following way:

Lemma 3. *Let A and B be 2×2 matrices satisfying $|A| = |B| \neq 0$ and $|A \nabla_\mu B| \neq 0$. Then*

$$A !_\mu B = \frac{|A|}{|A \nabla_\mu B|} A \nabla_\mu B. \tag{9}$$

Proof. We put $\nu = 1 - \mu$ and denote by $\tilde{X}$ the cofactor matrix of a matrix X, i.e., $\tilde{X} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ for $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then we have

$$\begin{aligned} A !_\mu B &= (\nu A^{-1} + \mu B^{-1})^{-1} = \left(\frac{\nu}{|A|}\tilde{A} + \frac{\mu}{|B|}\tilde{B} \right)^{-1} \\ &= \frac{|A|}{|\nu \tilde{A} + \mu \tilde{B}|} (\nu \tilde{A} + \mu \tilde{B})^{\tilde{}} = \frac{|A|}{|\nu A + \mu B|} (\nu A + \mu B), \end{aligned}$$

as required.

In the below, we fix matrices A and B for a given $M > 0$ as follows:

$$A = \begin{pmatrix} M+1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} M & 0 \\ 0 & 0 \end{pmatrix} + 1 \quad \text{and} \quad B = UAU^*, \tag{10}$$

where $U = \begin{pmatrix} u & v \\ w & z \end{pmatrix}$ is unitary.

Lemma 4. *Let A, B, μ and ν be as in above. Then the spectrum and determinant of $A \ \nabla_\mu \ B$ are as follows:*

$$\sigma(A \ \nabla_\mu \ B) = \{1 + \frac{M}{2}(1 \pm \sqrt{1 - 4\nu\mu|w|^2})\}$$
$$|A \ \nabla_\mu \ B| = 1 + M + M^2\nu\mu|w|^2.$$

Thus it follows that

$$\begin{aligned} A\nabla_\mu B - A \ !_\mu \ B &= \frac{|A\nabla_\mu B| - |A|}{|A\nabla_\mu B|} A\nabla_\mu B \\ &\leq \frac{M^2\nu\mu|w|^2}{1 + M + M^2\nu\mu|w|^2}(1 + \frac{M}{2}(1 + \sqrt{1 - 4\nu\mu|w|^2})) \\ &= \frac{M^2\nu\mu|w|^2}{1 + \frac{M}{2}(1 - \sqrt{1 - 4\nu\mu|w|^2})}. \end{aligned}$$

Summing up, we have

Lemma 5. *Let A, B, μ and ν be as in above. Then*

$$A\nabla_\mu B - A \ !_\mu \ B \leq \frac{M^2\nu\mu|w|^2}{1 + \frac{M}{2}(1 - \sqrt{1 - 4\nu\mu|w|^2})}.$$

Under such preparation, we have the following conclusion:

Theorem 6. *Let A, B, μ and ν be as in above. That is, they satisfy $1 \leq A, \ B \leq M+1$. If $M \geq \frac{4\delta}{(1-\delta)^2}$ for $\delta = \sqrt{1 - 4\nu\mu}$, then the optimal upper bound $(\sqrt{M+1} - 1)^2$ of $A\nabla_\mu B - A \ !_\mu \ B$ can be attained.*

In particular, if $\mu = \frac{1}{2}$, then the optimal upper bound $(\sqrt{M+1} - 1)^2$ can be attained for all $M > 0$.

Proof. For convenience, we put

$$t = 4\nu\mu|w|^2, s = \sqrt{1-t} \text{ and } N = M/2.$$

Hence Lemma 5 ensures that it suffices to estimate

$$\max\left\{\frac{(M/2)^2 t}{1 + M/2 \cdot (1 - \sqrt{1-t})}; t \in [0,1]\right\} = \max\left\{\frac{N^2(1-s^2)}{1 + N(1-s)}; s \in [0,1]\right\}.$$

Since $g'(s) = \frac{N^2(Ns^2-2(1+N)s+N)}{(1+N(1-s))^2}$ for $g(s) = \frac{N^2(1-s^2)}{1+N(1-s)}$ $(s \in [0,1))$, the solusion of $g'(s) = 0$ is $\{\frac{1}{N}(N+1 \pm \sqrt{2N+1})\}$. So we adopt

$$s_0 = \frac{1}{N}(N+1-\sqrt{2N+1}) \in [0,1].$$

Then it is easily seen that $g(s_0) = (\sqrt{2N+1}-1)^2 = (\sqrt{M+1}-1)^2$. Incidentally it is clear that $\delta \le s \le 1$ by $|w|^2 \in [0,1]$. So we need the condition $\delta \le s_0 (\le 1)$ to be attained the optimal constant, and it is equivalent to $M \ge \frac{4\delta}{(1-\delta)^2}$.

