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# Second-Order Logic of Paradox

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**Abstract** The logic of paradox, *LP*, is a first-order, three-valued logic that has been advocated by Graham Priest as an appropriate way to represent the possibility of acceptable contradictory statements. Second-order *LP* is that logic augmented with quantification over predicates. As with classical second-order logic, there are different ways to give the semantic interpretation of sentences of the logic. The different ways give rise to different logical advantages and disadvantages, and we canvass several of these, concluding that it will be extremely difficult to appeal to second-order *LP* for the purposes that its proponents advocate, until some deep, intricate, and hitherto unarticulated metaphysical advances are made.

## 1 Background on the Logic of Paradox

Over the past three or four decades, but importantly in his [10], Graham Priest has investigated a variety of paradoxical topics—the semantic paradoxes are the ones that come first to a logician's mind, but he has also studied puzzles arising from vagueness, motion, Buddhist philosophy, and, in his 2014 book [14], metaphysical perplexities arising out of the notion of parthood and in relation to the question of the unity of the proposition—all from a dialetheist perspective: one that considers it possible that there are true contradictions, that is, that some propositions are both true and false or, equivalently, that some true propositions have true negations. As a basic logical framework for his investigations, he has adopted the system he calls *LP* (for logic of paradox), which is perhaps the simplest modification of classical logic to allow nontrivial contradictions. This is a three-valued logic, with values **True**, **False**, and **Both**. It has the usual propositional connectives $\neg$ (negation), $\wedge$ (conjunction), $\vee$ (disjunction), and universal ($\forall$) and existential ($\exists$) quantifiers. The truth functions

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$\supset$ and $\equiv$ are usually treated as defined connectives:

$$(\varphi \supset \psi) =_{df} (\neg\varphi \vee \psi), \qquad (\varphi \equiv \psi) =_{df} \big((\varphi \supset \psi) \wedge (\psi \supset \varphi)\big).$$

Truth values of compound formulas are derived from those of their subformulas by the familiar "truth tables" of Kleene's [9, Section 64] (strong) three-valued logic, but whereas for Kleene (thinking of the "middle value" as truth-value*less*ness) only the top value (**True**) is designated, for Priest the top two values are both designated. As Priest might say: a formula which is both true and false is, after all, true. Valid formulas, then, are those which take (in every model, on every assignment of truth values to the atomic formulas) either **True** or **Both** as values; valid inferences are those whose conclusions never take **False** as their value when all their premisses take either **True** or **Both**. Validities, therefore, include the law of excluded middle (Priest assumes that every proposition has *at least one* of the two usual truth values), the classical principles of double negation, and the classical De Morgan equivalences and their quantificational analogues. (As a result, every valid formula of classical logic is valid in *LP* as well.) What is missing are some of the traditional rules of inference, such as modus ponens and the principle of ex falso quodlibet (or explosion): if $A$ is both true and false, then both $A$ and $\neg A$ are true (as well as false), so they cannot imply some (purely) false $B$. Logics of this general nature had been developed earlier, including, in particular, the investigations of Asenjo–Tamburino [2] and Asenjo [1], whose logic is essentially just *LP*.

The model-theoretic semantics for a predicate logic of *LP* is, again, a natural generalization of that familiar from classical logic. Classically, a predicate is assigned an arbitrary subset of the domain of a model (for monadic predicates; a subset of a Cartesian power of the domain for polyadic) as its extension, and an atomic formula is true just in case the object (or tuple of objects) denoted by its singular term (or terms) belongs to the extension of its predicate. For *LP*, a predicate is assigned instead a pair of an extension and an antiextension, whose union must exhaust the domain (or Cartesian power of the domain, for relational predicates), but which are allowed to have a nonempty intersection. Atomic formulas have the value **True** if and only if the denotations of their individual terms are in the extension, but not the antiextension, of their predicate, the value **False** if they are in the antiextension but not the extension, and the value **Both** if they are in the intersection of the extension and antiextension. Universal (existential) quantifiers act like generalized conjunctions (disjunctions), taking the lowest (highest) value attained by any of their instances in the ordering **False** $<$ **Both** $<$ **True**.

