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Dunn–Priest Quotients of Many-Valued Structures

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Abstract J. Michael Dunn's *Theorem in 3-Valued Model Theory* and Graham Priest's *Collapsing Lemma* provide the means of constructing first-order, three-valued structures from classical models while preserving some control over the theories of the ensuing models. The present article introduces a general construction that we call a *Dunn–Priest quotient*, providing a more general means of constructing models for arbitrary many-valued, first-order logical systems from models of any second system. This technique not only counts Dunn's and Priest's techniques as special cases, but also provides a generalized Collapsing Lemma for Priest's more recent plurivalent semantics in general. We examine when and how much control may be exerted over the resulting theories in particular cases. Finally, we expand the utility of the construction by showing that taking Dunn–Priest quotients of a family of structures commutes with taking an ultraproduct of that family, increasing the versatility of the tool.

1 Introduction

J. Michael Dunn and Graham Priest have each introduced related methods of constructing 3-valued, nonclassical structures from classical, bivalent structures. Dunn's technique (see [6]) introduces a method for generating models appropriate to Kleene's 3-valued K_3 introduced in [10], while Priest's (see [13]) introduces a method for generating models appropriate to his own LP.

Besides merely providing a method of constructing new K_3- or LP-models, the techniques also afford a level of control over the theories of the ensuing structures. Dunn's technique ensures that for no formula φ true in the original structure will its negation be true in the resulting K_3-model. This control over the theory underwrites Dunn's corollaries in [6] such as the nontriviality of relevant arithmetic. Priest, similarly, presents a means to ensure that the theory of the initial structure is a subset of

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the theory of the ensuing LP-model. Priest [15], [16] often made use of this technique in describing inconsistent models of true arithmetic.

These techniques, however, are limited to the cases in which one wishes to construct either a K_3- or LP-model from a classical one. The utility of these methods can be enhanced by having a more general method available by which one can, for any arbitrary many-valued logics L_0 and L_1, construct L_1-models from L_0-models with some control over the resulting L_1-model's theory. The aim of this article is to provide precisely this tool.

Where $\mathfrak{M}$ is an L_0-model, we introduce a general method of constructing a corresponding L_1-model $\mathfrak{M}^{\sim}$ by taking a quotient of $\mathfrak{M}$ modulo some congruence relation $\sim$ and interpreting it with respect to a function h. This latter structure will be called a *Dunn–Priest quotient along h* of $\mathfrak{M}$ modulo $\sim$. In particular, the selection of h can ensure the desired control over the theory of $\mathfrak{M}^{\sim}$ by, for example, ensuring that the formulae in the theory of $\mathfrak{M}$ remain in the theory of $\mathfrak{M}^{\sim}$.

This article is organized as follows. First we will outline the essential elements of many-valued model theory and introduce a general account of Dunn–Priest quotients. Then we will examine the correspondence between certain conditions on these quotients and the preservation theorems which hold for them. The Dunn and Priest results will be put into context and the proofs rehearsed against this general backdrop. Further applications will be introduced, for example, an instance for constructing paraconsistent models from Dmitri Bochvar's logic of nonsense and the general case of plurivalent quantification in the sense of Priest [17]. The final section will be devoted to showing an interesting relationship between Dunn–Priest quotients and the taking of ultraproducts of families of models, namely, that Dunn–Priest quotients and ultraproduct quotients commute.

2 Definitions and Preliminaries

To begin, we will first review some of the notational conventions observed in this article before offering a précis of many-valued model theory.

The notation peculiar to this article is as follows.

Definition 2.1 The *positive power set* of a set X, denoted by $\wp^+(X)$, is the set of all nonempty subsets of X, that is, $\wp(X) \setminus \{\varnothing\}$.

Definition 2.2 We denote the image of a set X under a function f by $f[X]$.

One slightly esoteric convention that will be employed in this article is the following.

Definition 2.3 The Hilbert epsilon notation $\epsilon x.\Phi(x)$ denotes *some arbitrary object* of which Φ is true.

This notation enables us to succinctly describe a choice function, for example, with respect to a family of sets $\{X_i \mid i \in I\}$. A choice function mapping each set $X_i \in \mathbf{X}$ to some element of X_i can be described as $f : X_i \mapsto \epsilon x.x \in X_i$.

Definition 2.4 For a family of sets $\mathbf{X} = \{X_i \mid i \in I\}$ indexed by a set I, let $\mathscr{S}_I(\mathbf{X})$ denote the set of choice functions, that is, the set $\{g \mid g : X_i \mapsto \epsilon x.x \in X_i\}$.

Definition 2.5 The *threaded image* $\mathsf{Thread}(\mathbf{X})$ of a family of sets $\mathbf{X} = \{X_i \mid i \in I\}$ is the set $\{g[\mathbf{X}] \mid g \in \mathscr{S}_I(\mathbf{X})\}$, that is, all sets choosing precisely one element from each set X_i.

Finally, we will use definite description operators as follows.

Definition 2.6 The Russell definite description operator $\iota x.\Phi(x)$ denotes the unique object of which Φ is true.

Importantly, when $\iota x.\Phi(x)$ fails to denote an object, either because $\Phi(x)$ is true of no object or because it is true of multiple objects, any formula $\Psi(\iota x.\Phi(x))$ is false.

It is necessary to describe a few essential definitions and observations for many-valued model theory in general before proceeding. First, we pin down what sorts of semantical presentations of logic with which we are concerned.

Definition 2.7 A *logical system* is an ordered 4-tuple $\langle \mathscr{V}, \mathscr{D}, \mathbf{S}, I \rangle$, where

- $\mathscr{V}$ is a nonempty set of *truth values*,
- $\mathscr{D} \subseteq \mathscr{V}$ is a nonempty set of *designated values*,
- $\mathbf{S}$ is a set of n-ary connective symbols $\odot$ and quantifier symbols Q,
- I maps $\mathbf{S}$ to interpretations where
  - I maps each n-ary $\odot$ to a function $f^{\odot} : \mathscr{V}^n \to \mathscr{V}$;
  - I maps each quantifier Q to a function $f^{\mathsf{Q}} : \wp^+(\mathscr{V}) \to \mathscr{V}$.

Note that the present interpretation of quantifier conforms to Walter Carnielli's *distribution quantifiers* as introduced in [4], in which quantifiers are functions taking "distributions" of values as arguments. More general definitions of quantifier are possible, but Carnielli's approach is elegant while remaining very general. We will thus settle for this account of quantifiers.

Some syntactical preliminaries are in order in defining a language.

Definition 2.8 A *signature* σ is a 3-tuple $\langle \mathbf{C}, \mathbf{F}, \mathbf{R} \rangle$, where

- $\mathbf{C}$ is a set of *constant symbols*,
- $\mathbf{F}$ is a set of n-ary *function symbols*,
- $\mathbf{R}$ is a set of m-ary *relation symbols*.

Definition 2.9 Let $\mathbf{Var}$ be a set of variable symbols. The set $\mathbf{Tm}_\sigma$ of *terms* of σ is the smallest set such that

- $\mathbf{Var} \subseteq \mathbf{Tm}_\sigma$;
- $\mathbf{C} \subseteq \mathbf{Tm}_\sigma$;
- if $f \in \mathbf{F}$ is n-ary and $t_0, \ldots, t_{n-1} \in \mathbf{Tm}_\sigma$, then $f(t_0, \ldots, t_{n-1})$.

Definition 2.10 The set $\mathbf{Var}(t)$ of variables in a term t is recursively defined:

- $\mathbf{Var}(t) = \{t\}$ when $t \in \mathbf{Var}$;
- $\mathbf{Var}(t) = \varnothing$ when $t \in \mathbf{C}$;
- $\mathbf{Var}(f(t_0, \ldots, t_{n-1})) = \mathbf{Var}[\{t_0, \ldots, t_{n-1}\}]$.

Definition 2.11 The set $\mathbf{Tm}_\sigma^{Cl}$ of *closed terms* of σ is the set of terms $t \in \mathbf{Tm}_\sigma$ such that $\mathbf{Var}(t) = \varnothing$.

Definition 2.12 The language $\mathscr{L}_\sigma$ is the smallest set such that

- if $R \in \mathbf{R}$ is n-ary and $t_0, \ldots, t_{n-1} \in \mathbf{Tm}_\sigma$, then $R(t_0, \ldots, t_{n-1}) \in \mathscr{L}_\sigma$;
- if $\odot \in \mathbf{S}$ is n-ary and $\varphi_0, \ldots, \varphi_{n-1} \in \mathscr{L}_\sigma$, then $\odot(\varphi_0, \ldots, \varphi_{n-1}) \in \mathscr{L}_\sigma$;
- if $\mathsf{Q} \in \mathbf{S}$ is a quantifier and $x \in \mathbf{Var}$, then $\mathsf{Q}x\varphi \in \mathscr{L}_\sigma$.

