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More Automorphism Groups of Countable, Arithmetically Saturated Models of Peano Arithmetic

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Abstract There is an infinite set $\mathcal{T}$ of Turing-equivalent completions of Peano Arithmetic (PA) such that whenever $\mathcal{M}$ and $\mathcal{N}$ are nonisomorphic countable, arithmetically saturated models of PA and $\mathrm{Th}(\mathcal{M})$, $\mathrm{Th}(\mathcal{N}) \in \mathcal{T}$, then $\mathrm{Aut}(\mathcal{M}) \ncong \mathrm{Aut}(\mathcal{N})$.

Investigating the extent to which (the isomorphism type of) a countable, recursively saturated model $\mathcal{M}$ of Peano Arithmetic (PA) is determined by (the isomorphism type of) its automorphism group $\mathrm{Aut}(\mathcal{M})$ has been of interest since the appearance of [2]. Recent progress was made in [8], where it was proved that if $\mathcal{M}$, $\mathcal{N}$ are countable, arithmetically saturated models of PA and $\mathrm{Aut}(\mathcal{M}) \cong \mathrm{Aut}(\mathcal{N})$, then $\mathrm{Th}(\mathcal{M})' \equiv_T \mathrm{Th}(\mathcal{N})'$. (As usual, X' is the Turing-jump of X and $\equiv_T$ is Turing-equivalence.) The following theorem affirmatively answers Question 5.8 in [8].

Theorem *There are infinitely many completions $T_0, T_1, T_2, \ldots$ of* PA *such that whenever $i < j < \omega$, then*

(1) $T_i \equiv_T T_j$,
(2) $\mathrm{Aut}(\mathcal{M}_i) \ncong \mathrm{Aut}(\mathcal{M}_j)$ *for all countable, arithmetically saturated $\mathcal{M}_i \models T_i$ and $\mathcal{M}_j \models T_j$.*

From Nurkhaidarov [5], one can get 4 completions T_0, T_1, T_2, T_3 of PA such that (2) of the theorem holds whenever $i < j < 4$ and (1) holds whenever $1 \leq i < j < 4$, with $T_0 = \mathrm{TA}$. With some more effort, one can get (1) to hold whenever $i < j < 4$. This result was improved in [8, Theorem 6], where the number 4 was increased to any finite n. It was then asked in [8, Question 5.8] if there are infinitely many such completions. The theorem confirms that there are.

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It should be remarked that for any countably many completions $T_0, T_1, T_2, \ldots$ of PA, there are (many) countable, arithmetically saturated models $\mathcal{M}_i \models T_i$ for which $\mathrm{SSy}(\mathcal{M}_i) = \mathrm{SSy}(\mathcal{M}_j)$ whenever $i < j < \omega$.

The proof of the Theorem is much more in the style of [5] than [8, Theorem 6]. In the next two lemmas, we give a quick overview of Nurkhaidarov's proof.

Let $\mathbb{N} = (\omega, +, \times, 0, 1, \leq)$ be the standard model of PA and let $\mathsf{TA} = \mathrm{Th}(\mathbb{N})$. For $n, k < \omega$, we let RT^n_k be infinite Ramsey's Theorem for n-sets and k colors as formalized in second-order arithmetic. Nurkhaidarov [5, Theorem 3.8] proved the following lemma.

Lemma 1 *If $2 \leq n < \omega$ and $\mathcal{M}, \mathcal{N}$ are countable, arithmetically saturated models of* PA *such that* $\mathrm{Aut}(\mathcal{M}) \cong \mathrm{Aut}(\mathcal{N})$*, then*

$$\big(\mathbb{N}, \mathrm{Rep}\big(\mathrm{Th}(\mathcal{M})\big)\big) \models \mathsf{RT}^n_2 \qquad \textit{iff} \qquad \big(\mathbb{N}, \mathrm{Rep}\big(\mathrm{Th}(\mathcal{N})\big)\big) \models \mathsf{RT}^n_2.$$

To get the four theories, Nurkhaidarov [5] made use of the existence of countable Scott sets $\mathfrak{X}_1, \mathfrak{X}_2$, and $\mathfrak{X}_3$ such that $(\mathbb{N}, \mathfrak{X}_1) \models \neg\mathsf{RT}^2_2$, $(\mathbb{N}, \mathfrak{X}_2) \models \mathsf{RT}^2_2 + \neg\mathsf{RT}^3_2$, and $(\mathbb{N}, \mathfrak{X}_3) \models \mathsf{RT}^3_2$, obtaining distinct completions T_0, T_1, T_2, T_3 such that $T_0 = \mathsf{TA}$ and $\mathrm{Rep}(T_i) = \mathfrak{X}_i$ for $i \in \{1, 2, 3\}$. To get that $T_1 \equiv_T T_2 \equiv_T T_3$, we use the following lemma, which is an immediate consequence of Marker [4, Theorem 1.27] and also of a theorem (see [8, Theorem 1.2]) due to Knight and Marker.