## 2 Second-Order *LP*: The Standard Semantics

For some applications, Priest wants to use a second-order version of *LP*. But here there are choices to be made, since second-order logics can be interpreted in more than one way! In the "standard" semantics for classical second-order logic, the ($n$-adic) predicate variables are taken to range over all the permissible extensions for ($n$-adic) predicates, that is, over all subsets of the ($n$-fold Cartesian power of the) domain of individuals. In the analogous interpretation of second-order *LP* (which, for short, we will call the *standard interpretation*), the predicate variables are again taken to range over all the permissible interpretations of predicates: over, in other words, all the appropriate $\langle$extension, antiextension$\rangle$ pairs. Priest [12, pp. 338–339] suggests this interpretation in his expository article. Second-order

*LP*, on this standard interpretation, has some surprising properties. One, which Priest notes, is that there will be semantically valid sentences with semantically valid negations: $\forall X \forall x(Xx \vee \neg Xx)$, the quantified law of excluded middle, is valid, but, since the second-order domain will contain "over-defined" elements ($\langle$extension, antiextension$\rangle$ pairs with nonempty intersections), the statement that there are such properties, $\exists X \exists x(Xx \wedge \neg Xx)$, will also be valid, and by De Morgan and quantifier negation this statement is equivalent to $\neg\forall X \forall x(Xx \vee \neg Xx)$. Such a sentence, then, can be taken as the definiens for a propositional constant, **b**, which will have the value **Both** in every model. Note for future reference that replacing an atomic subformula of any formula with **b**, though it may change the value of the whole formula from **True** (or **False**) to **Both**, will never change its value from a designated one to the nondesignated **False**.

First-order *LP*, like first-order classical logic, has complete proof procedures: with appropriate changes of rules, a system for *LP* can be given in the style of any textbook version of natural deduction. (For example, the double negation elimination and introduction rules, the negative introduction and elimination rules for $\wedge$, $\vee$, $\forall$, and $\exists$ (as in Fitch [7]), a rule allowing the inter substitution of $\supset$ with its definiens in *LP*, and the usual positive rules yield complete systems when supplemented by an axiom scheme of excluded middle.) Classical second-order logic (on the standard interpretation) does not: its class of valid formulas is not recursively enumerable and, indeed, is of a very high recursion-theoretic degree. In contrast, we have the following result.

**Theorem 1** *The class of valid second-order LP formulas on the standard interpretation is recursively enumerable.*

**Proof** Without loss of generality, we may assume that all formulas are in prenex form: all quantifiers are initial. (The same prenexing equivalences hold in *LP* and classical logic.) It can be shown that, in the quantifier prefix, all the second-order quantifiers may precede all the first-order ones. ($\forall x \forall X$ and $\exists x \exists X$ are equivalent, respectively, to $\forall X \forall x$ and $\exists X \exists x$. By appealing to the axiom of choice in the metatheory we can show that a formula with $\forall x \exists X$, with $X$ an $n$-adic predicate variable, in its prefix is equivalent to one having instead $\exists Y \forall x$, with $Y$ an $(n + 1)$-adic variable, and similarly for $\exists x \forall X$ and $\forall Y \exists x$. This equivalence, for classical logic, is regularly appealed to in the study of definability hierarchies.)

The range of a predicate variable contains the maximally overdefined $\langle$extension, antiextension$\rangle$ pair (the one in which each of the extension and antiextension is identical to the whole first-order domain or appropriate Cartesian power thereof). Any atomic formula with a predicate assigned this pair will have the truth value **Both**. Second-order existential quantifications are therefore trivial: any prenex formula is equivalent (in the sense that in any model one will have a designated value if and only if the other does) to that obtained by deleting its existential second-order quantifiers and replacing each atomic formula containing a variable bound by one of the deleted quantifiers with **b**. Thus, the only second-order formulas we have to consider are those starting with a block of second-order universal quantifiers followed by a first-order formula (some of whose atomic subformulas may have variables bound to the initial quantifiers as predicates). But such a formula is valid if and only if the first-order sentence obtained by dropping the initial second-order universal quantifiers and reconstruing the variables bound to them as predicate constants is valid.