Definition 2.13 For $\varphi \in \mathscr{L}_\sigma$ and $t, t' \in \mathbf{Tm}_\sigma$, the *uniform substitution* $\varphi(t'/t)$ is the formula arrived at by substituting all instances of t' in φ with t.

Definition 2.14 For $\varphi \in \mathscr{L}_\sigma$, $\vec{t} \in \mathbf{Tm}_\sigma^{<\omega}$ a string of each instance of closed terms in φ—possibly with repetitions—in the order in which they appear in φ, and $\vec{t}'$ a string of closed terms of equal arity to $\vec{t}$, the *scattered substitution* $\varphi(\vec{t}' /\!\!/ \vec{t})$ is the formula arrived at by substituting the appearance of the term t_i with t_i' in the order in which each t_i appears.

Definition 2.15 The set $\mathbf{Var}(\varphi)$ of *open variables* of a formula φ is defined recursively as follows.

- For a formula $R(t_0, \ldots, t_{n-1})$, $\mathbf{Var}[\{t_0, \ldots, t_{n-1}\}]$.
- For a formula $\odot(\varphi_0, \ldots, \varphi_{n-1})$, $\mathbf{Var}(\odot(\varphi_0, \ldots, \varphi_{n-1})) = \mathbf{Var}[\{\varphi_0, \ldots, \varphi_{n-1}\}]$.
- For a formula $\mathsf{Q}x\varphi$, $\mathbf{Var}(\mathsf{Q}x\varphi) = \mathbf{Var}(\varphi) \smallsetminus \{x\}$.

Definition 2.16 The set $\mathscr{L}_\sigma^0$ of *sentences* of σ is the set $\{\varphi \in \mathscr{L}_\sigma \mid \mathbf{Var}(\varphi) = \varnothing\}$.

Of course, we are primarily concerned with models that are suitable for evaluation by some logical system or other. The definition of a model relative to a particular logical system L is as follows.

Definition 2.17 With respect to a logical system $\mathsf{L} = \langle \mathscr{V}, \mathscr{D}, \mathbf{S}, I \rangle$, an L-*model* $\mathfrak{M}$ is a 4-tuple $\langle M, \mathbf{C}^{\mathfrak{M}}, \mathbf{F}^{\mathfrak{M}}, \mathbf{R}^{\mathfrak{M}} \rangle$, where

- M is a set of elements,
- for each $c \in \mathbf{C}$, $c^{\mathfrak{M}} \in M$,
- for each n-ary $f \in \mathbf{F}$, $f^{\mathfrak{M}} : M^n \to M$,
- for each n-ary $R \in \mathbf{R}$, $R^{\mathfrak{M}} : M^n \to \mathscr{V}$.

Definition 2.18 For an L-model $\mathfrak{M}$, the corresponding *Henkin model* $\mathfrak{M}(M)$ is the 4-tuple $\langle M, \mathbf{C}^{\mathfrak{M}} \cup \{\underline{a} \mid a \in M\}, \mathbf{F}^{\mathfrak{M}}, \mathbf{R}^{\mathfrak{M}} \rangle$, where $\underline{a}$ is a new constant such that $\underline{a}^{\mathfrak{M}(M)} = a$ for all elements of M.

Definition 2.19 For an L-model $\mathfrak{M}$, the valuation function $v_{\mathfrak{M}} : \mathscr{L}_\sigma^0 \to \mathscr{V}$ is defined so that for all n-ary connectives and quantifiers we have the following.

- For atomic sentences $\psi = R(t_0, \ldots, t_{n-1})$, $v_{\mathfrak{M}}(\psi) = R^{\mathfrak{M}}(t_0^{\mathfrak{M}}, \ldots, t_{n-1}^{\mathfrak{M}})$.
- For sentences $\psi = \odot(\varphi_0, \ldots, \varphi_{n-1})$, $v_{\mathfrak{M}}(\psi) = f_{\mathsf{L}}^{\odot}(v_{\mathfrak{M}}(\varphi_0), \ldots, v_{\mathfrak{M}}(\varphi_{n-1}))$.
- For sentences $\psi = \mathsf{Q}x\varphi$, $v_{\mathfrak{M}}(\psi) = f_{\mathsf{L}}^{\mathsf{Q}}(\{v_{\mathfrak{M}(M)}(\varphi(\underline{a}/x)) \mid a \in M\})$.

The classical notion of truth in a model can likewise be generalized as follows.

Definition 2.20 A sentence φ is *designated* in an L-model $\mathfrak{M}$—$\mathfrak{M} \vDash_{\mathsf{L}} \varphi$—if $v_{\mathfrak{M}}(\varphi) \in \mathscr{D}_{\mathsf{L}}$.

Definition 2.21 Two L-models $\mathfrak{M}_0$ and $\mathfrak{M}_1$ are *isomorphic*—$\mathfrak{M}_0 \cong \mathfrak{M}_1$—if there exists a bijection $g : M_0 \to M_1$ such that for all n-ary $f \in \mathbf{F}$ and $R \in \mathbf{R}$ and n-tuples $a_0, \ldots, a_{n-1} \in M_0$,

- $f^{\mathfrak{M}_1}(g(a_0), \ldots, g(a_{n-1})) = g(f^{\mathfrak{M}_0}(a_0, \ldots, a_{n-1}))$,
- $R^{\mathfrak{M}_1}(g(a_0), \ldots, g(a_{n-1})) = R^{\mathfrak{M}_0}(a_0, \ldots, a_{n-1})$.

To pin down the importance of isomorphism between models, we introduce a generalization of the *elementary diagram* of classical model theory with the following observation.

Observation 2.22 *The value of every sentence of a structure $\mathfrak{M}$ is determined solely by the set $\{\langle R(\underline{\vec{a}}), v_{\mathfrak{M}(M)}(R(\underline{\vec{a}}))\rangle \mid R \in \mathbf{R}$ and $\vec{a} \in M^{<\omega}\}$, that is, the values that $\mathfrak{M}(M)$ assigns to atomic sentences.*

Proof Noting that interpretations of connectives and quantifiers are deterministic functions of the values of formulae of lesser complexity, a simple induction on the complexity of formulae establishes this. □

Corollary 2.23 *If $\mathfrak{M} \cong \mathfrak{M}'$, then for all sentences φ, $v_{\mathfrak{M}}(\varphi) = v_{\mathfrak{M}'}(\varphi)$.*

Proof The set $\{\langle R(\underline{\vec{a}}), v_{\mathfrak{M}(M)}(R(\underline{\vec{a}}))\rangle \mid R \in \mathbf{R}$ and $\vec{a} \in M^{<\omega}\}$ is entirely determined by the interpretations of constants, function symbols, and relation symbols. Hence, two isomorphic structures will evaluate all atomic formulae identically. By Observation 2.22, for any sentence φ, $\mathfrak{M}$ and $\mathfrak{M}'$ will evaluate φ identically as well. □

3 Dunn–Priest Quotients

We now turn our attention to generalizing the Dunn–Priest constructions.

Let $\mathsf{L}_0 = \langle \mathscr{V}_0, \mathscr{D}_0, \mathbf{S}_0, I_0\rangle$ and $\mathsf{L}_1 = \langle \mathscr{V}_1, \mathscr{D}_1, \mathbf{S}_1, I_1\rangle$ be two logical systems sharing connectives and quantifiers, that is, such that $\mathbf{S}_0 = \mathbf{S}_1$. Also, recall the definition of a *threaded image* $\mathsf{Thread}(\mathbf{X})$ of a family of sets $\mathbf{X}$ from Definition 2.5.

Definition 3.1 Let h be a function such that $h : \wp^+(\mathscr{V}_0) \to \mathscr{V}_1$. Then we call h a *Dunn–Priest map* if it enjoys the following two properties with respect to all nonempty sets $X_i \subseteq \mathscr{V}_0$, families $\mathbf{X} = \{X_i \mid i \in I\}$ of nonempty subsets of $\mathscr{V}_0$, n-ary connectives $\odot$, and quantifiers Q:

- *Property 1*: $h(f^{\odot}_{\mathsf{L}_0}[X_0 \times \cdots \times X_{n-1}]) = f^{\odot}_{\mathsf{L}_1}(h(X_0), \ldots, h(X_{n-1}))$;
- *Property 2*: $h(f^{\mathsf{Q}}_{\mathsf{L}_0}[\mathsf{Thread}(\mathbf{X})]) = f^{\mathsf{Q}}_{\mathsf{L}_1}(h[\mathbf{X}])$.

