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On Some Mistaken Beliefs About Core Logic and Some Mistaken Core Beliefs About Logic

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Abstract This is in part a reply to a recent work of Vidal-Rosset, which expresses various mistaken beliefs about Core Logic. Rebutting these leads us further to identify, and argue against, some mistaken core beliefs about logic.

In his recent work titled "Why Intuitionistic Relevant Logic Cannot Be a Core Logic," Joseph Vidal-Rosset [18] raises some objections to that logical system. Here these objections are rebutted (in Section 1) by showing that they rest on some mistaken beliefs about the system. But because these mistaken beliefs derive from some mistaken core beliefs about logic *tout court*, some space will also be devoted (in Section 2) to identifying and refuting the latter.

The reader needs to be alerted, at the outset, to the fact that the system *IR* of intuitionistic relevant logic was renamed *Core Logic*.[1] The reasons for this will be explained below. Vidal-Rosset, however, uses the phrase "core logic" as a *common noun*, without explaining the criteria one ought to apply in order to tell whether a given system is *a* core logic.

1 Some Mistaken Beliefs About Core Logic

Vidal-Rosset writes that the purpose in formulating the system *IR* of Intuitionistic Relevant Logic was[2]

> to prove all *theorems* of Heyting logic without accepting the intuitionistic absurdity rule
>
> $$\frac{\bot}{\varphi}.$$

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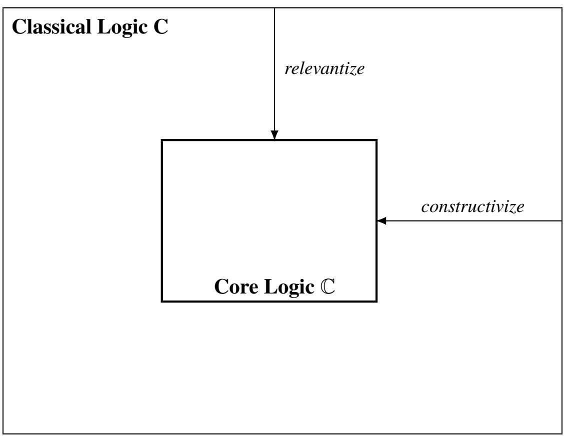
The focus in the process of discovering *IR*, however, has always been on *deducibility in general*, and not just deducibility from the empty set of premises.

The system *IR* does more than just deliver all the theorems of Intuitionistic Logic. It also proves all intuitionistic inconsistencies, and enables one to deduce all intuitionistic consequences of intuitionistically consistent sets of premises. Indeed, if by $\models$ we mean the semantic relation of intuitionistic consequence, and by $\vdash$ we mean the deducibility relation of *IR*, then we have the following:

If $\Delta \models \varphi$, then for some (finite) $\Gamma \subseteq \Delta$, either $\Gamma \vdash \bot$ or $\Gamma \vdash \varphi$.

The system was originally called *Intuitionistic Relevant Logic* because it stood at the intersection of two orthogonal lines of logical reform: *constructivizing*, and *relevantizing*, all passages of deductive reasoning from premises to conclusions. More recently, however, as already mentioned, the system was renamed *Core Logic*. When thought of as a result of sacrificial reforms, Core Logic $\mathbb{C}$ might *look* like a mere *residue* of Classical Logic **C**:



Two lines of logical reform.

Such a picture gives the impression that Core Logic is to be characterized only as what is left over when one eschews certain principles of classical reasoning (those that are not constructive, and those that embody deductive irrelevancies). But this impression, unsupplemented by any other perspective on how Core Logic might be the canon of choice, would be mistaken.

It was argued in [13] that the foregoing picture needs to be enriched in two ways. Core Logic earns its new name by being a genuinely core canon of rules any *additions to which* (e.g., the Absurdity Rule, or a strictly classical rule for negation) would need to be justified. Core Logic should not be represented as the result of *subtracting* features from Classical Logic, the latter presumed to be the "default choice," deviations *from which* would demand special justifications. The burden of proof is to be shifted: it is the classical logician's *additions* to Core Logic that stand in need of special justification.

How, then, does Core Logic deserve to be described as a "genuinely core canon of rules"?

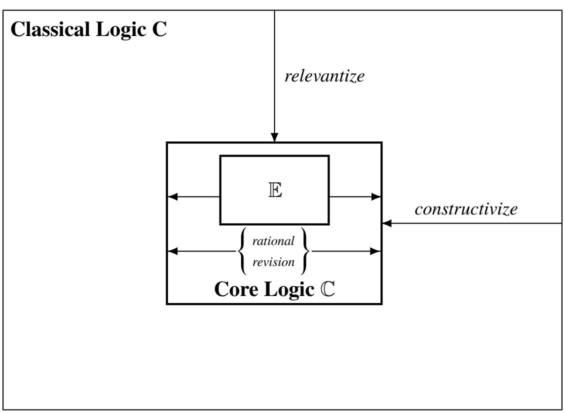
First, there is a very natural "logic of evaluation," which may be called $\mathbb{E}$, which traces just the inferences one makes when one evaluates a sentence as true or as false on the basis of *literals*—atomic sentences and/or negations of atomic sentences. The system $\mathbb{E}$ contains a *verification* rule and a *falsification* rule for each logical operator. These rules enable one to construct verifications of sentences from sets of literals as premises, and to construct falsifications of sentences *modulo* such sets. The falsifications are special kinds of *dis*proof, with absurdity ($\perp$) as their conclusions. The logic $\mathbb{E}$ is therefore limited in its field: it generates deducibilities only of the constrained forms $\Lambda : \varphi$ and $\Lambda, \varphi : \perp$, where Λ is a (consistent) set of *literals*. (A fuller account of these matters can be found in [9], [17].)[3]

It is natural to inquire after an extension of $\mathbb{E}$ that will generate deducibilities of the more general forms $\Delta : \varphi$ or $\Delta : \perp$ (where Δ is now a finite set of sentences, however complex). Core Logic $\mathbb{C}$ is just such an extension. It is reached by a very smooth extrapolation from $\mathbb{E}$, as one minimally generalizes its verification and falsification rules so that they become, respectively, the *introduction* and *elimination* rules for the construction of deductions in general. Moreover, an important constraint met by evaluation proofs and disproofs (verifications and falsifications) is preserved in the transition to a fully general deductive system of Core Logic: major premises for eliminations (analogues of sentences being falsified) *stand proud*—that is, they have no proof-work above them. This means that all Core proofs are in *normal form*. (This notion of "standing proud," and the proof-theoretic advantages of insisting that major premises of eliminations should stand proud, are to be found in [6].)

Second, as was argued in [10], at a level of detail that cannot be reprised here,

> Core Logic is the minimal inviolable core of logic without any part of which one would not be able to establish the rationality of belief-revision. [10, p. 261]

A fuller picture of the emerging situation is then as follows:



Pressure from without meets more pressure from within.

That was why the new *name* "Core Logic" seemed appropriate for the system. The system *IR* was *given an alias*—one that struck its discoverer as warranted, and obviously more succinct, and catchier, than the laborious "Intuitionistic Relevant Logic." Henceforth, the system will be referred to as *Core Logic*.