Thus, the problem of axiomatizing the valid formulas of second-order *LP* reduces to that of the first-order logic. □

This completeness result can, however, be seen as an undesirable feature of second-order *LP*. One of the things second-order logic is valued for is the analysis it provides of mathematical theories and structures. Thus, Frege and Dedekind showed that with each sentence of arithmetic we can associate a formula of second-order (classical) logic such that the sentence is true (as a description of the natural numbers) just in case the formula is valid (on the standard interpretation). The very fact that its valid formulas are recursively enumerable shows that no such correlation can be found for (standard-semantics) second-order *LP*.

Another difference between second-order *LP* and classical second-order logic is that classical second-order logic allows an axiom of infinity (a sentence, with no nonlogical vocabulary, true in all and only models with an infinite domain of individuals). However, we have the following result.

**Theorem 2** *In second-order LP, any (purely logical) sentence that is satisfiable at all (i.e., takes a designated value in some model) is satisfiable in a model with just a single individual.*

**Proof** The proof of this is easy enough once we note that the "delete existential second-order quantifiers" bit from Theorem 1 does not just yield a formula equivalent to the original in the sense of being satisfiable if and only if, but something a bit stronger: for every cardinal number, if the original is satisfiable in a model with that many individuals, the modified formula is also satisfiable with the same individuals. But now: a standard second-order *LP* model with just one individual has just three items in the range of any predicate variable: one that makes all atomic formulas with that variable **True**, one that makes them all **False**, and one that makes them all **Both**. But things that do this are present in *all* models, namely, the three ⟨extension, antiextension⟩ pairs, where $\mathcal{D}$ denotes the entire domain: $\langle \mathcal{D}, \emptyset \rangle$, $\langle \emptyset, \mathcal{D} \rangle$, and $\langle \mathcal{D}, \mathcal{D} \rangle$. So any sentence starting with second-order universal quantifiers if it is true (= **True** or **Both**) in some model with lots of individuals will a fortiori be true in the model with only one! □

This problem is related to a well-known expressive weakness of first-order *LP*, its lack of a full-service conditional connective. In *LP*, modus ponens fails for the conditional connective $(A \supset B)$ (as equivalent to $(\neg A \vee B)$ or $\neg(A \wedge \neg B)$). (If $A$ has the value **Both** and $B$ **False**, then $A$ and $(A \supset B)$ both have designated values but $B$ is undesignated.) Because of this, there is no obvious way of formulating the principle of mathematical induction in second-order *LP*. We would like to be able to infer $\forall n X n$ from $X0$ and $\forall n(Xn \supset X(n+1))$, but this inference is not in general valid. (Let $X$ be assigned $\langle \{0\}, \mathbb{N} \rangle$. Then $X0$ and $\forall n(Xn \supset X(n+1))$ will both have the value **Both**, the latter because $(X0 \supset X1)$ has **Both** and all other instances **True**, but $\forall x X x$ has the value **False** since all but the first of its instances have **False**.)

Another consequence of *LP*'s lack of a usable conditional is that it is not possible to encode the valid inferences of second-order *LP* in valid formulas. In classical logic, the inference from $A$ to $B$ is valid if and only if the formula $(A \supset B)$ is valid, but this does not hold for *LP*.

**Theorem 3** *Valid inference in second-order LP is not recursively enumerable.*

**Proof** Consider the conjunction, $P$, of Robinson's seven axioms for inductionless arithmetic with the usual second-order axiom of induction,

$$\forall X\big(X0 \wedge \forall n\big(Xn \supset X(n+1)\big)\big) \supset \forall n(Xn).$$

This conjunction is true (though also, because of its universally quantified second-order conjunct, also false: it takes the value **Both**) in second-order *LP* models with the genuine natural numbers as individuals and in many, though not all, of the finite inconsistent models described in Priest [11]: those in which the number series, perhaps after a linear initial segment, forms a loop. Robinson's axioms are true (i.e., have either the value **True** or the value **Both**) in all of these models. Now consider the instance of the induction axiom given by assigning some particular ⟨extension, antiextension⟩ pair to the variable $X$. The instance can only have the value **False** if its consequent does, that is, if some number falls in the antiextension but not the extension. If 0 falls in the antiextension, then the antecedent of the instance will have either the value **False** (if 0 is not also in the extension) or **Both** (if it is), and in either case the whole instance will have one of the designated values. So assume that $X0$ is true. Then in going along the number series from 0 to the $n$ for which $Xn$ takes the value **False**, we must at some step go from a number in the extension to one not in the extension. But this is enough to give the second conjunct of the antecedent one of the values **False** or **Both**, and again this will give the whole antecedent one of these values and so give the instance as a whole one of the values **True** or **Both**. (The induction axiom will, however, take the value false in models with nonstandard numbers, whether arranged in an infinite series as in classical nonstandard models of arithmetic or in a circle unconnected to 0 as in the finite models from [11].)