For convenience, we rephrase Theorem 6 in a general setting:

Theorem 7. *Suppose that $0 < r \le A,\ B \le R$ and $0 < \mu < 1$. If $\frac{R}{r} \ge \left(\frac{1+\delta}{1-\delta}\right)^2$ for $\delta = \sqrt{1-4\nu\mu}$, then the optimal upper bound $(\sqrt{R}-\sqrt{r})^2$ of $A\nabla_\mu B - A\,!_\mu\, B$ can be attained.*

In particular, if $\mu = \frac{1}{2}$, then the optimal upper bound $(\sqrt{R}-\sqrt{r})^2$ can be attained.

Remark. We mention that the second half of Theorem 7 has been discussed in the further observation after [2, Example 2.4].

3. Comparison with the geometric mean. The μ-geometric mean $A \sharp_\mu B$ for positive (invertible) operators A and B is defined by

$$A \sharp_\mu B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^\mu A^{\frac{1}{2}},$$

and we denote by $A \sharp B = A \sharp_{\frac{1}{2}} B$ simply, see [5].

It is well-known that

$$A\,!_\mu\, B \le A \sharp_\mu B \le A\nabla_\mu B$$

for positive operators A and B.

In this section, we compare the μ-geometric mean $A \sharp_\mu B$ with $A\nabla_\mu B$, $A\,!_\mu\, B$ and $\frac{1}{2}(A\nabla_\mu B + A\,!_\mu\, B)$ for positive operators A and B. We first discuss the following reverse inequalities:

Theorem 8. *If A and B are positive operators $0 < m \le A,\ B \le M$, $h = \frac{M}{m}$ and $\mu \in (0,1)$, then*

$$L(\mu)^{-1}A\nabla_\mu B \le A \sharp_\mu B \le L(1-\mu)A\,!_\mu\, B, \tag{11}$$

where

$$L(\beta) = \frac{1-\beta+\beta h}{h^\beta} \quad (0 < \beta \le 1/2), \quad = \frac{1-\beta+\beta h^{-1}}{h^{-\beta}} \quad (1/2 < \beta < 1).$$

Proof. The representing functions of ∇_μ, $\sharp_\mu$ and $!_\mu$ are

$$1-\mu+\mu t,\ t^\mu \text{ and } \frac{t}{(1-\mu)t+\mu}$$

respectively. So if we set

$$L_1 = \max_{h^{-1}\le t\le h}\{\frac{1-\mu+\mu t}{t^\mu}\} \quad \text{and} \quad L_2 = \max_{h^{-1}\le t\le h}\{\frac{(1-\mu)t+\mu}{t^{1-\mu}}\},$$

then we have

$$A\nabla_\mu B \le L_1 A \sharp_\mu B \quad \text{and} \quad A \sharp_\mu B \le L_2 A \,!_\mu\, B$$

and L_1, L_2 are optimal.

Next we determine them exactly. For this, we show that

$$g(t) = \frac{1-t+th^{-1}}{h^{-t}} - \frac{1-t+th}{h^t} \ (0<t<1)$$

satisfies $g(t) < 0$ for $0 < t < 1/2$ and $g(t) \ge 0$ if $1/2 \le t < 1$. Noting that $g(t) = t(1-t)(f(t) - f(1-t))$ for

$$f(t) = \frac{h^t - h^{-t}}{t} = \frac{k(t)}{t} \ (0<t<1),$$

it suffices to prove that $f(t)$ is an increasing function, which is exhibited as lemma:

Lemma 9. *The function* $f(t) = \frac{h^t-h^{-t}}{t} = \frac{k(t)}{t}$ *for* $0<t<1$ *is increasing.*

Proof. Since $k'(t) = (\log h)(h^t + h^{-t}) > 0$ and $k''(t) = (\log h)^2 k(t) > 0$, $k(t)$ is increasing and convex. Combining it with $k(0) = 0$, it follows that for $\alpha \in (0,1)$,

$$\alpha k(t) = \alpha k(t) + (1-\alpha)k(0) \ge k(\alpha t + (1-\alpha)0) = k(\alpha t).$$

Therefore we have

$$\frac{t}{t+\epsilon}k(t+\epsilon) \ge k(\frac{t}{t+\epsilon}(t+\epsilon)) = k(t)$$

for $\epsilon > 0$, so that $f(t) = \frac{k(t)}{t}$ is increasing, as desired.

Remark. In the proof of [1, Theorem 11], it is shown that if σ is an operator mean, $0 < m \le A, B < M$ and $h = \frac{M}{m}$, then

$$A\ \sigma\ B \ge \frac{g(h)-g(h^{-1})}{h-h^{-1}}B + \frac{hg(h^{-1}) - h^{-1}g(h)}{h-h^{-1}}A, \tag{12}$$

where g is the representing function of σ, $g(t) = 1\ \sigma\ t$ for $t \ge 0$. Therefore, if $f(t) = \frac{h^t-h^{-t}}{t}$ $(0<t<1)$ as in Lemma 9, then we have

$$A \sharp_\mu B \ge \frac{1}{h-h^{-1}}\{(1-\mu)f(1-\mu)A + \mu f(\mu)B\}. \tag{13}$$