"Core Logic" is a *name*, not a common noun. It would be hard to say exactly what sort of system "*a* Core Logic" would be, just as it would be hard to make completely clear sense of anyone other than Frank Sinatra or Édith Piaf, respectively, being "*a* Frank Sinatra" or "*an* Édith Piaf." We have gestured above, however, at what arguably makes Core Logic a "genuinely core canon of rules"; but it is by no means clear that *this* provides even a rough sense of what Vidal-Rosset himself might have in mind.

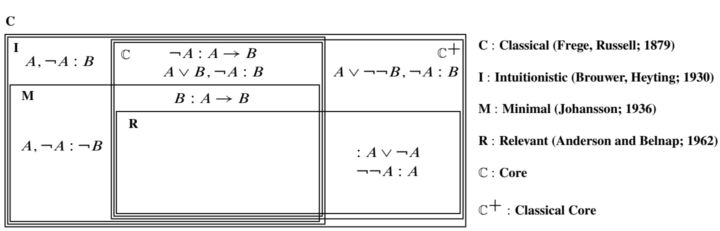
If Vidal-Rosset has a clear sense in mind for "core logic" construed as a common noun, then he is welcome to it; and it would be fascinating to learn how it is that Core Logic is *not* a core logic in that sense.[4] But Vidal-Rosset does not supply any such clear sense to his reader; so one is left wondering what his observations about Core Logic actually amount to. The best one can do here is examine his main complaints against Core Logic in an attempt to extract the set of whatever criteria are occasioning the allegations of shortcomings.

In order to forestall any possibility of misunderstanding, the reader is advised that I will use the definite description "the Core logician" to refer to the generic advocate of the system $\mathbb{C}$ of Core Logic, not to the sort of character who wishes merely to espouse *a* core logic. The right precedent for understanding this terminological convention is Dummett's use of "*the* semantic realist" or "*the* intuitionistic mathematician."

Vidal-Rosset frequently mentions features of the so-called *Minimal Logic* of Johansson [3]. This is a proper subsystem of Intuitionistic Logic, which was originally intended to capture the notion of relevance between the premises and the conclusions of deductions that commit no fallacies of relevance. Johansson did not quite succeed in this regard, however, since Minimal Logic contains the negative form of Lewis's First Paradox:

$$A, \neg A : \neg B.$$

For the benefit of the reader curious about the relationship between the various systems mentioned, here is a picture of how **Core Logic** $\mathbb{C}$ and its classicized counterpart, **Classical Core Logic** $\mathbb{C}^+$, sit in relation to the well-known systems of **Classical Logic C**, **Intuitionistic Logic I**, and **Minimal Logic M**, which have well-behaved natural-deduction formulations, and in relation to the **relevance logic R** of Anderson and Belnap (see [1]). Note that Core Logic is the intersection of Classical Core with Intuitionistic Logic.



$(\mathbf{I} \cap \mathbf{R}) \subset \mathbf{M} \subset \mathbf{I} \subset \mathbf{C}$

$(\mathbf{I} \cap \mathbf{R}) \subset \mathbb{C} = (\mathbb{C}^+ \cap \mathbf{I}) \subset \mathbf{I} \subset \mathbf{C}$

$\mathbf{R} \subset \mathbb{C}^+ \subset \mathbf{C}$

C is the "union" of $\mathbb{C}^+$ and I

Suppose that one is allowed to use the Absurdity Rule. Then judicious insertions of applications of it would allow the Classical Core logician to mimic any classical proof while otherwise reasoning in accordance with "only" the rules of $\mathbb{C}$ plus the "core" versions of the strictly classical rules Classical Reductio and/or Dilemma (for which, see the Appendix). That is why we comment in the diagram that **C** is the "union" of $\mathbb{C}^+$ and **I**.[5]

Core Logic contains the invaluable form of inference known as *Disjunctive Syllogism* ($A \vee B, \neg A : B$), but contains neither one of the two closely related Lewis paradoxes $A, \neg A : B$ and $A, \neg A : \neg B$. Hence, the Core logician contends, Core Logic does a better job of relevantizing Intuitionistic Logic than does Johansson's Minimal Logic.

Core Logic (and its classical extension) is not at all difficult to understand. It is, after all, presented as a system of natural deduction rules, and also as a system of sequent rules.

In the natural-deduction setting, the *only* rules of Core Logic are the introduction and elimination rules for the logical operators. *All* the elimination rules are in parallelized form, which endows the system with a pleasing uniformity. All major premises of eliminations (MPEs) must stand proud; so all proofs are in normal form. This is all very simple, and very easy to understand.

In the sequent setting, the *only* rules of Core Logic are the right- and left-rules for the logical operators. Apart from the rule of reflexivity $A : A$, *there are no structural rules*. All sequent proofs are not only cut-free, but dilution-free. Once again, this is all very simple, and very easy to understand.

Core Logic enjoys the proof-theoretic distinction that its proofs have exactly the same structure whether they are presented as natural deductions or as sequent proofs. This is a consequence of the fact that in the natural deductions of Core Logic all MPEs stand proud. Isomorphism between natural deductions and cut-free, thinning-free sequent proofs is then immediate. This isomorphism property would be enjoyed by any logical system that resembled Core Logic in these key respects (MPEs standing proud in natural deductions, and sequent proofs being available only in cut-free and thinning-free forms). But to the best of the author's knowledge there is no extant rival to Core Logic that matches it in these respects.

The sheer simplicity of Core Logic, by virtue of this isomorphism between natural deductions and sequent proofs, should not be underestimated. The previous history of proof theory was bedeviled by the often onerous task of having to devise complicated transformations that would turn a natural deduction into a sequent proof of the same result, and *vice versa*. *That* certainly made the systems in question more difficult to understand. All that troublesome work is now obviated by the Core systems, because of the direct correspondence between introduction rules and right-rules, and between elimination rules and left-rules.

1.1 A digression on transitivity of deduction, and reductions In Section 1.5 we will be making use of a notion of *reduction* (of pairs of core proofs) and this is an appropriate juncture at which to explain this notion. The reader who does not wish to tarry for these mild technicalities can proceed to the next subsection, and return to this one in due course if need be.

If one were to allow nontrivial proof-work to appear above an MPE, one would be allowing for *abnormal* proofs, in which the MPE in question stands as the conclusion of an application of the corresponding introduction rule. Consider, for example, the following proof, which is in Gentzen–Prawitz form (because it allows nontrivial proof-work above an MPE) but which uses the parallelized, rather than serial, form of $\wedge$-elimination:

$$
\dfrac{\dfrac{\begin{matrix}\Delta_1 & \Delta_2\\ \Pi_1 & \Pi_2\\ \varphi_1 & \varphi_2\end{matrix}}{\varphi_1\wedge\varphi_2}(\wedge I) \qquad \begin{matrix}\underbrace{\Gamma,\overline{\varphi_1}^{(i)},\overline{\varphi_2}^{(i)}}\\ \Sigma\\ \psi\end{matrix}}{\psi}(\wedge E)_{(i)}
$$

The occurrence of $\varphi_1 \wedge \varphi_2$ is a so-called *maximal* occurrence. It is a local "peak" of unnecessary complexity. It can be eliminated from the proof by means of a so-called $\wedge$-*reduction*. In the case at hand the reduct (following the method of Prawitz) would be

$$
\begin{matrix} \Delta_1 \quad \Delta_2\\ \Pi_1 \quad \Pi_2\\ \underbrace{\Gamma,\ (\varphi_1),\ (\varphi_2)}\\ \Sigma\\ \psi \end{matrix}
$$

A similar kind of problem could now arise with any of the "grafting occurrences" of φ_1 and φ_2. They too might be local peaks. But at least these newly formed peaks would be of lesser degree than $\varphi \wedge \varphi_2$. Hence they would present no obstacle to the required sequence of Prawitzian reductions terminating after finitely many steps. In such a fashion would the conventional Gentzen–Prawitz theorist establish the *Normalization Theorem* for the system of proof in which MPEs did *not* have to stand proud.