Definition 3.2 Let $\sim$ be an equivalence relation on the domain of a model $\mathfrak{M}$. Then the *$\sim$-spectrum* $v^{\mathsf{Out}}_{\mathfrak{M}}(\varphi)$ of a sentence φ in a model $\mathfrak{M}$ is recursively defined:

- $v^{\mathsf{Out}}_{\mathfrak{M}}(R(t_0, \ldots, t_{n-1})) = \{v_{\mathfrak{M}(M)}(R(t_0, \ldots, t_{n-1})(\underline{\vec{a}} \mathbin{/\!/} \vec{t})) \mid \forall i \in \omega, a_i \sim t_i^{\mathfrak{M}(M)}\}$;
- $v^{\mathsf{Out}}_{\mathfrak{M}}(\odot(\varphi_0, \ldots, \varphi_{n-1})) = f^{\odot}_{\mathsf{L}_0}[v^{\mathsf{Out}}_{\mathfrak{M}}(\varphi_0) \times \cdots \times v^{\mathsf{Out}}_{\mathfrak{M}}(\varphi_{n-1})]$;
- $v^{\mathsf{Out}}_{\mathfrak{M}}(\mathsf{Q}x\varphi) = f^{\mathsf{Q}}_{\mathsf{L}_0}[\mathsf{Thread}(\{v^{\mathsf{Out}}_{\mathfrak{M}}(\varphi(a := x)) \mid a \in M\})]$.

Especially important is to note the use of the *scattered substitution* in the case of $R(t_0, \ldots, t_{n-1})$. If we were to define this in terms of *uniform substitution*, we would get undesirable consequences. For example, the possibility of inconsistent identity in Priest's Collapsing Lemma relies on a sentence $c = c$ being interpreted by each instance of c. Were one to define $v^{\mathsf{Out}}_{\mathfrak{M}}(c = c)$ in terms of the truth values of uniform substitutions, no such substitution could turn out false. All equations true in a model would remain precisely true in the collapsed structure.

This provides us with sufficient machinery to define a *Dunn–Priest quotient*. Let $\mathfrak{M}$ be a model for L_0, let h be a Dunn–Priest map from L_0 to L_1, and let $\sim$ be a congruence relation on $\mathfrak{M}$, so that $[\![a]\!]$ is the congruence class of a modulo $\sim$.

Definition 3.3 The *Dunn–Priest quotient* of $\mathfrak{M}$ along h modulo $\sim$, denoted by $\mathfrak{M}^{\sim}$, is the L_1-model $\langle M^{\sim}, \mathbf{C}^{\mathfrak{M}^{\sim}}, \mathbf{F}^{\mathfrak{M}^{\sim}}, \mathbf{R}^{\mathfrak{M}^{\sim}}\rangle$, where

- $M^{\sim} = \{[\![a]\!] \mid a \in M\}$,

- $c^{\mathfrak{M}^{\sim}} = [\![c^{\mathfrak{M}}]\!]$,
- $f^{\mathfrak{M}^{\sim}}([\![a_0]\!], \ldots, [\![a_{n-1}]\!]) = [\![f(a_0, \ldots, a_{n-1})]\!]$,
- $R^{\mathfrak{M}^{\sim}}([\![a_0]\!], \ldots, [\![a_{n-1}]\!]) = h(v^{\mathsf{Out}}_{\mathfrak{M}(M)}(R(\underline{a_0}, \ldots, \underline{a_{n-1}})))$.

We finally prove two lemmas concerning the relationship between $v_{\mathfrak{M}}$, $v_{\mathfrak{M}^{\sim}}$, and $v^{\mathsf{Out}}_{\mathfrak{M}}$. Let h be a Dunn–Priest map from L_0 to L_1, and let $\mathfrak{M}$ be an L_0-model.

Lemma 3.4 *For all sentences φ, $v_{\mathfrak{M}}(\varphi) \in v^{\mathsf{Out}}_{\mathfrak{M}}(\varphi)$.*

Proof We prove this by a pair of inductions on the complexity of formulae. Initially, we assume that $\mathfrak{M}$ is a Henkin model, that is, that $\mathfrak{M}(M) = \mathfrak{M}$; once this is established, we can repeat the induction for non-Henkin models $\mathfrak{M}$.

As a basis step, in which φ is an atomic sentence $R(t_0, \ldots, t_{n-1})$, the reflexivity of $\sim$ entails that $t_i^{\mathfrak{M}} \sim t_i^{\mathfrak{M}}$. Hence, $R^{\mathfrak{M}}(t_0^{\mathfrak{M}}, \ldots, t_{n-1}^{\mathfrak{M}}) \in R^{\mathfrak{M}}[\{R^{\mathfrak{M}}(a_0, \ldots, a_{n-1}) \mid$ for all $i < n, a_i \sim t_i^{\mathfrak{M}}\}]$, which is just to say that $v_{\mathfrak{M}}(R(t_0, \ldots, t_{n-1})) \in v^{\mathsf{Out}}_{\mathfrak{M}}(R(t_0, \ldots, t_{n-1}))$. As the induction hypothesis, assume that for all sentences ψ of lesser complexity than φ, $v_{\mathfrak{M}}(\psi) \in v^{\mathsf{Out}}_{\mathfrak{M}}(\psi)$.

If φ is a sentence $\odot(\varphi_0, \ldots, \varphi_{n-1})$, then by the induction hypothesis, for all $i < n$, $v_{\mathfrak{M}}(\varphi_i) \in v^{\mathsf{Out}}_{\mathfrak{M}}(\varphi_i)$. Hence, $(v_{\mathfrak{M}}(\varphi_0), \ldots, v_{\mathfrak{M}}(\varphi_{n-1})) \in v^{\mathsf{Out}}_{\mathfrak{M}}(\varphi_0) \times \cdots \times v^{\mathsf{Out}}_{\mathfrak{M}}(\varphi_{n-1})$. Hence, $f^{\odot}_{\mathsf{L}_0}(v_{\mathfrak{M}}(\varphi_0), \ldots, v_{\mathfrak{M}}(\varphi_{n-1})) \in f^{\odot}_{\mathsf{L}_0}[v^{\mathsf{Out}}_{\mathfrak{M}}(\varphi_0) \times \cdots \times v^{\mathsf{Out}}_{\mathfrak{M}}(\varphi_{n-1})]$, entailing that $v_{\mathfrak{M}}(\odot(\varphi_0, \ldots, \varphi_{n-1})) \in v^{\mathsf{Out}}_{\mathfrak{M}}(\odot(\varphi_0, \ldots, \varphi_{n-1}))$.

If φ is a quantified sentence $\mathsf{Q}x\psi$, then $v_{\mathfrak{M}}(\varphi) = f^{\mathsf{Q}}_{\mathsf{L}_0}(\{v_{\mathfrak{M}(M)}(\psi(\underline{a} := x)) \mid a \in M\})$. Now, by the induction hypothesis, for each $a \in M$, $v_{\mathfrak{M}(M)}(\psi(\underline{a} := x)) \in v^{\mathsf{Out}}_{\mathfrak{M}(M)}(\psi(\underline{a} := x))$. So $\{v_{\mathfrak{M}(M)}(\psi(\underline{a} := x)) \mid a \in M\} \in \mathsf{Thread}(\{v^{\mathsf{Out}}_{\mathfrak{M}(M)}(\psi(\underline{a} := x)) \mid a \in M\})$. Hence, $f^{\mathsf{Q}}_{\mathsf{L}_0}(\{v_{\mathfrak{M}(M)}(\psi(\underline{a} := x)) \mid a \in M\}) \in f^{\mathsf{Q}}_{\mathsf{L}_0}[\mathsf{Thread}(\{v^{\mathsf{Out}}_{\mathfrak{M}(M)}(\psi(\underline{a} := x)) \mid a \in M\})]$, that is, $v_{\mathfrak{M}}(\mathsf{Q}x\psi) \in v^{\mathsf{Out}}_{\mathfrak{M}}(\mathsf{Q}x\psi)$. □

Lemma 3.5 *For all sentences φ, $v_{\mathfrak{M}^{\sim}}(\varphi) = h(v^{\mathsf{Out}}_{\mathfrak{M}}(\varphi))$.*

Proof We prove this by induction on the complexity of sentences. As a basis, note that this is precisely how we have defined the interpretation of predicate symbols R in the quotient $\mathfrak{M}^{\sim}$.