The only sentences of the language of first-order arithmetic taking designated values in *all* models of $P$ (since the model based on the genuine natural numbers is one of them) are those true of the genuine natural numbers. Thus, where $S$ is a sentence of first-order arithmetic, the inference from $P$ to $S$ is a semantically valid inference of second-order *LP* just in the case in which $S$ is true. Since, by Gödel's first incompleteness theorem the truths of first-order arithmetic are not recursively enumerable, it follows that the valid inferences of second-order *LP*, unlike its valid formulas, are not recursively enumerable. □

## 3 Identity

Our Theorems 1 and 2 are consequences of the limited expressive power of second-order *LP*. There are other consequences. One nonmathematical reason for philosophical interest in second-order logic, to which Priest [14] appeals, is the possibility of defining identity by Leibniz's law.

**Definition 1 (Identity)** $a = b \quad =_{df} \quad \forall X(Xa \equiv Xb)$.

Using this definition and the fact that $\equiv$ is reflexive and symmetric in *LP*, Priest concludes that identity will likewise be reflexive and symmetric, that is, that the following are semantically valid:

$$\vDash a = a, \qquad \vDash a = b \supset b = a.$$

However, Priest notes that if $A$ is **True** and $B$ is **Both**, then $A \equiv B$ is **Both** in *LP*; and if in addition $C$ is **False**, then $B \equiv C$ is **Both** in *LP*. However, $A \equiv C$ will be **False** in *LP*. Or in other words, $\equiv$ is not an equivalence relation in *LP*, because it is not transitive. From this Priest concludes that identity is not transitive, using Definition 1,

$$\not\models (a = b \wedge b = c) \supset (a = c).$$

Since the transitivity of identity is derivable from its reflexivity and symmetry by use of the rule of substitution of identicals, this rule will also not be valid when identity is defined in this way. Much of Priest's view concerning the various manifestations of the "problem of unity and oneness" in [14] derives from this foundation.

But the definition of identity in Definition 1 is dubious if second-order *LP* has its standard interpretation. One item in the range of (monadic) predicate variables is the totally overdefined pair $\langle \mathcal{D}, \mathcal{D} \rangle$. When this pair is assigned to the predicate variable $X$, $Xa$ and also $Xa \supset Xa$ will take the value **Both**, for any $a$ in the first-order domain. Thus, when identity is defined as above, $a = a$ will have the value **Both** for every $a$. It is part of Priest's philosophical program that some paradoxical objects may be analyzed as being both identical to and distinct from themselves, but he has always hoped that consistent accounts could be given of "ordinary" things: self-identity should not be universally dialetheic! (Replacing $\supset$ in the above definition of identity with the new conditional $\Rightarrow$ discussed below makes no difference: $a = a$ will still always have the value **Both**.) It would seem preferable, in systems formulated in second-order *LP*, to include identity of individuals as an additional primitive rather than defining it as can be done in second-order classical logic. However, given the standard interpretation of second-order quantifiers, Definition 1 succeeds, semantically, in defining identity of individuals. (Proof: any two distinct objects in the domain are separated by a classical property—one whose anti-extension is exactly the complement of its extension—and this, since all the classical properties are included in the second-order domain, suffices to give $\forall X(Xa \equiv Xb)$ the value **False** for distinct $a$ and $b$. So the defined identity statement takes a designated value—**Both**, as argued above—precisely in the case of genuinely identical $a$ and $b$.) Hence, although the rule of substitution is not *LP derivable* from Definition 1, it is semantically valid and could be added as an additional postulate.