Lemma 2 *If $\mathfrak{X}_0, \mathfrak{X}_1, \mathfrak{X}_2, \ldots$ are countably many countable Scott sets, then there are distinct completions $T_0, T_1, T_2, \ldots$ of* PA *such that whenever $i < j < \omega$, then $T_i \equiv_T T_j$ and* $\mathrm{Rep}(T_i) = \mathfrak{X}_i$.

The proof of the Theorem makes essential use of a result of Patey [7, Theorem 4.14] as a replacement for Ramsey's Theorem in Lemma 1. If $n < \omega$ and X is a subset of an ordered set (such as ω or some M where $\mathcal{M}$ is a model of PA), then $[X]^n$ is the set of all strictly increasing n-tuples from X. Let TS^n_k be the Thin Set Theorem for n-sets and k colors, which asserts the following: for any function $f : [\omega]^n \longrightarrow k$, there is an infinite set $X \subseteq \omega$ such that $|\{f(x) : x \in [X]^n\}| < k$. In particular, $\mathsf{TS}^n_2 = \mathsf{RT}^n_2$. Also, notice that $\mathsf{RCA}_0 \vdash \mathsf{TS}^n_k \to \mathsf{TS}^n_{k+1}$. Patey [7] proved that whenever $2 \leq k < \omega$, there is a countable Scott set $\mathfrak{X}_k$ such that $(\mathbb{N}, \mathfrak{X}_k) \models \mathsf{TS}^2_{k+1} + \neg\mathsf{TS}^2_k$. Thus, with Lemma 2 available, it suffices to prove the following generalization of Lemma 1.

Lemma 3 *If $2 \leq k, n < \omega$ and $\mathcal{M}$, $\mathcal{N}$ are countable, arithmetically saturated models of* PA *such that* $\mathrm{Aut}(\mathcal{M}) \cong \mathrm{Aut}(\mathcal{N})$*, then*

$$\big(\mathbb{N}, \mathrm{Rep}\big(\mathrm{Th}(\mathcal{M})\big)\big) \models \mathsf{TS}^n_k \qquad \textit{iff} \qquad \big(\mathbb{N}, \mathrm{Rep}\big(\mathrm{Th}(\mathcal{N})\big)\big) \models \mathsf{TS}^n_k.$$

Proof We will freely use terminology from [8]. For example, if $G \leq \mathrm{Aut}(\mathcal{K})$ and $a \in K$, then $G_{(a)}$ is the stabilizer of a in G. A basic open subgroup of $\mathrm{Aut}(\mathcal{K})$ is one having the form $\mathrm{Aut}(\mathcal{K})_{(a)}$. Let $\mathcal{M}$, $\mathcal{N}$ be countable, arithmetically saturated models of PA, and let $\alpha : \mathrm{Aut}(\mathcal{M}) \longrightarrow \mathrm{Aut}(\mathcal{N})$ be an isomorphism. Recall (from [8, Section 4] or [6, Corollary 3.14]) that basic open subgroups are recognizable. This means, in particular, that if $a \in M$ and $H = \mathrm{Aut}(\mathcal{M})_{(a)}$, then there is $b \in N$ such that $\alpha[H] = \mathrm{Aut}(\mathcal{N})_{(b)}$.

Suppose that $\mathcal{K}$ is any countable, arithmetically saturated model of PA, and let $G = \mathrm{Aut}(\mathcal{K})$. We will say that a basic open subgroup $H < G$ is *maximal* if whenever $a \in K$ is such that $H \leq G_{(a)} < G$, then $H = G_{(a)}$. We let $\Omega_\omega(\mathcal{K})$ (or simply Ω_ω when no confusion is likely) be the smallest interstice of $\mathcal{K}$; that is, Ω_ω is the set of nonstandard elements that are less than all the definable nonstandard elements of $\mathcal{K}$. The arithmetic saturation of $\mathcal{K}$ implies that $\Omega_\omega \neq \varnothing$. We now define a subgroup $H < G$ to be (n,k)-*Ramsey*, where $2 \leq k, n < \omega$, if each of the following holds.