For the Core logician, however, matters stand differently. These abnormal proofs simply cannot be formed. There is, however, the question what one is to do when in possession of, say, two (core) proofs of the overall forms

$$
(\wedge I)\dfrac{\begin{matrix}\Delta_1 & \Delta_2\\ \Pi_1 & \Pi_2\\ \varphi_1 & \varphi_2\end{matrix}}{\varphi_1\wedge\varphi_2} \qquad\qquad (\wedge E)\dfrac{\varphi_1\wedge\varphi_2 \qquad \begin{matrix}\underbrace{\Gamma,\overline{\varphi_1}^{(i)},\overline{\varphi_2}^{(i)}}\\ \Sigma\\ \psi\end{matrix}}{\psi}{}_{(i)}
$$

In such a situation, when one is seeking the fruits of transitivity of deduction, the Core logician applies (in the notation of [11]) a binary operation [,] of *reduction* to the two proofs in question. The reduct

$$
\left[\ \dfrac{\begin{matrix}\Delta_1 & \Delta_2\\ \Pi_1 & \Pi_2\\ \varphi_1 & \varphi_2\end{matrix}}{\varphi_1\wedge\varphi_2}\ ,\ \dfrac{\varphi_1\wedge\varphi_2 \qquad \begin{matrix}\underbrace{\Gamma,\overline{\varphi_1}^{(i)},\overline{\varphi_2}^{(i)}}\\ \Sigma\\ \psi\end{matrix}}{\psi}{}_{(i)}\ \right]
$$

is defined, recursively, to be

$$
\left[\begin{matrix}\Delta_1\\ \Pi_1\\ \varphi_1\end{matrix}\ ,\ \left[\begin{matrix}\Delta_2\\ \Pi_2\\ \varphi_2\end{matrix}\ ,\ \begin{matrix}\underbrace{\Gamma,\varphi_1,\varphi_2}\\ \Sigma\\ \psi\end{matrix}\right]\right]
$$

By these methods, illustrated here with respect to $\wedge$, the Core logician can enjoy the fruits of normalization within a setting where MPEs are required to stand proud. We have confined ourselves here to an illustration with $\wedge$. Similar reductions are available for all the familiar logical operators (see [14] for details).

1.2 Vidal-Rosset's objections to Core Logic We turn now to the alleged drawbacks of Core Logic that Vidal-Rosset has tried to describe. There appear to be at least four:

(1) [Core Logic] changes the meaning of disjunction elimination.
(2) The law of substitution of equivalents fails in [Core Logic].
(3) $(A \wedge \neg A) \wedge ((A \wedge \neg A) \leftrightarrow (B \wedge \neg B)) : B$ is provable in [Core Logic].
(4) "Nowhere has Tennant clearly explained the difference between a provable inference 'at the level of the turnstile' and a provable conditional (i.e., an implication).... Why [should $A, \neg A : B$] be unprovable, while the conditional $(\neg A \wedge A) \rightarrow B$ is a theorem of [Core Logic]?" [18, p. 246]

1.3 Reply to objection (1)

[Core Logic] changes the meaning of disjunction elimination.

Vidal-Rosset appears not to have entertained the rebuttal-possibility that the usual deductive rules governing the connectives arguably do not quite specify their individually separable meanings correctly. This is because ramifying logical interrelationships might adversely affect one's grasp of what those meanings really are. There are the ramifications mediated by the Absurdity Rule. These are aided and abetted by the standard license issued to thinkers who blithely put together proofs so as to form proofs *not in normal form*. Has anyone ever paused to think what *deformations* of the meanings of the connectives this might entail?

The Core logician urges, by contrast, that "the" genuine meanings of the connectives are to be captured by rules governing only *them*; and that *those* rules should then suffice to negotiate any deductive passage for which one may demand detailed justification. Given *that* methodological orientation at the outset, one comes to see the practices of the Intuitionistic logician and of the Classical logician in a rather more jaundiced light. They both subscribe to the Absurdity Rule, the prime source of deductive irrelevance. They both subscribe to *absolutely unrestricted* chainings-together of proofs, without pausing to consider a very troublesome possibility. This is the possibility that the ever-growing set of premises on which their overall conclusion "rests" might have become *inconsistent*—and therefore unable to afford any rational grounding at all for the conclusion supposedly "based on" them, *via* their proof-by-accumulation.

1.4 Reply to objection (2)

The law of substitution of equivalents fails in [Core Logic].

The objection here is that the law of substitution of equivalents fails in [Core Logic]. A reply can be given in two parts.

(i) This is not new.
(ii) So what?

Ad (i). Vidal-Rosset's counterexample to the law of substitution of equivalents is an alternative to the one furnished in [8, Observation 3, p. 256], to establish Observation 3 in that paper (for which, see below). The advantage of this earlier counterexample was that it did not involve the conditional $\rightarrow$.

The notion $\overset{\Box}{\vdash}$ of *perfect* relevant deducibility was introduced, on the basis of a relevant deducibility relation $\vdash$:

$\Delta \overset{\square}{\vdash} \varphi$ if and only if $\Delta \vdash \varphi$, and no proper subsequent of $\Delta : \varphi$ is provable.

The second conjunct in this definiens amounts to:

$\Delta \nvdash \bot$ and for no proper subset Δ' of Δ do we have $\Delta' \vdash \varphi$.

An *entailment* was defined to be any substitution instance of a perfect deducibility.

Observation 3 had three parts:

1. Even perfect (relevant) interdeducibility of P and Q is insufficient to guarantee their interreplaceability, *salva veritate*, in all statements of (relevant) deducibility. *A fortiori*:
2. the straightforward (relevant) interdeducibility of P and Q is insufficient to the same end; as is their mutual entailment. Finally, since every perfect deducibility is an entailment:
3. the insufficiencies in question extend to interreplaceability, *salva veritate*, in all statements of entailment.

The example furnished to establish these three points consisted of the following facts:

$$B \overset{\square}{\vdash} (A \wedge \neg A) \vee B$$

$$(A \wedge \neg A) \vee B \overset{\square}{\vdash} B$$

$$A \wedge \neg A \vdash (A \wedge \neg A) \vee B$$

$$A \wedge \neg A \nvdash B$$

These facts immediately establish point (1) above: despite their perfect relevant interdeducibility, the sentences B and $(A \wedge \neg A) \vee B$ are not interreplaceable, *salva veritate*, in all statements of (relevant) deducibility. Points (2) and (3) follow easily upon further reflection.