## 4 Second-Order *LP*: General Semantics

In discussions of classical second-order logic, the standard interpretation is often (though not always) the most interesting. The considerations given above suggest that standardly interpreted second-order *LP* is less interesting and less useful. It would seem that Graham Priest has also come to this view: in discussing second-order *LP* in [14, pp. 28–29], he recommends instead what is called a "general" interpretation. For classical second-order logic this is one on which the predicate variables are taken to range over nonempty subsets of the powerset of the individual domain (or of the powersets of its Cartesian powers, for polyadic predicates), but not necessarily the full powerset. For second-order *LP*, correspondingly, it would be one on which they range over nonempty sets of ⟨extension, antiextension⟩ pairs.

"General" semantics is thus a cover term for a very diverse range of interpretations, validating a diverse range of logics. Further specification is needed if we are to state any interesting conclusions, and depending on the goals sought, logicians have specified different kinds of "general" models. For example, in Henkin [8], which introduced general interpretations of classical higher-order logic, there were specified closure conditions that have to be imposed on the ranges of higher-order variables in order to validate comprehension principles. We would like to ask whether the results of Section 2 carry over to the logic given by a semantics acceptable from the point of view of Priest's philosophical project. Priest, however, leaves this as a topic for future work, suggesting that a full characterization of the semantics of second-order logic ought to depend on a "robust theory of properties." In this he is making contact with the work of other Australian metaphysicians like David Armstrong and John Bigelow.

Until the metaphysicians have provided an appropriate theory of universals we can make only tentative comments. The first part of the proof of Theorem 1—the equivalence of every sentence of second-order *LP* to one whose second-order quantifiers are all initial—assumes that the logic validates choice principles, so it may fail on many proposals. The second part of the proof—the eliminability of second-order existential quantifiers—and the results on identity (Section 3) require only the presence of a few special items, the everywhere-**Both** predicates of each adicity, in the second-order domains. They are likely to hold, therefore, on many possible versions of the general semantics. In particular, if the models validate the principle of comprehension, these items seem forced on us. (Since *LP* does not have a usable conditional connective, comprehension cannot be stated as an axiom scheme. It can, however, be embodied in rules of substitution for second-order variables, as in Church [6, Chapter 5].) Comprehension in effect postulates that every formula of the language defines a property in the appropriate second-order domain. Now suppose that at least one sentence (closed formula) takes the value **Both**—after all, a dialetheist would hardly be interested in a logic where this is not the case! Conjoining this sentence with an arbitrary valid formula with $n$ free individual variables yields a formula defining an $n$-ary relation taking the value **Both** for every $n$-tuple individual. This may not be a conclusive objection, however, since some proponents of sparse theories of universals have held that the domain of properties is not closed under conjunction, and the second-order logics corresponding to their theories would therefore not validate comprehension.

## 5 Enriching Second-Order *LP* with New Operators

Many logicians, including many interested in the foundations of mathematics, are interested in second-order logic specifically *because* of the expressive power that is lost when we move from standard to general semantics. The categorical axiomatizations it allows of, for example, the arithmetic of natural numbers do not allow of complete proof procedures, but they do provide definitions of interesting mathematical structures, and first-order axiomatizations are seen as approximations to them. So perhaps there is some interest in trying to add second-order quantification to *LP* in a way which allows us to keep the full expressive power of standardly interpreted second-order logic and avoid our anomalous Theorem 1. One possibility is to enrich the language of *LP* with a new conditional operator. Leave the interpretations of $\neg$, $\wedge$, and $\vee$ and of the quantifiers as in *LP*, but add a new operator $\Rightarrow$ with the interpretation that $A \Rightarrow B$ takes the value **True** when $A$ has the value **False**, and otherwise takes the same value as $B$ (see Avron [3]). (This is a genuine addition to the language of *LP*: it is easy to see that modus ponens is valid for this conditional: if $A$ and $A \Rightarrow B$ both have designated values, so will $B$, but Beall, Forster, and Seligman [5] show that there is no operator that is definable *within LP* that has such a property.) Note that the interpretation of $\Rightarrow$ differs from the defined $\supset$ in only one case: when $A$ has the value **Both** and $B$ has the value **False**, $A \supset B$ takes the designated value **Both** but $A \Rightarrow B$ takes the undesignated **False**. Call *LP* augmented with this new connective *LPA*. Tedder [16] has studied first-order *LPA*: it has a natural and efficient proof procedure and can serve as the logical framework for a number of interesting inconsistent axiomatic extensions of first-order arithmetic: *LP* itself, not having a full-service conditional operator, is not suitable for formulating axioms from which deductions are to be made. Tedder's [16] interest is proof-theoretic, but we can look at it semantically. Second-order *LPA* is expressively powerful, as we shall see, but in ways that may make it inappropriate for use in the context of a dialetheist philosophy. For a start, the logic of $\Rightarrow$ in *LPA* is precisely identical to that of $\supset$ in classical logic: a formula (inference) in which $\Rightarrow$ is the only connective occurring is valid in *LPA* if and only if the corresponding formula (inference) with $\supset$ everywhere replacing $\Rightarrow$ is classically valid. (Proof: Squint at the truth table for $\Rightarrow$ until the cells for the two designated values **True** and **Both** blur together, and it will look just like the classical truth table for $\supset$.) As remarked, this means that $\Rightarrow$ supports modus ponens: it is a "detachable" conditional. It also means that $\Rightarrow$ supports contraction:

$$p \Rightarrow (p \Rightarrow q) \vdash_{LPA} (p \Rightarrow q).$$

Although a conditional with these properties might be suitable for certain dialetheic applications, there will be many for which it is inappropriate. One of the main motivating goals of dialetheism has been that of recovering the naturalness of naïve set theory and a similarly naïve theory of truth and satisfaction. This will not be attainable with *LPA* because $\Rightarrow$ will allow the derivation of Curry's paradox for set theory and the similar Löb's paradox in the theory of truth.

Second-order *LPA* has even more (disturbing?) classical-like features and can, in fact, be seen as including full classical logic. By using second-order quantifiers, we can define propositional constants. We have already seen the constant **b**, which takes the value **Both** in any model. Since the domain over which monadic predicate variables range contains the pair of the null set with the whole first-order domain, that is, $\langle \emptyset, \mathcal{D} \rangle$, and since if $X$ is assigned this pair, any formula $Xa$ will have the value **False**, it follows that the sentence $\forall x \forall X(Xx)$ will have the value **False** in every model: unlike any formula of first-order *LP*, it is unsatisfiable. It may thus be taken as the *definiens* of a propositional falsum constant, **f**:

$$\mathbf{f} =_{df} \forall x \forall X(Xx).$$

Dually, since the domain over which monadic predicate variables range contains the pair of the whole first-order domain with the null set, that is, $\langle \mathcal{D}, \emptyset \rangle$, and since if $X$ is assigned this pair, any formula $Xa$ will have the value **True**, it follows that the sentence $\exists x \exists X(Xx)$ will have the value true in every model and can be taken as the definiens of a propositional verum constant **t**.

Not only is **t** valid, it, unlike any formula of first-order *LP*, takes the value **True** in every model. (Note that the full language of second-order logic is not required for these definitions. If we just add propositional quantifiers to the language of propositional *LP* and so move to the language of what Church [6, Section 28] calls the *extended propositional calculus*, we can define **f**, **b**, and **t** by $\forall p(p)$, $\exists p(p \wedge \neg p)$, and $\exists p(p)$, respectively.)

Since **f** never takes a designated value, any formula whatsoever may validly be inferred from it. But the logic of $\wedge$, $\vee$, and $\Rightarrow$ is exactly the same as the *classical* logic of $\wedge$, $\vee$, and $\supset$. (If one thinks of **True** and **Both** as subspecies of the classical truth value true, then the truth tables for $\wedge$, $\vee$, and $\Rightarrow$ can be seen as simply complicated versions of the classical tables for $\wedge$, $\vee$, and $\supset$.) Defining a new negation operator by

$$\mathbf{N}A =_{df} (A \Rightarrow \mathbf{f}),$$

therefore, we obtain full classical logic with its principle of explosion: $A$ and $\mathbf{N}A$ together validly imply $B$, where $B$ can be any formula whatsoever. (Before using this logic, then, a defender of a dialetheist treatment of semantic or metaphysical paradoxes would have to explain why $\neg A$, rather than $\mathbf{N}A$, is the "real" negation of $A$.) $\mathbf{N}p$ takes the value **True** if and only if $p$ has the value **False,** and it takes the value **False** otherwise. In particular, $\mathbf{N}p$ takes the value **False** when $p$ has the value **Both**. $(\mathbf{N}p \vee \mathbf{N}\neg p)$ is therefore **True** if $p$ has one of the two classical truth values (and **False** otherwise): so we can abbreviate