(1) H is a maximal basic open subgroup.
(2) $H = G_{(a)}$ for some $a \in \Omega_\omega$.
(3) Suppose that $H_{i,j}$ are conjugates of H , for $i < n$ and $j < k$, such that $H_{0,j}, H_{1,j}, \ldots, H_{n-1,j}$ are pairwise distinct for each $j < k$. Then there are $r < s < k$, a permutation $\pi : n \longrightarrow n$, and $h \in G$ such that $hH_{i,r}h^{-1} = H_{\pi(i),s}$ for each $i < n$.

As already noted, those subgroups H satisfying (1) are recognizable. It is obvious that those subgroups $H \leq G$ satisfying (3) are recognizable. We next will prove that (n,k)-Ramsey subgroups are recognizable by showing that those subgroups $H \leq G$ satisfying (2) are recognizable. A cut I of $\mathcal{K}$ is an *icut* if $I < \Omega_\omega \cup \omega$ and it is closed under all $\varnothing$-definable functions $f : K \longrightarrow K$ such that $f(x) < \omega$ whenever $x < \omega$. A set $\gamma \subseteq K$ is an *igap* if there is $a \in \Omega_\omega$ such that if $I \subseteq K$ is the largest icut for which $a \notin I$ and $J \subseteq K$ is the smallest icut for which $a \in J$, then $\gamma = J \setminus I$. It follows from [8, Lemma 4.4(d)] that setwise stabilizers of igaps are recognizable. But a consequence of Bamber and Kotlarski [1, Theorem 3.8] is that a subgroup $H \leq G$ satisfies (2) iff it is the setwise stabilizer of an igap.

Hence, (n,k)-Ramsey subgroups are recognizable, so, to complete the proof, we need only prove the following:

($*$) For a countable, arithmetically saturated $\mathcal{K} \models \mathsf{PA}$, $\mathrm{Aut}(\mathcal{K})$ has an (n,k)-Ramsey subgroup iff $(\mathbb{N}, \mathrm{Rep}(\mathrm{Th}(\mathcal{K}))) \models \mathsf{TS}^n_k$.

Let $T = \mathrm{Th}(\mathcal{K})$ and $G = \mathrm{Aut}(\mathcal{K})$.

($\Longrightarrow$): Let $f : [\omega]^n \longrightarrow k$ be a function in $\mathrm{Rep}(T)$. We want an infinite $X \in \mathrm{Rep}(T)$ such that $|\{f(x) : x \in [X]^n\}| < k$.

Let $H < G$ be (n,k)-Ramsey, and let $a \in \Omega_\omega$ be such that $H = G_{(a)}$. Whenever $B \subseteq K$ is $\varnothing$-definable and $a \in B$, then $B \cap \omega$ is an infinite set in $\mathrm{Rep}(T)$. We now claim that there is a $\varnothing$-definable $D \subseteq K$ such that $a \in D$ and $|\{f(x) : x \in [\omega \cap D]^n\}| < k$. For, if not, then by recursive saturation, for each $j < k$, there are $b_{0,j} < b_{1,j} < \cdots < b_{n-1,j}$ such that $\mathrm{tp}(b_{0,j}) = \mathrm{tp}(b_{1,j}) = \cdots = \mathrm{tp}(b_{n-1,j}) = \mathrm{tp}(a)$ and $\hat{f}(b_{0,j}, b_{1,j}, \ldots, b_{n-1,j}) = j$, where $\hat{f}$ is the definitional extension of f to all of $[K]^n$. Then $H_{i,j} = G_{(b_{i,j})}$ is a counterexample to (3).

($\Longleftarrow$): Suppose that $(\mathbb{N}, \mathrm{Rep}(T)) \models \mathsf{TS}^n_k$. We next construct a complete type. Let $\langle \varphi_m(\overline{x}, y) : m < \omega \rangle$ be a computable sequence of $\mathcal{L}_{\mathsf{PA}}$-formulas that define all the $\varnothing$-definable functions from K^n into K and nothing else. Let $\langle \psi_m(x, y) : m < \omega \rangle$ be a computable sequence of $\mathcal{L}_{\mathsf{PA}}$-formulas that define all the $\varnothing$-definable functions from K into K and nothing else. (These two sequences can be obtained independently of T.) Let f_m be the function defined by $\varphi_m(\overline{x}, y)$, and let g_m be the function defined by $\psi_m(x, y)$. We obtain a T-arithmetic sequence $\langle \theta_m(x) : m < \omega \rangle$ of

$\mathcal{L}_{\mathsf{PA}}$-formulas such that the following hold for all $m < \omega$, where we let X_m be the set defined by $\bigwedge_{\ell<m} \theta_\ell(x)$ (so that $K = X_0 \supseteq X_1 \supseteq X_2 \supseteq \cdots$).