Noting that $f(t)$ is increasing by Lemma 9, it follows from (13) that

$$A \sharp_\mu B \geq \begin{cases} \dfrac{f(\mu)}{h-h^{-1}} A\nabla_\mu B & (0<\mu\leq \dfrac{1}{2}), \\ \dfrac{f(1-\mu)}{h-h^{-1}} A\nabla_\mu B & (\dfrac{1}{2}\leq\mu\leq 1). \end{cases}$$

Unfortunately the above estimation is not better than that of Theorem 8 by Lemma 9. As a matter of fact, if $0<\mu\leq\frac{1}{2}$, then

$$(L(\mu)^{-1} - \frac{f(\mu)}{h-h^{-1}})\mu(h-h^{-1})(1-\mu+\mu h) = \mu(1-\mu)(f(1-\mu)-f(\mu)) \geq 0.$$

Similarly, if $\frac{1}{2}\leq\mu\leq 1$, then

$$h(L(\mu)^{-1} - \frac{f(1-\mu)}{h-h^{-1}})(1-\mu)(h-h^{-1})(1-\mu+\mu h^{-1}) = \mu(1-\mu)(f(\mu)-f(1-\mu)) \geq 0.$$

Next we compare $A \sharp B$ and $\frac{A\nabla B + A\ !\ B}{2}$.

Theorem 10. *If A and B are positive operators, then*

$$\frac{A\nabla B + A\ !\ B}{2} \geq A \sharp B. \tag{14}$$

On the other hand, if $0<m\leq A,\ B\leq M$ and $K=\frac{m\nabla M + m\ !\ M}{2(m\ \sharp\ M)}$, then

$$\frac{A\nabla B + A\ !\ B}{2} \leq K\ A \sharp B. \tag{15}$$

Proof. We put

$$f(t) = 4\cdot\frac{1\nabla t + 1\ !\ t}{2\sqrt{t}} \quad \text{for } t\in[h^{-1},h], \text{ where } h=\frac{M}{m}.$$

Then $f'(t) = \frac{(t-1)^3}{2t\sqrt{t}(t+1)^2}$ and $f''(t)>0$, so that $\min f(t) = f(1) = 1$. Therefore we have the former.

To prove the latter, we note that $f'(t)<0$ for $0<t<1$, $f'(t)>0$ for $t>1$ and $f(h) = f(h^{-1})$. It follows that

$$\max f(t) = \max\{f(h^{-1}),\ f(h)\} = f(h),$$

which implies (15).

Next we consider the weighted version of the above, in which they are not ordered in the sense that

$$f_\mu(t) = \frac{1\nabla_\mu t + 1\ !_\mu\ t}{2(1\ \sharp_\mu\ t)} \not\geq 1.$$

So one of inequalities we can discuss is a reverse one as follows:

Theorem 11. *If A and B are positive operators $0 < m \leq A,\ B \leq M$ and $h = \frac{M}{m}$, then*

$$k_\mu\ A \natural_\mu B \leq \frac{A \nabla_\mu B + A\, !_\mu\, B}{2} \leq K_\mu\ A \natural_\mu B, \tag{16}$$

where $k_\mu = f_\mu((\frac{\mu}{1-\mu})^2)$, and $K_\mu = f_\mu(h)$ if $0 < \mu < \frac{1}{2}$, and $K_\mu = f_\mu(h^{-1})$ if $\frac{1}{2} < \mu < 1$.

Proof. We put $\nu = 1 - \mu$ for convenience. Then

$$f'_\mu(t) = \frac{\nu\mu(t-1)^2(\nu^2 t - \mu^2)}{2t^{\mu+1}(\mu + \nu t)^2}.$$

Noting that $f'_\mu(1) = f'_\mu(\frac{\mu^2}{\nu^2}) = 0$, $f_\mu(t)$ is decreasing in $(0, \frac{\mu^2}{\nu^2})$ and $f_\mu(t)$ is increasing in $(\frac{\mu^2}{\nu^2}, \infty)$. Hence it follows that

$$\max\{f_\mu(t); t \in [h^{-1}, h]\} = \max\{f_\mu(h^{-1}), f_\mu(h)\}.$$

If $0 < \mu < \frac{1}{2}$, then $f_\mu(h^{-1}) \leq f_\mu(h)$, and, if $\frac{1}{2} < \mu < 1$, then $f_\mu(h^{-1}) \geq f_\mu(h)$. As a matter of fact, it is assured as follows: We put $g(h) = \frac{f_\mu(h)}{f_\mu(h^{-1})}$ for $h \geq 1$. Then $g(1) = 1$. Since

$$\log g(h) = \log \frac{\mu h + \nu}{\mu + \nu h} + (1 - 2\mu) \log h,$$

it follows that

$$\{\log g(h)\}' = \frac{\mu}{\mu h + \nu} - \frac{\nu}{\mu + \nu h} + \frac{1 - 2\mu}{h} = \frac{\mu\nu(\nu - \mu)(h-1)^2}{h(\mu h + \nu)(\mu + \nu h)}.$$

Finally the left hand side of (16) follows from $k_\mu = \min f_\mu(t)$.

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Received June 13, 2006
Revised August 12, 2006