As the induction hypothesis, suppose that this holds for all formulae of lesser complexity than φ. Then, in the case of connectives $\odot$, we observe that

$$\begin{aligned} v_{\mathfrak{M}^{\sim}}\big(\odot(\varphi_0, \ldots, \varphi_{n-1})\big) &= f^{\mathsf{Q}}_{\mathsf{L}_1}\big(v_{\mathfrak{M}^{\sim}}(\varphi_0), \ldots, v_{\mathfrak{M}^{\sim}}(\varphi_{n-1})\big) \\ &= f^{\odot}_{\mathsf{L}_1}\big(h\big(v^{\mathsf{Out}}_{\mathfrak{M}}(\varphi_0)\big), \ldots, h\big(v^{\mathsf{Out}}_{\mathfrak{M}}(\varphi_{n-1})\big)\big) \\ &= h\big(f^{\odot}_{\mathsf{L}_0}\big[v^{\mathsf{Out}}_{\mathfrak{M}}(\varphi_0) \times \cdots \times v^{\mathsf{Out}}_{\mathfrak{M}}(\varphi_{n-1})\big]\big) \\ &= h\big(v^{\mathsf{Out}}_{\mathfrak{M}}\big(\odot(\varphi_0, \ldots, \varphi_{n-1})\big)\big). \end{aligned}$$

In the case of quantifiers Q, the following equivalences establish the result:

$$\begin{aligned} v_{\mathfrak{M}^{\sim}}(\mathsf{Q}x\psi) &= f^{\mathsf{Q}}_{\mathsf{L}_1}\big(\big\{v_{\mathfrak{M}^{\sim}}\big(\psi(\underline{a} := x)\big) \mid a \in M^{\sim}\big\}\big) \\ &= f^{\mathsf{Q}}_{\mathsf{L}_1}\big(\big\{h\big(v^{\mathsf{Out}}_{\mathfrak{M}}\big(\psi(\underline{a} := x)\big)\big) \mid a \in M^{\sim}\big\}\big) \\ &= h\big(f^{\mathsf{Q}}_{\mathsf{L}_0}\big[\mathsf{Thread}\big(\big\{v^{\mathsf{Out}}_{\mathfrak{M}}\big(\psi(\underline{a} := x)\big) \mid a \in M\big\}\big)\big]\big). \end{aligned}$$ □

We are now prepared to prove the primary theorem concerning Dunn–Priest quotients, that is, sufficient conditions for preservation between models.

Theorem 3.6 *Let $\mathscr{X}_0 \subseteq \mathscr{V}_0$ and $\mathscr{X}_1 \subseteq \mathscr{V}_1$ be nonempty sets. If h is a Dunn–Priest map with the property that,*

$$\textit{for all } X \subseteq \mathscr{V}_0, \textit{ if } X \cap \mathscr{X}_0 \neq \varnothing, \textit{ then } h(X) \in \mathscr{X}_1,$$

then for all L_0-models $\mathfrak{M}$, if $v_{\mathfrak{M}}(\varphi) \in \mathscr{X}_0$, then $v_{\mathfrak{M}^{\sim}}(\varphi) \in \mathscr{X}_1$.

Proof Suppose that $v_{\mathfrak{M}}(\varphi) \in \mathscr{X}_0$. Then $v^{\mathsf{Out}}_{\mathfrak{M}}(\varphi) \cap \mathscr{X}_0 \neq \varnothing$. Hence, by hypothesis, $h(v^{\mathsf{Out}}_{\mathfrak{M}}(\varphi)) \in \mathscr{X}_1$. However, by Lemma 3.5, for all formulae φ, $v_{\mathfrak{M}^{\sim}}(\varphi) = h(v^{\mathsf{Out}}_{\mathfrak{M}}(\varphi))$, whence $v_{\mathfrak{M}^{\sim}}(\varphi) \in \mathscr{X}_1$. $\square$

4 Some Examples

In this section, we will examine a few examples. The theorems of Dunn and Priest will be demonstrated by appeal to Theorem 3.6 and a similar, novel result—the preservation of nonsensical formulae in quotients of models of Bochvar's Σ_0—will be demonstrated by the same method. Finally, we will proceed to look at the interaction of the foregoing observations and the more recent work of Priest on *plurivalent semantics*.

4.1 The Dunn–Priest results As mentioned in Section 3, Dunn's Theorem in 3-Valued Model Theory of [6] and Priest's Collapsing Lemma of [13] are special cases of the above technique. In this section, we show how these results fall out of the foregoing observations.

We will introduce the relevant logical systems before proceeding to the proofs of the results themselves. Fix a set $\mathbf{S} = \{\neg, \wedge, \forall\}$ common to classical logic (CL), the strong Kleene logic K_3 of [10], and the paraconsistent logic of paradox LP, introduced by Priest [12].

Classical logic, according to our scheme, may be represented as follows.

Definition 4.1 Classical logic is the 4-tuple $\langle \mathscr{V}_{\mathsf{CL}}, \mathscr{D}_{\mathsf{CL}}, \mathbf{S}, I_{\mathsf{CL}} \rangle$, where

- $\mathscr{V}_{\mathsf{CL}} = \{\mathfrak{t}, \mathfrak{f}\}$,
- $\mathscr{D}_{\mathsf{CL}} = \{\mathfrak{t}\}$.

The function I_{CL} interprets connectives $\neg$ and $\wedge$ and the quantifier $\forall$:

$f^{\neg}_{\mathsf{CL}}$	
$\mathfrak{t}$	$\mathfrak{f}$
$\mathfrak{f}$	$\mathfrak{t}$

$f^{\wedge}_{\mathsf{CL}}$	$\mathfrak{t}$	$\mathfrak{f}$
$\mathfrak{t}$	$\mathfrak{t}$	$\mathfrak{f}$
$\mathfrak{f}$	$\mathfrak{f}$	$\mathfrak{f}$

$$f^{\forall}_{\mathsf{CL}}(X) = \begin{cases} \mathfrak{t} & \text{if } \mathfrak{f} \notin X, \\ \mathfrak{f} & \text{if } \mathfrak{f} \in X. \end{cases}$$

Definition 4.2 The strong Kleene logic K_3 is the 4-tuple $\langle \mathscr{V}_{\mathsf{K}_3}, \mathscr{D}_{\mathsf{K}_3}, \mathbf{S}, I_{\mathsf{K}_3} \rangle$, where

- $\mathscr{V}_{\mathsf{K}_3} = \{\mathfrak{t}, \mathfrak{n}, \mathfrak{f}\}$,
- $\mathscr{D}_{\mathsf{K}_3} = \{\mathfrak{t}\}$.

The function I_{K_3} interprets connectives $\neg$ and $\wedge$ and the quantifier $\forall$:

$f^{\neg}_{\mathsf{K}_3}$	
$\mathfrak{t}$	$\mathfrak{f}$
$\mathfrak{n}$	$\mathfrak{n}$
$\mathfrak{f}$	$\mathfrak{t}$

$f^{\wedge}_{\mathsf{K}_3}$	$\mathfrak{t}$	$\mathfrak{n}$	$\mathfrak{f}$
$\mathfrak{t}$	$\mathfrak{t}$	$\mathfrak{n}$	$\mathfrak{f}$
$\mathfrak{n}$	$\mathfrak{n}$	$\mathfrak{n}$	$\mathfrak{f}$
$\mathfrak{f}$	$\mathfrak{f}$	$\mathfrak{f}$	$\mathfrak{f}$

$$f^{\forall}_{\mathsf{K}_3}(X) = \begin{cases} \mathfrak{t} & \text{if } \mathfrak{n} \notin X \text{ and } \mathfrak{f} \notin X, \\ \mathfrak{n} & \text{if } \mathfrak{n} \in X \text{ and } \mathfrak{f} \notin X, \\ \mathfrak{f} & \text{if } \mathfrak{f} \in X. \end{cases}$$

Definition 4.3 The paraconsistent logic LP is the 4-tuple $\langle \mathscr{V}_{\mathsf{LP}}, \mathscr{D}_{\mathsf{LP}}, \mathbf{S}, I_{\mathsf{LP}} \rangle$, where

- $\mathscr{V}_{\mathsf{LP}} = \{\mathfrak{t}, \mathfrak{b}, \mathfrak{f}\}$,
- $\mathscr{D}_{\mathsf{LP}} = \{\mathfrak{t}, \mathfrak{b}\}$.