(4) The set $X_m \cap \omega$ is infinite.
(5) $|\{f_m(\overline{x}) : \overline{x} \in [X_{m+1}]^n\}| < k$.
(6) $g_m \restriction X_{m+1}$ is either constant or injective.

We can obtain such a sequence $\langle \theta_m(x) : m < \omega \rangle$ by recursion as follows. Let $X_0 = K$. Now suppose that we have X_m and that $X_m \cap \omega$ is infinite and $X_m \cap \omega \in \mathrm{Rep}(T)$. The function $f_m \restriction [X_m]^n$ is also in $\mathrm{Rep}(T)$; therefore, since $(\mathbb{N}, \mathrm{Rep}(T)) \models \mathsf{TS}^n_k$, there is an infinite $Y \in \mathrm{Rep}(T)$ such that $Y \subseteq X_m$ and $|\{f_m(\overline{x}) : \overline{x} \in [Y]^n\}| < k$. Then there is an infinite $Z \in \mathrm{Rep}(T)$ such that $Z \subseteq Y$ and $g_m \restriction Z$ is either constant or injective. Let $\theta(x)$ be an $\mathcal{L}_{\mathsf{PA}}$-formula that defines a set $X \subseteq X_m$ such that $Z = X \cap \omega$. Let $a \in K \cup \{\infty\}$ be the greatest such that $g_m \restriction \{x \in X : x < a\}$ is constant or injective and that $|\{f_m(\overline{x}) : \overline{x} \in [\{x \in X : x < a\}]^n\}| < k$, and then let $\theta_{m+1}(x) = \theta(x) \wedge x < a$. Note that a is nonstandard and definable in $\mathcal{K}$. Thus, $X_{m+1} \cap \omega$ is infinite and $X_{m+1} \cap \omega \in \mathrm{Rep}(T)$.

One easily checks that the $\theta_m(x)$'s satisfy (4)–(6). The set $\{\theta_m(x) : m < \omega\}$ determines a complete type of T. To see why it does, consider a $\varnothing$-definable $D \subseteq K$. Let $m < \omega$ be such that g_m is the characteristic function of D. Then, since X_{m+1} is infinite, (6) implies that g_m is either constantly 0 or constantly 1 on X_{m+1}. Thus, either $D \supseteq X_{m+1}$ or $K \setminus D \supseteq X_{m+1}$.

We have yet to say anything about the effectiveness of the construction of this sequence, but it should be clear that it can be obtained by a construction that is arithmetic in T, so we assume that that is the case. Now, by the arithmetic saturation of $\mathcal{K}$, there is $a \in \bigcap\{X_m : m < \omega\}$. (In fact, a is unique up to automorphic images.) Clearly, $a \in \Omega_\omega$. Let $p(x) = \mathrm{tp}(a)$.

We claim that there are at most $k - 1$ n-types $q(x_0, x_1, \ldots, x_{n-1})$ such that the formula $x_0 < x_1 < \cdots < x_{n-1}$ is in $q(\overline{x})$ and $p(x_i) \subseteq q(\overline{x})$ for all $i < n$. For, suppose that $q_0(\overline{x}), q_1(\overline{x}), \ldots, q_{k-1}(\overline{x})$ form a counterexample. Let $\alpha_0(\overline{x}), \alpha_1(\overline{x}), \ldots, \alpha_{k-1}(\overline{x})$ be pairwise contradictory formulas such that $\alpha_j(\overline{x}) \in q_j(\overline{x})$ for $j < k$. Let $m < \omega$ be such that $f_m(\overline{b}) = j$ whenever $\mathcal{K} \models \alpha_j(\overline{b})$. Let $j < k$ be such $f_m(\overline{x}) \neq j$ whenever $\overline{x} \in [X_{m+1}]^n$. But then it cannot be that $p(x_i) \subseteq q_j(\overline{x})$ for all $i < n$. This contradiction proves the claim.

Now let $H = G_{(a)}$. We show that H is (n, k)-Ramsey.