Ad (ii). Vidal-Rosset bases his objection on the assumption that Core Logic would have to deal with disjunctions of the special form $\bot \vee B$. But let us be mindful of the fact that $\bot$ is given no such role in either Core Logic or its classicized extension. In neither of these systems does $\bot$ ever occur as a (sub)sentence. Rather, it is a mere punctuation-marker within proofs. In fact, it is *eliminable* from the proof theory altogether, along lines explained in [7]. One can define, in a coinductive fashion, the notions of *proof* and *disproof*, neither of them containing any occurrences of $\bot$. The sole reason for having $\bot$ for the convenient formulation of rules of inference is to allow one to treat *dis*proofs as forming a special kind of *proof*, namely proofs "of the conclusion $\bot$". But even then $\bot$ does not feature as a sentence in its own right, subject to the iterable rules of grammatical formation. It is *certainly* not allowed to feature as a proper *subsentence* of any sentence; the would-be sentence $\bot \vee B$ is simply ill-formed.

Here, no doubt, Vidal-Rosset would appeal to both authority and tradition by pointing out that intuitionists often define $\neg A$ as (short for) $A \rightarrow \bot$, and therefore must be allowed to use $\bot$ as a subsentence. But this tradition obtains only within mathematics (where $\bot$ is actually taken to be the sentence $0 = 1$) and there are good arguments (see [7]) *against* its over-ambitious extension to other discourses in general. (Moreover, Cook and Cogburn [2] argue persuasively that defining $\neg A$ as $A \rightarrow \bot$ is a mistake *even within the confines of mathematics*.)

For Vidal-Rosset, two sentences P and Q are logically equivalent just in case the biconditional sentence $P \leftrightarrow Q$ is a logical theorem (i.e., is deducible from the empty set of premises):

$$\vdash P \leftrightarrow Q.$$

His version of the law of substitution of such logical equivalents is

$$\text{if } \vdash P \leftrightarrow Q, \text{ then for } \textit{any } \Gamma, \Gamma \vdash P \text{ if and only if } \Gamma \vdash Q.$$

Of course, it suffices to have "only if" in place of "if and only if" in the foregoing:

$$\text{if } \vdash P \leftrightarrow Q, \text{ then for } \textit{any } \Gamma, \Gamma \vdash P \text{ only if } \Gamma \vdash Q.$$

This in turn is equivalent to

$$\text{if } \Gamma \vdash P \text{ and } \vdash P \leftrightarrow Q, \text{ then } \Gamma \vdash Q,$$

with each of the parameters given a wide-scope universal interpretation. Indeed, this law of substitution in conclusion-position can be prima facie strengthened even further:

$$\text{if } \Gamma \vdash P \text{ and } \vdash P \rightarrow Q, \text{ then } \Gamma \vdash Q.$$

Vidal-Rosset's counterexample involves taking $\{\bot\}$ for Γ, $(\bot \vee B)$ for P, and B for Q, so that one has

$$\bot \vdash (\bot \vee B) \text{ and } \vdash (\bot \vee B) \rightarrow B$$

while yet

$$\bot \nvdash B.$$

This, according to Vidal-Rosset, is somehow objectionable.

The Core logician's reply is "*Au contraire*, it is perfectly in order." Note that this reply does not rest on a *concession* allowing Vidal-Rosset to use $\bot$ as a subsentence (which, for independent reasons, one should not do). For it would be perfectly acceptable to both sides of this particular debate to run Vidal-Rosset's argument using the genuine, and absurd, sentence $0 = 1$ in place of the would-be "(sub)sentence" $\bot$. His counterexample would then involve taking $\{0 = 1\}$ for Γ, $(0 = 1 \vee B)$ for P, and B for Q, so that one has

$$0 = 1 \vdash (0 = 1 \vee B) \text{ and } \vdash (0 = 1 \vee B) \rightarrow B$$

while yet

$$0 = 1 \nvdash B.$$

But that is as it should be. For, even if we were to discover a contradiction in Peano Arithmetic, by proving $0 = 1$, we would not be justified in concluding that the moon is made of green cheese (arbitrary B, remember!). Nay, we might receive a token of the proof just before a power blackout on a night with a full moon, and work feverishly in the moonlight, trying to find a fallacy in the proof of $0 = 1$. And even if we found no fallacy, and had to admit the proof as licit, we would not be able to conclude that the illumination enabling our vain search for a fallacy came from a piece of green cheese. It is apparent that we have here a very clear and compelling counterexample to the general principle

$$\text{if } \Gamma \vdash P \text{ and } \vdash P \rightarrow Q, \text{ then } \Gamma \vdash Q.$$

Note also that with the instance

$$A, B \vdash A \wedge B \text{ and } \vdash (A \wedge B) \rightarrow A$$

the principle dictates that

$$A, B \vdash A,$$

which is actually overkill, since we have

$$A \vdash A.$$

So, clearly, there would be no objection to re-framing the principle as follows:

if $\Gamma \vdash P$ and $\vdash P \to Q$, then *for some subset* Δ of Γ, we have $\Delta \vdash Q$.

That is, we can allow for "subsetting down on the left." Why not, then, also allow for "subsetting down on the *right*"? Why not, that is, frame the principle as follows:

if $\Gamma \vdash P$ and $\vdash P \to Q$, then for some subset Δ of Γ, either $\Delta \vdash Q$ or $\Delta \vdash \bot$.

This principle holds for Core Logic. Vidal-Rosset's counterexample is neutralized by pointing out that, if he wishes to regard $0 = 1$ *as* $\bot$, we have $0 = 1 \vdash 0 = 1$; whereas, if $0 = 1$ is not taken to *be* $\bot$, then certainly $0 = 1 \vdash \bot$. And even if we allow him to insist on having $\bot$ as a (sub)sentence, we can point out that $\bot \vdash \bot$.

1.5 Reply to objection (3)

$(A \wedge \neg A) \wedge ((A \wedge \neg A) \leftrightarrow (B \wedge \neg B))$: B is provable in [core logic].

Vidal-Rosset does not say whether he takes the biconditional as primitive or defined (on behalf of the Core Logician), so it is worth emphasizing that it does not matter which approach one takes. It is convenient, though, to have introduction and elimination rules for dealing with the biconditional as a primitive connective. The following are the four possible ways in Core Logic that one can introduce a biconditional as a conclusion. So we are about to state a four-part rule of $\leftrightarrow$-Introduction. (Note that a box affixed to a discharge stroke means that the assumption in question must have been used; whereas a diamond means that it need not have been used.)

$$\dfrac{\begin{array}{cc} (i)\underline{\quad}\Box & (i)\underline{\quad}\Box \\ \varphi & \psi \\ \vdots & \vdots \\ \bot & \bot \end{array}}{\varphi \leftrightarrow \psi}\,(i) \qquad \dfrac{\begin{array}{cc} (i)\underline{\quad}\Diamond & (i)\underline{\quad}\Diamond \\ \varphi & \psi \\ \vdots & \vdots \\ \psi & \varphi \end{array}}{\varphi \leftrightarrow \psi}\,(i) \qquad \dfrac{\begin{array}{cc} (i)\underline{\quad}\Box & (i)\underline{\quad}\Diamond \\ \varphi & \psi \\ \vdots & \vdots \\ \bot & \varphi \end{array}}{\varphi \leftrightarrow \psi}\,(i) \qquad \dfrac{\begin{array}{cc} (i)\underline{\quad}\Diamond & (i)\underline{\quad}\Box \\ \varphi & \psi \\ \vdots & \vdots \\ \psi & \bot \end{array}}{\varphi \leftrightarrow \psi}\,(i)$$