$$\mathbf{C}p =_{df} (\mathbf{N}p \vee \mathbf{N}\neg p)$$

as a "classicality" operator, meaning that $p$ has a classical value. With this we can faithfully interpret classical logic in our logic. For the propositional case, the conditional $((\mathbf{C}p_1 \wedge \mathbf{C}p_2 \wedge \cdots \wedge \mathbf{C}p_n) \Rightarrow \varphi)$ (where the antecedent asserts the classicality of all the atoms that occur in the consequent $\varphi$) is *LPA*-with-**f** valid if and only if the formula $\varphi$ is classically valid. For the case of first-order logic, the antecedent must also contain conjuncts asserting the classicality of the predicates occurring in $\phi$: $\forall x \mathbf{C} F x$, and similarly for polyadic predicates. For the case of second-order logic, we have to modify the consequent by restricting its second-order quantifiers to classical properties and relations: by replacing each subformula in it of the form

$$\forall X(\psi) \quad \text{with} \quad \forall X\big((\forall x \mathbf{C} X x) \Rightarrow \psi\big)$$

(and similarly for polyadic $X$) and replacing each subformula of the form

$$\exists X(\psi) \quad \text{with} \quad \exists X\big((\forall x \mathbf{C} X x) \wedge \psi\big).$$

This embedding of full classical second-order logic in second-order *LPA* gives us, in striking contrast to our Theorem 1, the following result.

**Theorem 4** *The class of valid second-order LPA formulas on the standard interpretation is not recursively enumerable and is of the same complexity as the class of valid second-order classical formulas.*

Other writers have also explored the possibilities of supplementing *LP* with a conditional operator (e.g., Beall [4] and Priest [13]). Our trivialization results for *LP* with $\Rightarrow$ can be seen as illustrating just how difficult the task they have undertaken really is: a simple and obvious conditional operator is not adequate for their purposes. Perhaps some nonsimple and nonobvious conditional operator will prove to be suitable, but the choices available for (truth-functional) connectives in a three-valued setting that are sufficiently like a conditional are quite limited: besides the *LP* conditional, there seems to be $\Rightarrow$, the RM3 conditional $\rightarrow_{\text{RM3}}$, and a Łukasiewicz conditional $\rightarrow_{\text{Ł}}$.[1] Although we have not discussed the RM3- and Ł-conditionals, it is not so clear to us that they will fare any better than $\Rightarrow$ has fared; it can be seen from the truth tables in Table 1 that the conditionals all share certain features that may lead all of them to manifesting these difficulties. But that is a topic for future discussion.

**Table 1** Conditionals from *LP*, *LPA*, RM3, and Ł3.

| $\varphi$ | $\psi$ | $(\varphi \rightarrow_{LP} \psi)$ | $(\varphi \Rightarrow \psi)$ | $(\varphi \rightarrow_{\mathrm{RM3}} \psi)$ | $(\varphi \rightarrow_{Ł} \psi)$ |
|---|---|---|---|---|---|
| $T$ | $T$ | $T$ | $T$ | $T$ | $T$ |
| $T$ | $B$ | $B$ | $B$ | $F$ | $B$ |
| $T$ | $F$ | $F$ | $F$ | $F$ | $F$ |
| $B$ | $T$ | $T$ | $T$ | $T$ | $T$ |
| $B$ | $B$ | $B$ | $B$ | $B$ | $T$ |
| $B$ | $F$ | $B$ | $F$ | $F$ | $B$ |
| $F$ | $T$ | $T$ | $T$ | $T$ | $T$ |
| $F$ | $B$ | $T$ | $T$ | $T$ | $T$ |
| $F$ | $F$ | $T$ | $T$ | $T$ | $T$ |