The function I_{LP} interprets connectives $\neg$ and $\wedge$ and the quantifier $\forall$:

$f^{\neg}_{\mathsf{LP}}$	
$\mathfrak{t}$	$\mathfrak{f}$
$\mathfrak{b}$	$\mathfrak{b}$
$\mathfrak{f}$	$\mathfrak{t}$

$f^{\wedge}_{\mathsf{LP}}$	$\mathfrak{t}$	$\mathfrak{b}$	$\mathfrak{f}$
$\mathfrak{t}$	$\mathfrak{t}$	$\mathfrak{b}$	$\mathfrak{f}$
$\mathfrak{b}$	$\mathfrak{b}$	$\mathfrak{b}$	$\mathfrak{f}$
$\mathfrak{f}$	$\mathfrak{f}$	$\mathfrak{f}$	$\mathfrak{f}$

$$f^{\forall}_{\mathsf{LP}}(X) = \begin{cases} \mathfrak{t} & \text{if } \mathfrak{b} \notin X \text{ and } \mathfrak{f} \notin X, \\ \mathfrak{b} & \text{if } \mathfrak{b} \in X \text{ and } \mathfrak{f} \notin X, \\ \mathfrak{f} & \text{if } \mathfrak{f} \in X. \end{cases}$$

The apparatus introduced in Section 3 provides a direct way to prove Dunn's Theorem in 3-Valued Model Theory. First, we define the particular Dunn–Priest map employed in Dunn's theorem.

Definition 4.4 The *Dunn map* from CL to K_3 is the Dunn–Priest map h_D, where

$$h_D(X) = \begin{cases} \mathfrak{t} & \text{if } X = \{\mathfrak{t}\}, \\ \mathfrak{n} & \text{if } X = \{\mathfrak{t}, \mathfrak{f}\}, \\ \mathfrak{f} & \text{if } X = \{\mathfrak{f}\}. \end{cases}$$

Observation 4.5 *The Dunn–Priest map h_D satisfies Properties 1 and 2.*

Proof In the case of negation, first consider the case in which X is a singleton. Without loss of generality, let $X = \{\mathfrak{t}\}$. Then $h(f^{\neg}_{\mathsf{CL}}[X]) = \mathfrak{f}$ and $f^{\neg}_{\mathsf{K}_3}(h(X)) = \mathfrak{f}$. If $X = \{\mathfrak{t}, \mathfrak{f}\}$, then $f^{\neg}_{\mathsf{CL}}[X] = X$ and $h(X) = \mathfrak{n}$. But $f^{\neg}_{\mathsf{K}_3}(h[X]) = \mathfrak{n}$.

In the case of conjunction, we can examine three cases.

- Suppose that either $\mathfrak{t} \notin X_0$ or $\mathfrak{t} \notin X_1$. Then $f^{\wedge}_{\mathsf{CL}}[X_0 \times X_1] = \{\mathfrak{f}\}$ and $h(f^{\neg}_{\mathsf{CL}}[X_0 \times X_1]) = \mathfrak{f}$. But in this case, either $h(X_0)$ or $h(X_1)$ is equal to $\mathfrak{f}$, whence $f^{\wedge}_{\mathsf{K}_3}(h(X_0), h(X_1)) = \mathfrak{f}$.
- If $X_0 = X_1 = \{\mathfrak{t}\}$, then $h(f^{\wedge}_{\mathsf{CL}}[X_0 \times X_1]) = \mathfrak{t}$, as is $f^{\neg}_{\mathsf{K}_3}(h(X_0), h(X_1))$.
- Otherwise, $\mathfrak{t}$ is a member of both X_0 and X_1 and at least one of these sets counts $\mathfrak{f}$ as a member as well. Then $f^{\wedge}_{\mathsf{CL}}[X_0 \times X_1] = \{\mathfrak{t}, \mathfrak{f}\}$, whence $h(f^{\wedge}_{\mathsf{CL}}[X_0 \times X_1]) = \mathfrak{n}$. But one of $h(X_0)$ and $h(X_1)$ is $\mathfrak{n}$ and neither is $\mathfrak{f}$, whence $f^{\wedge}_{\mathsf{K}_3}(h(X_0), h(X_1)) = \mathfrak{n}$.

Finally, to examine the case of the universal quantifier, let $\mathbf{X}$ denote the family $X_0, \ldots, X_{m-1}$. There are again three cases.

- Suppose that $\{\mathfrak{f}\} \in \mathbf{X}$. Then for all $X' \in \mathsf{Thread}(\mathbf{X})$ it follows that $\mathfrak{f} \in X'$, whence $h(f^{\forall}_{\mathsf{CL}}[\mathsf{Thread}(\mathbf{X})]) = \mathfrak{f}$. But $\mathfrak{f} \in h[\mathbf{X}]$, whence $f^{\forall}_{\mathsf{K}_3}(h[\mathbf{X}]) = \mathfrak{f}$.
- Suppose that $\{\mathfrak{f}\} \notin \mathbf{X}$ but $\{\mathfrak{t}, \mathfrak{f}\} \in \mathbf{X}$. Then either $\mathsf{Thread}(\mathbf{X}) = \{\{\mathfrak{t}\}, \{\mathfrak{t}, \mathfrak{f}\}\}$ or $\{\{\mathfrak{t}\}, \{\mathfrak{f}\}\}$; in both cases, $f^{\forall}_{\mathsf{CL}}[\mathsf{Thread}(\mathbf{X})] = \{\mathfrak{t}, \mathfrak{f}\}$, whence we conclude that $h(f^{\forall}_{\mathsf{CL}}[\mathsf{Thread}(\mathbf{X})]) = \mathfrak{n}$. But $\mathfrak{n} \in h[\mathbf{X}]$ and $\mathfrak{f} \notin h[\mathbf{X}]$, whence $f^{\forall}_{\mathsf{K}_3}(h[\mathbf{X}]) = \mathfrak{n}$.
- Finally, suppose that $\mathbf{X} = \{\{\mathfrak{t}\}\}$. Then $\mathsf{Thread}(\mathbf{X}) = \{\{\mathfrak{t}\}\}$, whence $f^{\forall}_{\mathsf{CL}}[\mathsf{Thread}(\mathbf{X})] = \{\mathfrak{t}\}$ and $h(f^{\forall}_{\mathsf{CL}}[\mathsf{Thread}(\mathbf{X})]) = \mathfrak{t}$. But $h[\mathbf{X}] = \{\mathfrak{t}\}$, and we thus conclude that $f^{\forall}_{\mathsf{K}_3}(h[\mathbf{X}]) = \mathfrak{t}$.

$\square$

Observation 4.6 *The Dunn–Priest map h_D has the property that if $X \cap \{\mathfrak{t}\} \neq \varnothing$, then $h(X) \in \{\mathfrak{t}, \mathfrak{n}\}$.*

Proof It may be confirmed by examining the definition of h_D that whenever $\mathfrak{t} \in X$, either $h(X) = \mathfrak{t}$ or $h(X) = \mathfrak{n}$. $\square$

Corollary 4.7 *If $\mathfrak{M} \vDash_{\mathsf{CL}} \varphi$, then $\mathfrak{M}^{\sim} \nvDash_{\mathsf{K}_3} \neg\varphi$.*

Proof Suppose that $\mathfrak{M} \vDash_{\mathsf{CL}} \varphi$. Then $v_{\mathfrak{M}}(\varphi) = \mathfrak{t}$, whence $\mathfrak{t} \in v_{\mathfrak{M}}^{\mathsf{Out}}(\varphi)$. Then $h(v_{\mathfrak{M}}^{\mathsf{Out}}) \in \{\mathfrak{t}, \mathfrak{n}\}$, whence $v_{\mathfrak{M}^{\sim}}(\varphi) \in \{\mathfrak{t}, \mathfrak{n}\}$ by Lemma 3.5. But were $\mathfrak{M}^{\sim} \vDash_{\mathsf{K}_3} \neg\varphi$ to hold, then $v_{\mathfrak{M}^{\sim}}(\varphi) = \mathfrak{f}$. Hence, $\mathfrak{M}^{\sim} \nvDash_{\mathsf{K}_3} \neg\varphi$. □

Priest's collapsing lemma follows just as easily from the above observations.