Corresponding to this four-part introduction rule is the following two-part elimination rule:

$$\dfrac{\begin{array}{ccc} & & (i)\underline{\quad}\Box \\ & & \psi \\ & \vdots & \vdots \\ \varphi \leftrightarrow \psi & \varphi & \theta \end{array}}{\theta}\,(i) \qquad \dfrac{\begin{array}{ccc} & & (i)\underline{\quad}\Box \\ & & \varphi \\ & \vdots & \vdots \\ \varphi \leftrightarrow \psi & \psi & \theta \end{array}}{\theta}\,(i)$$

Note that the major premise $\varphi \leftrightarrow \psi$ is to "stand proud"—that is, there is to be no proof-work above it. The *minor* premise φ (in the left half) or ψ (in the right half), however, is allowed to have proof-work above it, as indicated by the vertical dots. In all our rule-presentations of this graphic form, we will follow this convention. We remind the reader: the *absence* of vertical dots above a sentence-occurrence means that it stands proud, with no proof-work above it.

We repeat: by requiring MPEs to stand proud, one ensures that all proofs are in normal form. This way of ensuring normal form of proofs was first treated in [6], which essayed the advantages, for computational logic, of having very exigent normal-form theorems, which allow one to dramatically constrain the search-space when seeking proofs from given premises to a given conclusion. In general, the problem in automated deduction is to provide a proof of the given argument if one exists, otherwise eventually to report that there is none.

Vidal-Rosset seeks to embarrass the Core Logician by pointing out that the sequent

$$(A \wedge \neg A) \wedge \big((A \wedge \neg A) \leftrightarrow (B \wedge \neg B)\big) : B$$

is provable in Core Logic. The point of the example becomes clearer if one offers it in the following form:

$$A \wedge \neg A, (A \wedge \neg A) \leftrightarrow (B \wedge \neg B) : B.$$

For then, the simple thought would go, one can *suppress* the second premise, since it is a logical theorem, thereby establishing (surely?)

$$A \wedge \neg A : B$$

as a sequent of Core Logic. And this would indeed be not just an embarrassment for the Core logician, but a catastrophe.

Let us look more closely, however, at the moves in this argument. To be sure, the second premise $(A \wedge \neg A) \leftrightarrow (B \wedge \neg B)$ is a theorem of Core Logic. Here is a proof:

$$\Pi : \cfrac{\cfrac{{}^{(3)}\overline{A \wedge \neg A} \qquad \cfrac{{}^{(1)}\overline{\neg A} \quad \overline{A}^{(1)}}{\bot}}{\bot}{\scriptstyle (1)} \qquad \cfrac{{}^{(3)}\overline{B \wedge \neg B} \qquad \cfrac{{}^{(2)}\overline{\neg B} \quad \overline{B}^{(2)}}{\bot}}{\bot}{\scriptstyle (2)}}{(A \wedge \neg A) \leftrightarrow (B \wedge \neg B)}{\scriptstyle (3)}$$

The remainder of the passage to the embarrassing conclusion B is supposed to be effected by the following further proof:

$$\Sigma : \cfrac{(A \wedge \neg A) \leftrightarrow (B \wedge \neg B) \qquad A \wedge \neg A \qquad \cfrac{{}^{(5)}\overline{B \wedge \neg B} \quad \overline{B}^{(4)}}{B}{\scriptstyle (4)}}{B}{\scriptstyle (5)}$$

But the conclusion of Π stands, within Π, as the conclusion of an application of $\leftrightarrow$-introduction; while within Σ it stands also as the major premise for an application of $\leftrightarrow$-elimination. So a $\leftrightarrow$-reduction is called for, in the sense of "reduction" that was explained in Section 1.5. In the notation of [11], the relevant reduct $[\Pi, \Sigma]$ for proofs Π, Σ of the respective forms

$$\cfrac{\begin{matrix}\underbrace{\Delta_1, \quad {}^{(i)}\overline{\varphi}^{\,\square}} \\ \Pi_1 \\ \bot\end{matrix} \qquad \begin{matrix}\underbrace{\Delta_2, \quad {}^{(i)}\overline{\psi}^{\,\square}} \\ \Pi_2 \\ \bot\end{matrix}}{\varphi \leftrightarrow \psi}{\scriptstyle (i)} \qquad \cfrac{\varphi \leftrightarrow \psi \qquad \begin{matrix}\Gamma_1 \\ \Sigma_1 \\ \varphi\end{matrix} \qquad \begin{matrix}\underbrace{\Gamma_2, \quad {}^{(j)}\overline{\psi}^{\,\square}} \\ \Gamma_2 \\ \theta\end{matrix}}{\theta}{\scriptstyle (j)}$$

is

$$\left[\begin{matrix}\Gamma_1 \\ \Sigma_1 \\ \varphi\end{matrix}, \begin{matrix}\underbrace{\Delta_1, \varphi} \\ \Pi_1 \\ \bot\end{matrix}\right]$$

In the case at hand, Σ_1 is trivial—a single occurrence of $A \wedge \neg A$—so we are left with Π_1, which is

$$\dfrac{A \wedge \neg A \qquad \dfrac{{}^{(1)}\overline{\neg A} \quad \overline{A}^{(1)}}{\bot}}{\bot}\,{\scriptstyle (1)}\ ,$$

showing us that $A \wedge \neg A$ is *inconsistent*,[6] not that B follows from it. And this is as things should be.

1.6 Reply to objection (4)

> Nowhere has Tennant clearly explained the difference between a provable inference "at the level of the turnstile" and a provable conditional (i.e., an implication)…. Why [should $A, \neg A : B$] be unprovable, while the conditional $(\neg A \wedge A) \to B$ is a theorem of [Core Logic]? [18, p. 246]

Vidal-Rosset could also have asked why $A, \neg A : B$ should be unprovable, while $\neg A : A \to B$ is provable.

The answer is that the relevantist investigations leading to the discovery of Core Logic and of its classicized extension concentrated on the relation of *deducibility*, while seeking to preserve as much as possible of the logical behavior of the familiar connectives and quantifiers.[7] I regarded the object-language connectives as governed, essentially, by their usual truth tables.