(1) By definition, H is a basic open subgroup. Since a is not definable, $H \neq G$. Although the argument that H is maximal basic open is well known, we include it. Suppose that $H \leq G_{(b)} < G$. Then there is an $m < \omega$ such that $g_m(a) = b$. Then $g_m \restriction X_{m+1}$ is either constant or injective. If $g_m \restriction X_{m+1}$ were constant, then b would be definable so that $G_{(b)} = G$. Hence, $g_m \restriction X_{m+1}$ is injective, so there is $\ell < \omega$ such that such that $g_\ell g_m(x) = x$ for all $x \in X_{m+1}$. Therefore, $g_\ell(b) = a$. Hence, $G_{(b)} \leq G_{(a)} = H$.

(2) This is obvious as $a \in \Omega_\omega$.

(3) Suppose that the $H_{i,j}$'s are as given in (3). Let $h_{i,j} \in G$ be such that $H_{i,j} = h_{i,j} H h_{i,j}^{-1}$. Let $a_{i,j} = h_{i,j}(a)$, so that $H_{i,j} = G_{(a_{i,j})}$. Since, for each $j < k$, the subgroups $H_{0,j}, H_{1,j}, \ldots, H_{n-1,j}$ are pairwise distinct, it must be that

$a_{0,j}, a_{1,j}, \ldots, a_{n-1,j}$ are pairwise distinct. Without loss of generality, assume that $a_{0,j} < a_{1,j} < \cdots < a_{n-1,j}$, and let $\overline{a}_j = \langle a_{0,j}, a_{1,j}, \ldots, a_{n-1,j} \rangle$.

By the claim proved earlier, there are $r < s < k$ such that $\mathrm{tp}(\overline{a}_r) = \mathrm{tp}(\overline{a}_s)$. By the homogeneity of $\mathcal{K}$, there is $h \in G$ such that $h(a_{i,r}) = a_{i,s}$ for all $i < n$. Then, $hH_{i,r}h^{-1} = H_{i,s}$ for each $i < n$. □

We end with two corollaries of the Theorem. The first is from the abstract.

Corollary 4 *There is an infinite set $\mathcal{T}$ of Turing-equivalent completions of* PA *such that whenever $\mathcal{M}$ and $\mathcal{N}$ are nonisomorphic countable, arithmetically saturated models and* $\mathrm{Th}(\mathcal{M})$, $\mathrm{Th}(\mathcal{N}) \in \mathcal{T}$, *then* $\mathrm{Aut}(\mathcal{M}) \ncong \mathrm{Aut}(\mathcal{N})$.

Proof Let $\mathcal{T} = \{T_0, T_1, T_2, \ldots\}$, where the T_i's are as in the Theorem. Let $\mathcal{M} \models T_i$ and $\mathcal{N} \models T_j$ be nonisomorphic countable, arithmetically saturated models. If $i \neq j$, then the Theorem implies that $\mathrm{Aut}(\mathcal{M}) \ncong \mathrm{Aut}(\mathcal{N})$. However, if $i = j$, then necessarily $\mathrm{SSy}(\mathcal{M}) \neq \mathrm{SSy}(\mathcal{N})$, so $\mathrm{Aut}(\mathcal{M}) \ncong \mathrm{Aut}(\mathcal{N})$ by [3]. □

The next corollary improves [8, Theorem 6.3].

Corollary 5 *There are infinitely many completions $T_0, T_1, T_2, \ldots$ of* PA *such that whenever $i < j < \omega$, then*

(1) $T_i \equiv_T T_j$,
(2) $\mathrm{Aut}(\mathcal{M}_i) \ncong \mathrm{Aut}(\mathcal{M}_j)$ *for all saturated $\mathcal{M}_i \models T_i$ and $\mathcal{M}_j \models T_j$.*

Proof Let $T_0, T_1, T_2, \ldots$ be as in the Theorem, so that (1) holds. For (2), suppose that $\mathcal{M}_i \models T_i$ and $\mathcal{M}_j \models T_j$ are saturated. Let $\mathcal{N}_i \prec \mathcal{M}_i$ and $\mathcal{N}_j \prec \mathcal{M}_j$ be minimal arithmetically saturated. Then, $\mathrm{Aut}(\mathcal{N}_i) \ncong \mathrm{Aut}(\mathcal{N}_j)$ by the Theorem. Then [8, Lemma 6.4] implies that $\mathrm{Aut}(\mathcal{M}_i) \ncong \mathrm{Aut}(\mathcal{M}_j)$. □

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