Definition 4.8 The *Priest map* from CL to LP is the Dunn–Priest map h_P, where

$$h_P(X) = \begin{cases} \mathfrak{t} & \text{if } X = \{\mathfrak{t}\}, \\ \mathfrak{b} & \text{if } X = \{\mathfrak{t}, \mathfrak{f}\}, \\ \mathfrak{f} & \text{if } X = \{\mathfrak{f}\}. \end{cases}$$

Observation 4.9 *The Dunn–Priest map h_P satisfies Properties 1 and 2.*

Proof Note that Observation 4.15 makes no use of which values are designated and that h_P is essentially h_D with $\mathfrak{b}$ replacing $\mathfrak{n}$.[1] Hence, substituting $\mathfrak{b}$ for $\mathfrak{n}$ suffices to prove the observation. □

Observation 4.10 *The Dunn–Priest map h_P has the property that if $X \cap \mathscr{D}_{\mathsf{CL}} \neq \varnothing$, then $h(X) \in \mathscr{D}_{\mathsf{LP}}$.*

Proof Inspection confirms that if $\mathfrak{t} \in X$, then either $h(X) = \mathfrak{t}$ or $h(X) = \mathfrak{b}$. □

Corollary 4.11 *If $\mathfrak{M} \vDash_{\mathsf{CL}} \varphi$, then $\mathfrak{M}^{\sim} \vDash_{\mathsf{LP}} \varphi$.*

Proof If $\mathfrak{M} \vDash_{\mathsf{CL}} \varphi$, then $v_{\mathfrak{M}}(\varphi) = \mathfrak{t}$. By Theorem 3.6, Observation 4.10 entails that $v_{\mathfrak{M}^{\sim}}(\varphi) \in \mathscr{D}_{\mathsf{LP}}$, whence $\mathfrak{M}^{\sim} \vDash_{\mathsf{LP}} \varphi$. □

4.2 Maps between the logics of Bochvar/Kleene and Deutsch/Oller A further example can be drawn from the realm of logics of nonsense. In particular, a novel preservation theorem analogous to those of Dunn and Priest can be established for quotients of structures taken along a map between a logic of nonsense and a paraconsistent containment logic. In the first case, we consider the internal logic of nonsense Σ_0 of Dmitri Bochvar introduced in [2] (translated as [3]), also independently described by Kleene in [10]. The target will be the paraconsistent containment logic $\mathsf{S}_{\mathtt{fde}}$ described first by Harry Deutsch in [5] and later independently discovered by Carlos Oller in [11], in which the system was called AL.

We will begin by introducing natural first-order extensions of these systems. Retain the interpretation found in Section 4.1 of $\mathbf{S}$ as the set $\{\neg, \wedge, \forall\}$.

Definition 4.12 The internal first-order nonsense logic Σ_0 is semantically defined as the 4-tuple $\langle \mathscr{V}_{\Sigma_0}, \mathscr{D}_{\Sigma_0}, \mathbf{S}, I_{\Sigma_0} \rangle$, where

- $\mathscr{V}_{\Sigma_0} = \{\mathfrak{t}, \mathfrak{n}, \mathfrak{f}\}$,
- $\mathscr{D}_{\Sigma_0} = \{\mathfrak{t}\}$.

The function I_{Σ_0} interprets connectives $\neg$ and $\wedge$ and the quantifier $\forall$:

$f_{\Sigma_0}^{\neg}$	
$\mathfrak{t}$	$\mathfrak{f}$
$\mathfrak{n}$	$\mathfrak{n}$
$\mathfrak{f}$	$\mathfrak{t}$

$f_{\Sigma_0}^{\wedge}$	$\mathfrak{t}$	$\mathfrak{n}$	$\mathfrak{f}$
$\mathfrak{t}$	$\mathfrak{t}$	$\mathfrak{n}$	$\mathfrak{f}$
$\mathfrak{n}$	$\mathfrak{n}$	$\mathfrak{n}$	$\mathfrak{n}$
$\mathfrak{f}$	$\mathfrak{f}$	$\mathfrak{n}$	$\mathfrak{f}$

$$f_{\Sigma_0}^{\forall}(X) = \begin{cases} \mathfrak{t} & \text{if } \mathfrak{n} \notin X \text{ and } \mathfrak{f} \notin X, \\ \mathfrak{n} & \text{if } \mathfrak{n} \in X, \\ \mathfrak{f} & \text{if } \mathfrak{n} \notin X \text{ and } \mathfrak{f} \in X. \end{cases}$$

Bochvar's intended interpretation of Σ_0 is that $\mathfrak{n}$ is a "nonsense" value afforded to formulae regarded as meaningless or nonsense.

Oller's paraconsistent containment logic is the following.

Definition 4.13 The paraconsistent containment logic $\mathsf{S}_{\mathtt{fde}}$ is semantically defined as the 4-tuple $\langle \mathscr{V}_{\mathsf{S}_{\mathtt{fde}}}, \mathscr{D}_{\mathsf{S}_{\mathtt{fde}}}, \mathbf{S}, I_{\mathsf{S}_{\mathtt{fde}}} \rangle$, where

- $\mathscr{V}_{\mathsf{S}_{\mathtt{fde}}} = \{\mathfrak{t}, \mathfrak{b}, \mathfrak{n}, \mathfrak{f}\}$,
- $\mathscr{D}_{\mathsf{S}_{\mathtt{fde}}} = \{\mathfrak{t}, \mathfrak{b}\}$.

The function $I_{\mathsf{S}_{\mathtt{fde}}}$ interprets connectives $\neg$ and $\wedge$ and the quantifier $\forall$:

$f^{\neg}_{\mathsf{S}_{\mathtt{fde}}}$	
$\mathfrak{t}$	$\mathfrak{f}$
$\mathfrak{b}$	$\mathfrak{b}$
$\mathfrak{n}$	$\mathfrak{n}$
$\mathfrak{f}$	$\mathfrak{t}$

$f^{\wedge}_{\mathsf{S}_{\mathtt{fde}}}$	$\mathfrak{t}$	$\mathfrak{b}$	$\mathfrak{n}$	$\mathfrak{f}$
$\mathfrak{t}$	$\mathfrak{t}$	$\mathfrak{b}$	$\mathfrak{n}$	$\mathfrak{f}$
$\mathfrak{b}$	$\mathfrak{b}$	$\mathfrak{b}$	$\mathfrak{n}$	$\mathfrak{f}$
$\mathfrak{n}$	$\mathfrak{n}$	$\mathfrak{n}$	$\mathfrak{n}$	$\mathfrak{n}$
$\mathfrak{f}$	$\mathfrak{f}$	$\mathfrak{f}$	$\mathfrak{n}$	$\mathfrak{f}$

$$f^{\forall}_{\mathsf{S}_{\mathtt{fde}}}(X) = \begin{cases} \mathfrak{t} & \text{if } \mathfrak{b} \notin X, \mathfrak{n} \notin X, \text{and } \mathfrak{f} \notin X, \\ \mathfrak{b} & \text{if } \mathfrak{b} \in X, \mathfrak{n} \notin X, \text{and } \mathfrak{f} \notin X, \\ \mathfrak{n} & \text{if } \mathfrak{n} \in X, \\ \mathfrak{f} & \text{if } \mathfrak{n} \notin X \text{ and } \mathfrak{f} \in X. \end{cases}$$

We are now able to define a construction by which one can build $\mathsf{S}_{\mathtt{fde}}$-models from Σ_0-models while retaining control over the ensuing theories. We first introduce the map along which the quotients will be taken.

Definition 4.14 The Dunn–Priest map $h_{\boldsymbol{B}} : \wp^{+}(\mathscr{V}_{\Sigma_0}) \to \mathscr{V}_{\mathsf{S}_{\mathtt{fde}}}$ is defined so that

$$h_{\boldsymbol{B}}(X) = \begin{cases} \mathfrak{n} & \text{if } \mathfrak{n} \in X, \\ \mathfrak{b} & \text{if } X = \{\mathfrak{t}, \mathfrak{f}\}, \\ \mathfrak{t} & \text{if } X = \{\mathfrak{t}\}, \\ \mathfrak{f} & \text{if } X = \{\mathfrak{f}\}. \end{cases}$$

The map can be shown to have the necessary properties.

Lemma 4.15 *The Dunn–Priest map $h_{\boldsymbol{B}}$ enjoys Properties 1 and 2*

Proof The lemma can be proved by examining two cases: those in which the value $\mathfrak{n}$ is found in one of the sets X and those in which it is not.

In the first case, the "infectiousness" of nonsense values ensures that Properties 1 and 2 hold. In the case of negation, both $f^{\neg}_{\mathsf{S}_{\mathtt{fde}}}(h_{\boldsymbol{B}}(X)) = \mathfrak{n}$ and $h(f^{\neg}_{\Sigma_0}[X]) = \mathfrak{n}$. Likewise, if either X_0 or X_1 contains $\mathfrak{n}$, then both $f^{\wedge}_{\mathsf{S}_{\mathtt{fde}}}(h_{\boldsymbol{B}}(X_0), h_{\boldsymbol{B}}(X_1)$ and $h(f^{\wedge}_{\Sigma_0}[X_0 X_1]))$ each have the value of $\mathfrak{n}$. In the case of the quantifier, if $\mathfrak{n}$ appears in some set $X \in \mathbf{X}$, then $\mathfrak{n}$ will appear in some element of $\mathsf{Thread}(\mathbf{X})$ and $\mathfrak{n}$ will appear in $h[\mathbf{X}]$. The former entails that $h(f^{\forall}_{\Sigma_0}[\mathsf{Thread}(\mathbf{X})]) = \mathfrak{n}$, while the latter entails that $f^{\forall}_{\mathsf{S}_{\mathtt{fde}}}(h[\mathbf{X}]) = \mathfrak{n}$.