It was in [6] that the rules of Core Logic satisfactorily incorporated the conditional $\to$. According to the truth table for $\to$, if the consequent is true, then so is the conditional. That means we must have the deducibility $B : A \to B$. According to the truth table it is also the case that if the antecedent is false, then the conditional is true. That means we must also have the deducibility $\neg A : A \to B$. In addition, there is the standard method of *conditional proof*, according to which if one has proved B from the assumption A along with perhaps other assumptions forming the set Δ, then one can discharge the assumption A and conclude $A \to B$ from Δ alone. As the familiar sequent rule ($\to$:) has it:

$$\frac{\Delta, A : B}{\Delta : A \to B}, \qquad \text{where } A \notin \Delta.$$

The net effect of these straightforward considerations is that the introduction rule for $\to$ is formulated in Core Logic as follows (using the usual Greek letters for placeholders for sentences):

$$(\to\text{-I}) \qquad \dfrac{\begin{array}{c} \Box\,\overline{\ \ \ \ }^{(i)} \\ \varphi \\ \vdots \\ \bot \end{array}}{\varphi \to \psi}\,{\scriptstyle (i)} \qquad \dfrac{\begin{array}{c} \Diamond\,\overline{\ \ \ \ }^{(i)} \\ \varphi \\ \vdots \\ \psi \end{array}}{\varphi \to \psi}\,{\scriptstyle (i)}$$

The corresponding (parallelized) elimination rule (for applications of which the major premise must stand proud) takes the form

$$(\to\text{-E})\qquad \dfrac{\varphi\to\psi \qquad \begin{array}{c}\vdots\\ \varphi\end{array} \qquad \begin{array}{c}\Box\,\overline{\quad}^{\,(i)}\\ \psi\\ \vdots\\ \theta\end{array}}{\theta}\,{\scriptstyle (i)}$$

We saw above how this leads to the rules for the biconditional in Core Logic, by simply dealing with each of the two directions in like manner.

Once the *structural* rules of Thinning (Dilution) and of *absolutely unrestricted* Cut have been identified as the primary culprits responsible for deductive irrelevancies (see [16]), the task of relevantizing *deduction* (at the "level of the turnstile") faces the following four challenges.

(1) Maintain completeness.
(2) Preserve enough in the way of deductive resources to meet all the logical demands of mathematics (either intuitionistic or classical), and the hypothetico-deductive method in the natural sciences.
(3) Ensure relevance (in the case of propositional logic) by means of a strongest possible form of a "variable-sharing" result that would arguably surpass any other such result then available for a great variety of systems of the Anderson–Belnap type, where the reforming focus has always been on changing the behavior of the *connective* $\to$, while uncritically accepting the usual structural rules governing $\vdash$ as sacrosanct.
(4) Show that achieving the proper analysis or systematic explication of the relevance of premises to conclusions of deductive arguments leads to genuine advantages in *automated deduction*, whose job (at the propositional level) is to take problems of the form Δ ?- φ and to find, efficiently, a proof of φ from (premises lying in) Δ, if there is one, and otherwise to report that there is no such proof to be found.

We submit that these four aims have been met—by Core Logic in the intuitionistic case, and by Classical Core Logic in the classical case. (Each of these logics is relevantized in the same way.) While achieving aim (1) would ipso facto accomplish (2), it is worth stating separately the results that respectively fulfill these aims.

Regarding aim (1). We have the following metatheorems.

(1.i) If Δ logically implies φ in Intuitionistic Logic, then in Core Logic there is a proof either of φ or of $\perp$ whose premises lie in Δ.
(1.ii) If Δ logically implies φ in Classical Logic, then in Classical Core Logic there is a proof either of φ or of $\perp$ whose premises lie in Δ.

Regarding aim (2). We have the following metatheorems in the intuitionistic case.

(2.i)(a) Every theorem of Intuitionistic Logic is a theorem of Core Logic.
(2.i)(b) Every set of sentences that is inconsistent in Intuitionistic Logic is inconsistent in Core Logic.
(2.i)(c) If φ is an intuitionistic consequence of an intuitionistically consistent set Δ of sentences, then φ is deducible in Core Logic from premises lying in Δ.

And we have the corresponding metatheorems in the classical case.

(2.ii)(a) Every theorem of Classical Logic is a theorem of Classical Core Logic.
(2.ii)(b) Every set of sentences that is inconsistent in Classical Logic is inconsistent in Classical Core Logic.
(2.ii)(c) If φ is a classical consequence of a classically consistent set Δ of sentences, then φ is deducible in Classical Core Logic from premises lying in Δ.

The conclusion justified by these metatheorems is that Core Logic is adequate for all the methodological demands of the intuitionist, and Classical Core Logic is adequate for all the methodological demands of the classicist.

While these considerations should in themselves suffice to allay any worries that the method of relevantizing employed in Core Logic and in its classicized extension might rob us of patterns of deductive reasoning that we *need*, there is one further worry that might be raised at this point, and which we are also concerned to emphasize can be definitively disposed of. The worry might take the form:

> It's all very well for you, the Core logician, to tell us that proofs are "there, to be had" if and when we need them; but our concern is actually to *have* them, so that we can check them and learn from them. One apparent drawback of your formal system of proof is that major premises for eliminations must always "stand proud," with no proof-work above them. But imagine now a situation of the following kind. One has found a (Core) proof Π of a lemma φ from axioms Δ, and the conclusion φ is obtained by an application of the introduction rule for the dominant operator of φ at the terminal step of Π:
>
> $$\begin{array}{c}\Delta\\ \Pi\\ \varphi\ ^{(I)}\end{array}$$
>
> Later one finds a (Core) proof Σ of a deep theorem ψ from yet other axioms Γ, along with lemma φ serving now as a *premise*. Within this proof Σ, at least one premise-occurrence of φ stands as the major premise for an application of the elimination rule for the operator dominant in φ:
>
> $$\begin{array}{c}(E)\ \underbrace{\varphi\ ,\ \Gamma}\\ \Sigma\\ \psi\end{array}$$
>
> *You cannot put these two Core proofs together!* For your formal definition of Core proof forbids that. So how on earth is one to enjoy the fruits of the deductive progress that we routinely assume we are entitled to? Can we really no longer "divide and conquer" by interpolating lemmas between our axioms and the deductively distant theorems that we seek to deduce from them?

The worry is well taken, and can be allayed as follows. As shown in [11], there is an inductively definable, effective operation on Core proofs, denoted

$$[\Pi, \Sigma]$$

which in general produces as its output (*from* the two inputs Π and Σ) a Core proof *either* of the sought conclusion ψ, *or* (surprisingly and informatively, perhaps) of $\bot$, from premises drawn from $\Delta \cup \Gamma$. In the mathematical case, of course, the confident belief will be that $\Delta \cup \Gamma$ is consistent—it consists, after all, of only mathematical axioms. So in this case we can rest assured that $[\Pi, \Sigma]$ will actually be a proof *of the sought theorem* ψ, rather than of $\bot$. We stress again: this proof of ψ from (premises in) $\Delta \cup \Gamma$ can be effectively determined from the two proofs Π and Σ that were produced by the worried mathematician's division of deductive labor *to* the lemma φ

and then *from* it to the theorem ψ. The worry expressed above has been disposed of. (The analogous result for *Classical* Core Logic is established in [14].)

Aims (3) and (4) were accomplished in [6], with aim (3) even more definitively accomplished in [15].

Regarding aim (3): In automated deduction the aim is to write programs to solve deductive problems of the form "Is there a proof of φ from Δ?". It is very useful to have "relevance filters" to weed out as many "no-hopers" as possible before even embarking on a serious search for a proof.

Core Logic, in both its constructive and its classical forms, is a *relevant* logic, in an interesting and deeper sense than that provided merely by the assurance that the logic does not allow derivation of the Lewis Paradox (in either its positive or its negative form). We shall confine ourselves here to the propositional system in explaining the formal explication of deductive relevance that is to be had from Core Logic. This task involves spelling out the details of a very exigent form of "variable-sharing" (or, in our terminology, sharing of *atoms*). To this end, we need to supply the following definitions.