## 6 *LPA*+: A Functionally Complete Three-Valued Logic

A simple observation may highlight the way in which the additional expressive power of second-order (or propositional) quantification can change the character of the underlying propositional logic. Neither *LP* nor *LPA* is a functionally complete three-valued logic: on an assignment giving classical values (**True** or **False**) to every sentence letter, every formula of either propositional *LP* or propositional *LPA* will receive a classical value; thus, no connective producing a formula with the value **Both** when applied to arguments with classical values is definable in either. When the constants **f**, **b**, and **t**[2] (which we have seen are definable by means of higher-order quantifiers) are added as primitives to first-order *LPA*, however, we get a functionally complete system. We call this system *LPA*+. The new negation, **N**, is what is called, in the standard terminology of multivalued logic (see Rosser–Turquette [15]), a $J$*-function* for the value **False**: an operator which, when applied to a formula with that value, yields one with the value **True** and, when applied to anything else, yields one with the value **False**. More generally, a $J$-operator of value **n** in many-valued logic is an operator which, when applied to a formula with the truth value **n**, yields the top, truest, truth value and, when applied to anything else, yields the bottom, falsest, truth value. Thus,

$$[\![J_n(\varphi)]\!] = \begin{cases} \textbf{True} & \text{if } [\![\varphi]\!] = \mathbf{n}, \\ \textbf{False} & \text{otherwise.} \end{cases}$$

As just mentioned, **N** is a $J$-function for the value **False**; $\mathbf{N}\neg\varphi$ has the value **True** if $\varphi$ has the value **True** and the value **False** otherwise, and so it can be taken as a $J$-function for **True**. The only way $p$ and $\neg p$ can both have designated values is if $p$ has the value **Both**, so $(\mathbf{NN}p \wedge \mathbf{NN}\neg p)$ gives a $J$-function for **Both**. So we have

$$[\![J_{\mathbf{False}}(\varphi)]\!] = \mathbf{N}\varphi, \qquad [\![J_{\mathbf{Both}}(\varphi)]\!] = (\mathbf{NN}\varphi \wedge \mathbf{NN}\neg\varphi), \qquad [\![J_{\mathbf{True}}(\varphi)]\!] = \mathbf{N}\neg\varphi.$$

Moving to a more general notion of $J$-operator, we note that $\mathbf{NN}p$ is **True** if and only if $p$ has one of the two designated values (and is **False** otherwise):

$$[\![J_{\mathbf{Designated}}(\varphi)]\!] = \mathbf{C}\varphi = \mathbf{NN}\varphi, \qquad [\![J_{\mathbf{Classical}}(\varphi)]\!] = (\mathbf{N}\varphi \vee \mathbf{N}\neg\varphi).$$

**Theorem 5** *LPA*+ *is a functionally complete three-valued logic.*

**Proof** The proof is a straightforward generalization of the well-known two-valued method of constructing a formula in disjunctive normal form that exhibits some arbitrary truth table. In the many-valued case, we construct the formulas that each describe a row of the many-valued truth table as follows. For each of the $n$ propositional variables ($\varphi_1, \varphi_2, \ldots, \varphi_n$) in the formula, every row will have assigned one of the values **True**, **Both,** or **False** to that variable. Whatever the value of that assignment is, it is described by the $J$-operator of that value as applied to the relevant $\varphi_i$. These $n$ $J$-formulas are conjoined, and the result is conjoined with whatever constant corresponds to the value of the formula at that row. So, each row will be described by a formula of the form

$$(J_a\varphi_1 \wedge J_b\varphi_2 \wedge \cdots \wedge J_i\varphi_n \wedge \mathbf{k}),$$

where $a, b, \ldots, i$ are the truth values (**t**, **b**, or **f**) assigned to the propositional variables $\varphi_1, \varphi_2, \ldots, \varphi_n$ in that row and **k** is the value (one of **t**, **b**, or **f**) that the formula takes with those values assigned to the atomic propositions. We now disjoin all these sentences that describe a row, and we have a (generalized) disjunctive normal form sentence that has the arbitrarily chosen many-valued truth table. □

## Notes

1. Priest [13] has sought to supplement *LP* with a conditional $\rightarrow$ that satisfies identity ($\varphi \rightarrow \varphi$) and modus ponens. However, because of Curry's paradox, this $\rightarrow$ could not satisfy contraction or any of a variety of generalized contraction principles (see [4] also for discussion). But it seems that we would then not have a *truth-functional* conditional connective, as initially imagined for *LP*.

2. We do not actually need **t**, of course, since that is definable by $\neg$**f**.

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