In the latter case, note that $\mathsf{S}_{\mathtt{fde}}$ restricted to the truth values $\{\mathfrak{t}, \mathfrak{f}, \mathfrak{b}\}$ is essentially LP. In these cases, Lemma 4.9 entails that Properties 1 and 2 hold.

As these cases exhaust the possibilities, we conclude that $h_{\boldsymbol{B}}$ enjoys these properties. □

Lemma 4.16 *For all $\mathscr{X} \subseteq \mathscr{V}_{\Sigma_0}$, if $\mathscr{X} \cap \{\mathfrak{n}\} \neq \varnothing$, then $h(X) \in \{\mathfrak{n}\}$.*

Proof This can be confirmed by examining the extension of h_B. □

Theorem 4.17 *For a Σ_0-model $\mathfrak{M}$ and a Dunn–Priest quotient along h_B $\mathfrak{M}^\sim$, nonsensical formulae remain nonsensical; that is,*

$$\text{if } \mathfrak{M} \nvDash_{\Sigma_0} \varphi \text{ and } \mathfrak{M} \nvDash_{\Sigma_0} \neg\varphi, \text{ then } \mathfrak{M}^\sim \nvDash_{\mathsf{S_{fde}}} \varphi \text{ and } \mathfrak{M}^\sim \nvDash_{\mathsf{S_{fde}}} \neg\varphi.$$

Proof By Theorem 3.6, Lemmas 4.15 and 4.16 jointly entail that if $\mathfrak{n} \in v^{\mathsf{Out}}_{\mathfrak{M}}(\varphi)$, then $h(v^{\mathsf{Out}}_{\mathfrak{M}}(\varphi)) = \mathfrak{n}$. By Lemma 3.5, however, this entails that whenever $v_{\mathfrak{M}}(\varphi) = \mathfrak{n}$, also $v_{\mathfrak{M}^\sim}(\varphi) = \mathfrak{n}$.

Hence, if $\mathfrak{M} \nvDash_{\Sigma_0} \varphi$ and $\mathfrak{M} \nvDash_{\Sigma_0} \neg\varphi$, then $v_{\mathfrak{M}}(\varphi) = \mathfrak{n}$. By the above reasoning, this entails that $v_{\mathfrak{M}^\sim}(\varphi) = \mathfrak{n}$, whence $\mathfrak{M}^\sim \nvDash_{\mathsf{S_{fde}}} \varphi$ and $\mathfrak{M}^\sim \nvDash_{\mathsf{S_{fde}}} \neg\varphi$. □

This type of preservation theorem is thus not peculiar to the Dunn–Priest results.

4.3 Plurivalent semantics In [17], Priest introduces a technique for constructing new logical systems from an arbitrary many-valued logic L, in which a formula is not necessarily assigned a *single* truth value, but may be assigned a *set* of truth values, motivating the term *plurivalent semantics*. The technique follows from taking the positive power set of truth values of a logic L—$\wp^+(\mathscr{V}_{\mathsf{L}})$—and considering this set *itself* as the set of truth values. On this set, a general scheme is employed to define the new system's designated values and its interpretations of the connectives and quantifiers of L. In the resulting system, for an element $X \in \wp^+(\mathscr{V}_{\mathsf{L}})$ that a formula φ is assigned, the value X can alternately be read as "X is the set of truth values assigned to φ" (this is the "plurivalent" reading) or "X is the truth value assigned to φ" (the "univalent" reading). As [17] shows, a plurivalent system can be read without loss of generality as a univalent system. For the sake of elegance, we will opt to consider a corresponding plurivalent semantics as a univalent system.

Where L is a logical system $\langle \mathscr{V}, \mathscr{D}, \mathbf{S}, I\rangle$, Priest defines the elements of the corresponding plurivalent system, denoted by $\widehat{\mathsf{L}}$,[2] as the system whose truth values are nonempty subsets of $\mathscr{V}$ and whose designated values $\widehat{\mathscr{D}}$ are determined by the scheme:

- For all $X \in \widehat{\mathscr{V}}$, $X \in \widehat{\mathscr{D}}$ if and only if there exists a $\mathfrak{v} \in X$ such that $\mathfrak{v} \in \mathscr{D}$.

To determine the corresponding interpretations of an n-ary connective $\odot$, Priest defines the value of $f^{\odot}_{\widehat{\mathsf{L}}}(X_0, \ldots, X_{n-1})$ piecemeal by the scheme:

- For all $\mathfrak{v} \in \mathscr{V}$ and $X_0, \ldots, X_{n-1} \in \widehat{\mathscr{V}}$, $\mathfrak{v} \in f^{\odot}_{\widehat{\mathsf{L}}}(X_0, \ldots, X_{n-1})$ if and only if there are $\mathfrak{v}_0 \in X_0, \ldots, \mathfrak{v}_{n-1} \in X_{n-1}$ such that $\mathfrak{v} = f^{\odot}_{\mathsf{L}}(\mathfrak{v}_0, \ldots, \mathfrak{v}_{n-1})$.

In the present notation, this is to say that $f^{\odot}_{\widehat{\mathsf{L}}}(X_0, \ldots, X_{n-1}) = f^{\odot}_{\mathsf{L}}[X_0 \times \cdots \times X_{n-1}]$.

Although [17] considers only propositional logics, Priest's approach suggests a very natural way of extending plurivalent semantics to the case of distribution quantifiers:

- For all $\mathfrak{v} \in \mathscr{D}$ and nonempty sets $\{X_0, \ldots, X_{n-1}\} \subseteq \widehat{\mathscr{V}}$, $\mathfrak{v} \in f^{\mathsf{Q}}_{\widehat{\mathsf{L}}}(\{X_0, \ldots, X_{n-1}\})$ if and only if there exist $\mathfrak{v}_0 \in X_0, \ldots, \mathfrak{v}_{n-1} \in X_{n-1}$ such that $\mathfrak{v} = f^{\mathsf{Q}}_{\mathsf{L}}(\{\mathfrak{v}_0, \ldots, \mathfrak{v}_{n-1}\})$.

Again, this easily can be made to conform to the notation of the present article in that $f^{\mathsf{Q}}_{\widehat{\mathsf{L}}}(\{X_0, \ldots, X_{n-1}\}) = f^{\mathsf{Q}}_{\mathsf{L}}[\mathsf{Thread}(\{X_0, \ldots, X_{n-1}\})]$.

With this latter observation, the relatedness of plurivalent semantics to the general Dunn–Priest quotient technique becomes very clear. In this section, we will observe that the identity map serves as a Dunn–Priest map between any L and $\widehat{\mathsf{L}}$, with the upshot that a general Collapsing Lemma holds in all such instances.

To begin, we are able to extend the material in [17] to define a version of Priest's plurivalent semantics suited to deal with quantifiers.

Definition 4.18 With respect to a logical system $\mathsf{L} = \langle \mathscr{V}, \mathscr{D}, \mathbf{S}, I \rangle$, the *corresponding plurivalent semantics* $\widehat{\mathsf{L}}$ is $\langle \widehat{\mathscr{V}}, \widehat{\mathscr{D}}, \mathbf{S}, \widehat{I} \rangle$, where

- $\widehat{\mathscr{V}} = \wp^{+}(\mathscr{V})$,
- $\widehat{\mathscr{D}} = \{X \in \widehat{\mathscr{V}} \mid X \cap \mathscr{D} \neq \varnothing\}$,
- for connectives $\odot$ and quantifiers Q,
  - $f^{\odot}_{\widehat{\mathsf{L}}}(X_0, \ldots, X_{n-1}) = f^{\odot}_{\mathsf{L}}[X_0 \times \cdots \times X_{n-1}]$,
  - $f^{\mathsf{Q}}_{\widehat{\mathsf{L}}}(\mathbf{X}) = f^{\mathsf{Q}}_{\mathsf{L}}[\mathsf{Thread}(\mathbf{X})]$.

Comparing the above definition to Properties 1 and 2 of Definition 3.1 reveals an immediate symmetry between the two.

Let Id denote the identity function, that is, a function such that $\mathsf{Id}(x) = x$. Then between a system and its corresponding plurivalent semantics, Id itself always serves as a Dunn–Priest map between the two.

Observation 4.19 *For all logical systems* L *and corresponding* $\widehat{\mathsf{L}}$*,* Id *is a Dunn–Priest map from* L *to* $\widehat{\mathsf{L}}$*.*

Proof When $h = \mathsf{Id}$, we may merely drop instances of "h" from the definitions of Properties 1 and 2. This immediately can be seen to yield the definitions of the connectives of $\widehat{\mathsf{L}}$ as outlined in Definition 4.18. □

This effectively yields a generalized Collapsing Lemma in the case of plurivalent logics as follows.