Definition 1.1 $\pm\,\varphi \equiv_{\rm df}$ some atom occurs both positively and negatively in φ. (Note that $\pm$ is a metalinguistic predicate, not a function sign.)

Definition 1.2 $\varphi \approx \Delta \equiv_{\rm df}$ some atom has the *same* parity (positive or negative, at some occurrence) in φ as it has in some member of Δ.

Definition 1.3 Suppose $\varphi \neq \psi$. Then $\varphi \bowtie \psi \equiv_{\rm df}$ some atom has the *opposite* parity at some occurrence in φ from that which it has at some occurrence in ψ.

Definition 1.4 $\varphi_1, \ldots, \varphi_n$ $(n > 1)$ is a $\bowtie$*-path connecting* φ_1 *to* φ_n *in* $\Delta \equiv_{\rm df}$ for $1 \leq i \leq n$, φ_i is in Δ, and for $1 \leq i < n$, $\varphi_i \bowtie \varphi_{i+1}$.

Definition 1.5 A set Δ of formulas is $\bowtie$*-connected* $\equiv_{\rm df}$ for all φ, ψ in Δ if $\varphi \neq \psi$, then there is a $\bowtie$-path connecting φ to ψ in Δ.

Definition 1.6 A *component* of Δ is an inclusion-maximal $\bowtie$-connected subset of Δ (where the $\bowtie$-connections are established via members of Δ).

Relevance metatheorem about Core Logic. A Classical Core proof of a conclusion φ from a set Δ of undischarged assumptions establishes that Δ is *relevantly connected both within itself and to* φ, in the sense that exactly one of the following three conditions holds:

(1) Δ is nonempty, φ is $\perp$, and:
if Δ is a singleton $\{\delta\}$, then $\pm\,\delta$; otherwise, Δ is $\bowtie$*-connected.*
(2) Δ is nonempty, φ is not $\perp$, and:
the components $\Delta_1, \ldots, \Delta_m$ $(m \geq 1)$ of Δ are such that for $1 \leq i \leq m$, we have $\varphi \approx \Delta_i$.
(3) Δ is empty, φ is not $\perp$, and $\pm\,\varphi$.

Cases (1) and (3) cover the two logical extremes. In case (1) we have a proof of the joint inconsistency of the premises in Δ. In that case Δ itself is the only component of Δ. In case (3) we have a proof of a logical theorem φ. In that case φ will contain some atom both positively and negatively.

Case (2) covers the "middle range," so to speak, and it is this case that reveals the most interesting structure involving both Δ and φ. First, the set Δ of premises is partitioned into components $\Delta_1, \ldots, \Delta_n$ ($n \geq 1$), each of which, if not a singleton, is $\bowtie$-connected. Moreover, each component Δ_i bears a special relation to φ, to wit: some atom occurs with the same parity in φ as it does in *some member* of Δ_i.

Regarding aim (4): Framing elimination rules in parallelized form is the natural and obvious thing to do when one is concerned with a framework for efficient *proof search*. When one is given a sequent to be proved (call this the "original problem"), one has to break it down into subproblems. One can do this by attending to the conclusion of the sequent, and framing the subproblem as that of finding appropriate subordinate proofs that would enable one to apply the *introduction* rule for the dominant operator in the conclusion. Alternatively, one can attend instead to one of the premises, and frame the subproblem as that of finding appropriate subordinate proofs that would enable one to apply the *elimination* rule for the dominant operator in the premise in question, which will then be the major premise for that elimination.

Using elimination rules in parallelized form makes for *shorter* formal proofs. This is because one can avoid the need to repeat whole fragments of proof every time one wishes to apply an elimination rule. This makes automated proof-search much more efficient. These motivations for using parallelized forms of elimination rules were treated in depth in [6].

Proof-search is a "bottom-up" process, by contrast with the inductive definition of proof, which is "top-down." At every stage in building a proof *upward* from the concluding, or *target* sequent, one does well to be on the lookout for results (subproofs) that establish a *proper subsequent* of the target sequent at the stage in question. For *opting to prove a proper subsequent of a given sequent never leads to loss*. In the search for a proof of any given sequent $\mathcal{S}$, one should *always* be able to rest content with a proof (if one is found) of a *proper subsequent* of $\mathcal{S}$. And this should be the case *iteratively*:

> If the call for a proof of $\mathcal{S}$ arises as a *sub*problem during a search for a proof of some other sequent $\mathcal{S}'$, say, then any proof of a *proper subsequent* of $\mathcal{S}$ should be just as valuable as a proof of $\mathcal{S}$ itself, in so far as the eventual construction of a proof of (a subsequent of) $\mathcal{S}'$ is concerned.

Call this the *Iterative Requirement that Proofs of Proper Subsequents Be Deployable*. Core Logic meets this requirement; indeed, *it was designed to do so*.

2 Some Mistaken Core Beliefs About Logic

The core beliefs about logic in general that the foregoing discussion shows are candidates for re-examination are the following.

(1) The interdeducibility of two sentences should be sufficient for their intereplaceability, *salva veritate*, in all statements of deducibility.

(2) The "deduction theorem" should hold, not only in the obvious direction determined by the rule of conditional proof:

$$\Delta, A \vdash B \Rightarrow \Delta \vdash A \to B,$$

but also in the *converse* direction:

$$\Delta \vdash A \to B \Rightarrow \Delta, A \vdash B.$$

(3) Deducibility should be *unrestrictedly transitive*:

$$(\Delta \vdash \varphi \text{ and } \varphi, \Gamma \vdash \psi) \Rightarrow \Delta, \Gamma \vdash \psi.$$

Ad (1): We owe to Smiley [4, p. 427] the explication of "sentences A and B are synonymous" as "A and B are interreplaceable *salva veritate* in all statements of provability and deducibility." This explication, of course, is provided for sentences of a particular logical system. So core belief (1) above is to the effect that interdeducible sentences are synonymous (within the system in question).

It is not at all clear that this is a reasonable belief. After all, sentences with very different logical structures can be interdeducible, while yet their differences of logical structure might provide reasonable grounds for refusing to regard them as synonymous. Indeed, finding some true statement of deducibility involving one of them, which becomes false upon its replacement with the other one, would be a good way to justify that refusal. The groundbreaking insight behind Smiley's suitably exigent notion of synonymy is that interdeducibility within a logical system is not always sufficient for synonymy within that system. In some systems, it is; but in many a system it is not. Core logic happens to be one of the latter.

Ad (2): The deduction theorem, in its dubious direction

$$\Delta \vdash A \to B \Rightarrow \Delta, A \vdash B$$

is hardly a feature onto which one must hold at all costs. After all, the simple instance

$$\neg A \vdash A \to B \Rightarrow \neg A, A \vdash B$$

provides a resounding counterexample for anyone who respects the truth table for $\to$ but who also has the (compatible!) "relevantist intuition" or conviction that it is not the case that any proposition whatsoever follows *logically* from a contradictory set of premises.

Ad (3): The addiction to unrestricted transitivity of deduction is one of the saddest afflictions from which logicians can suffer. A striking counterexample to

$$(\Delta \vdash \varphi \text{ and } \varphi, \Gamma \vdash \psi) \Rightarrow \Delta, \Gamma \vdash \psi$$

is the following:

$$(A \vdash A \vee B \text{ and } A \vee B, \neg A \vdash B); \text{ yet } A, \neg A \nvdash B.$$

Vidal-Rosset appeals to Leibniz as an authority on the matter of interreplaceability of interdeducibles, in connection with core belief (1). He could also have made similar appeals, for core beliefs (2) and (3), to authorities such as Aristotle, Frege, Russell, Tarski, Gödel, and even Gentzen. But appeals to authority and tradition have little traction here. All the fundamental principles of logic deserve continual re-examination. To commit to them all dogmatically, as a single package, is to forfeit the opportunity to think through their interconnections, and to discover subtle ways in which they might be revised for the better. This is very much the case with the matter of *basic rules of proof-formation*, and the associated issue of *unrestricted* transitivity of deduction.

Ironically, if any of the great figures came close to "getting it right" in this regard, it was Aristotle. As shown in [12], his syllogistic is a fragment of Core Logic; he did not even need his rule of Classical Reductio.

Appendix. The Rules of Core Logic $\mathbb{C}$ and Classical Core Logic $\mathbb{C}^{+}$

(¬-I)
$$\dfrac{\begin{array}{c}\square\overline{}^{(i)}\\ \varphi\\ \vdots\\ \bot\end{array}}{\neg\varphi}\,(i)$$

(¬-E)
$$\dfrac{\neg\varphi \qquad \begin{array}{c}\vdots\\ \varphi\end{array}}{\bot}$$

(∧-I)
$$\dfrac{\begin{array}{c}\vdots\\ \varphi\end{array} \qquad \begin{array}{c}\vdots\\ \psi\end{array}}{\varphi\wedge\psi}$$

(∧-E)
$$\dfrac{\varphi\wedge\psi \qquad \begin{array}{c}(i)\overline{}\square\overline{}(i)\\ \underbrace{\varphi\ ,\ \psi}\\ \vdots\\ \theta\end{array}}{\theta}\,(i)$$

(∨-I)
$$\dfrac{\begin{array}{c}\vdots\\ \varphi\end{array}}{\varphi\vee\psi} \qquad \dfrac{\begin{array}{c}\vdots\\ \psi\end{array}}{\varphi\vee\psi}$$

(∨-E)
$$\dfrac{\varphi\vee\psi \quad \begin{array}{c}\square\overline{}^{(i)}\\ \varphi\\ \vdots\\ \theta\end{array} \quad \begin{array}{c}\square\overline{}^{(i)}\\ \psi\\ \vdots\\ \theta\end{array}}{\theta}\,(i) \qquad \dfrac{\varphi\vee\psi \quad \begin{array}{c}\square\overline{}^{(i)}\\ \varphi\\ \vdots\\ \bot\end{array} \quad \begin{array}{c}\square\overline{}^{(i)}\\ \psi\\ \vdots\\ \theta\end{array}}{\theta}\,(i) \qquad \dfrac{\varphi\vee\psi \quad \begin{array}{c}\square\overline{}^{(i)}\\ \varphi\\ \vdots\\ \theta\end{array} \quad \begin{array}{c}\square\overline{}^{(i)}\\ \psi\\ \vdots\\ \bot\end{array}}{\theta}\,(i)$$

(→-I)
$$\dfrac{\begin{array}{c}\square\overline{}^{(i)}\\ \varphi\\ \vdots\\ \bot\end{array}}{\varphi\rightarrow\psi}\,(i) \qquad \dfrac{\begin{array}{c}\diamond\overline{}^{(i)}\\ \varphi\\ \vdots\\ \psi\end{array}}{\varphi\rightarrow\psi}\,(i)$$

(→-E)
$$\dfrac{\varphi\rightarrow\psi \qquad \begin{array}{c}\vdots\\ \varphi\end{array} \qquad \begin{array}{c}\square\overline{}^{(i)}\\ \psi\\ \vdots\\ \theta\end{array}}{\theta}\,(i)$$

(∃-I)
$$\dfrac{\begin{array}{c}\vdots\\ \varphi^{x}_{t}\end{array}}{\exists x\varphi}$$

(∃-E)
$$\dfrac{\exists x\varphi\ @ \qquad \begin{array}{c}\square\overline{}^{(i)}\\ \underbrace{@\ \ldots\varphi^{x}_{a}\ldots\ @}\\ \vdots\\ \psi\ @\end{array}}{\psi}\,(i)$$

(∀-I)
$$\dfrac{\begin{array}{c}@\\ \vdots\\ \varphi\end{array}}{\forall x\varphi^{a}_{x}}$$

(∀-E)
$$\dfrac{\forall x\varphi \qquad \begin{array}{c}(i)\overline{}\cdots\square\cdots\overline{}(i)\\ \underbrace{\varphi^{x}_{t_1}\ ,\ \ldots\ ,\ \varphi^{x}_{t_n}}\\ \vdots\\ \theta\end{array}}{\theta}\,(i)$$

Adding to $\mathbb{C}$ suitably relevantized versions of *Classical Reductio* or *Dilemma*

$$(\mathrm{CR})\quad \dfrac{\begin{array}{c}\square\overline{\quad}^{(i)}\\ \neg\varphi\\ \vdots\\ \bot\end{array}}{\varphi}\,(i) \qquad\qquad (\mathrm{Dil})\quad \dfrac{\begin{array}{c}\square\overline{\quad}^{(i)}\\ \varphi\\ \vdots\\ \psi\end{array}\qquad \begin{array}{c}\square\overline{\quad}^{(i)}\\ \neg\varphi\\ \vdots\\ \psi/\bot\end{array}}{\psi}\,(i)$$

yields the system $\mathbb{C}^+$ of Classical Core Logic.

Notes

1. The rules of Core Logic $\mathbb{C}$ and its classicized extension $\mathbb{C}^+$ are to be found in the Appendix at the end.

2. This quote is taken from [18, p. 241]; the emphasis is in the original.

3. Vidal-Rosset's critique is confined to the level of propositional, not first-order, logic; so in rebutting it one need pay attention only to propositional logic. It is worth mentioning, however, that the "logic of evaluation" $\mathbb{E}$ is available at first order, and not only at the propositional level. When confined to the propositional level, $\mathbb{E}$ can be thought of as the bare "logic of the truth tables."

4. He could also, then, have used the title "Why Core Logic cannot be a core logic." But it is unwise for the target of a critique to suggest ways to make the critique rhetorically more effective.

5. This answers positively a welcome query from an anonymous referee.

6. It might sound far-fetched, but we have to consider here the possibility that someone contemplating the premise $A \wedge \neg A$ does not see right away that it is inconsistent. If this is difficult to imagine, it is only because the example is so degenerate. In general, an inconsistency on the part of a single premise or a set of premises is not so readily detectable; and it is with such cases that the teasing out of a proof of $\bot$ (rather than of some irrelevant conclusion that allegedly "follows" from the inconsistent set of premises in question) provides us with epistemic gain.

7. This motivation was made very clear in [5], the study with which these investigations began.

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