Observation 4.20 *For all* L*-models* $\mathfrak{M}$ *and Dunn–Priest quotients* $\mathfrak{M}^{\sim}$ *along* Id*, if* $\mathfrak{M} \vDash_{\mathsf{L}} \varphi$*, then* $\mathfrak{M}^{\sim} \vDash_{\widehat{\mathsf{L}}} \varphi$*.*

Proof Priest's definition of $\widehat{\mathscr{D}}$ demands that $X \in \widehat{\mathscr{D}}$ whenever there exists a $\mathfrak{v} \in X$ such that $\mathfrak{v} \in \mathscr{D}$. But this is just to say that,

$$\text{for all } X \subseteq \mathscr{V},\ \text{if } X \cap \mathscr{D} \neq \varnothing,\ \text{then } \mathsf{Id}(X) \in \widehat{\mathscr{D}}.$$

By Theorem 3.6, if $v_{\mathfrak{M}}(\varphi) \in \mathscr{D}$, then $v_{\mathfrak{M}^{\sim}}(\varphi) \in \widehat{\mathscr{D}}$, that is, that $\mathfrak{M} \vDash_{\mathsf{L}} \varphi$ entails that $\mathfrak{M}^{\sim} \vDash_{\widehat{\mathsf{L}}} \varphi$. □

Just as Priest's plurivalent semantics opens the door to construct LP-like systems from, for example, Gödel–Dummett logic, so does the above observation permit constructions along the lines of the Collapsing Lemma to the models in such systems.

5 Dunn–Priest Quotients and Ultraproducts

The goal of the foregoing sections has been to provide a versatile technique for producing new models from old in a controlled fashion. The versatility of the technique can be extended—and the amount of control increased—if the Dunn–Priest quotient can be deployed in parallel with other, similar techniques. One such construction—which will be seen to play well with Dunn–Priest quotients—is the *ultraproduct*.

Classically, ultraproducts are an extremely elegant and powerful tool in mathematics in general and logic in particular. They provide a means of building nonstandard models with some particularly compelling applications in models of Peano arithmetic, nonstandard analysis, and set theory. In mathematical logic, ultraproducts provide a particularly elegant and algebraic technique for proving the compactness of a first-order logic; for example, in [8] I modified the proof of Malcev to prove all finitely valued, nondeterministic logics with first-order quantifiers to be compact. In short, access to ultraproducts is a very valuable tool.

In [7], I examined a particular case of commutativity with respect to the Dunn and Priest constructions. Namely, it was shown with respect to a family $\{\mathfrak{M}_i\}$ of classical structures each yielding alternatively a K_3- or LP-model by means of a congruence relation $\sim_i$, that a further congruence relation $\sim_{I/\mathscr{U}}$ can be naturally defined so that the ultraproduct of, for example, the Priest collapses of $\mathfrak{M}_i$ was isomorphic to the Priest collapse of the ultraproduct modulo $\sim_{I/\mathscr{U}}$, symbolically, $\prod_{i\in I}\mathfrak{M}_i^{\sim_i}/\mathscr{U} \cong [\prod_{i\in I}\mathfrak{M}_i/\mathscr{U}]^{\sim_{I/\mathscr{U}}}$.

This is essentially the claim that the order in which one takes a Dunn–Priest quotient and an ultraproduct quotient is immaterial. Not surprisingly, an analogous result holds generally for Dunn–Priest quotients generating L_1-models from L_0-models. In this section, we will outline general definitions relevant to the theory of ultraproducts before proving that this holds for arbitrary Dunn–Priest quotients and many-valued logics.

We must work with a bit less generality than we did in Section 3 when considering ultraproducts. Consider this definition.

Definition 5.1 A first-order logical system $\mathsf{L} = \langle \mathscr{V}_{\mathsf{L}}, \mathscr{D}_{\mathsf{L}}, \mathbf{S}, I_{\mathsf{L}}\rangle$ is finite when $|\mathscr{V}_{\mathsf{L}}| \in \mathbb{N}$.

As will be made clear in the remainder of the paper: that ultraproducts of L-models are themselves models in the sense of Definition 2.17 relies on the finitude of $\mathscr{V}_{\mathsf{L}}$. We thus restrict our attention in this section to finite logical systems.

5.1 Ultraproducts Initially, we define an intermediate construction from which ultraproducts are drawn.

Definition 5.2 With respect to a family of models $\{\mathfrak{M}_i \mid i \in I\}$, the *product premodel* $\prod_{i\in I}\mathfrak{M}_i$ is a tuple $\langle \prod_{i\in I} M_i, \mathbf{C}^{\prod_{i\in I}\mathfrak{M}_i}, \mathbf{F}^{\prod_{i\in I}\mathfrak{M}_i}, \mathbf{R}^{\prod_{i\in I}\mathfrak{M}_i}\rangle$, where

- $\prod_{i\in I} M_i = \{f \mid f : i \mapsto \epsilon a.a \in M_i\}$,
- $c^{\prod_{i\in I}\mathfrak{M}_i} = f : i \mapsto c^{\mathfrak{M}_i}$,
- $f^{\prod_{i\in I}\mathfrak{M}_i}(a_0,\ldots,a_{n-1}) = f : i \mapsto f^{\mathfrak{M}_i}(a_0(i),\ldots,a_{n-1}(i))$,
- $R^{\prod_{i\in I}\mathfrak{M}_i}(a_0,\ldots,a_{n-1}) = f : i \mapsto R^{\mathfrak{M}_i}(a_0(i),\ldots,a_{n-1}(i))$.

Importantly, note that the product is *not* in general a model as defined in Definition 2.17. The interpretation of, for example, a unary relation symbol R is not a function mapping elements of $\prod_{i\in I} M_i$ to truth values, but rather is a function mapping $a \in \prod_{i\in I} M_i$ to *further functions* mapping each $i \in I$ to the truth value to which $R^{\mathfrak{M}_i}$ maps $a(i)$. Classically, product structures are treated as models by adopting the convention that

$$R^{\prod_{i\in I}\mathfrak{M}_i}(a) = \begin{cases} \mathfrak{t} & \text{if, for all } i \in I,\ R^{\mathfrak{M}_i}(a(i)) = \mathfrak{t}, \\ \mathfrak{f} & \text{otherwise.} \end{cases}$$

In the many-valued case, however, there is no reason to privilege any one convention over the other. This will be overcome by taking a particular type of quotient of the product premodel.

Ultraproducts are defined as quotients of a product premodel $\prod_{i\in I}\mathfrak{M}_i$ modulo an equivalence relation induced by an ultrafilter $\mathscr{U} \subset \wp(I)$. Recall the definition of an ultrafilter.

Definition 5.3 An *ultrafilter* on a set $\wp(I)$ is a subset $\mathscr{U} \subset I$ such that:

- $I \in \mathscr{U}$;
- $\varnothing \notin \mathscr{U}$;
- if $X, Y \in \mathscr{U}$ are sets, then $X \cap Y \in \mathscr{U}$;
- if $X \in \mathscr{U}$ and $X \subseteq Y \subseteq I$, then $Y \in \mathscr{U}$.

In the sequel, we will employ a few notational conventions that will be prudent to introduce here.

Definition 5.4 With respect to a product premodel $\prod_{i\in I}\mathfrak{M}_i$, and ultrafilter $\mathscr{U} \subset \wp(I)$, and truth value $\mathfrak{v} \in \mathscr{V}$,

- $\|\varphi\|_{\mathfrak{v}} = \{i \in I \mid v_{\mathfrak{M}_i}(\varphi) = \mathfrak{v}\}$.

When discussing a metalanguage statement Φ, we will also employ the unqualified notation

- $\|\Phi\| = \{i \in I \mid \Phi \text{ is true }\}$.

For example, $\|a(i) = b(i)\|$ is the set of indices of those models $\mathfrak{M}_i$ in which $a(i)$ and $b(i)$ pick out the same element of M_i.

With respect to a product premodel $\prod_{i\in I}\mathfrak{M}_i$, an ultrafilter $\mathscr{U} \subset \wp(I)$ induces a congruence relation $\sim_{\mathscr{U}}$ between elements of $\prod_{i\in I}\mathfrak{M}_i$.

Definition 5.5 With respect to a product premodel $\prod_{i\in I}\mathfrak{M}_i$ and ultrafilter $\mathscr{U} \subset \wp(I)$,

$$a \sim_{\mathscr{U}} b \quad \text{if } \big\|a(i) = b(i)\big\| \in \mathscr{U}.$$

The equivalence class of an element $a \in \prod_{i\in I} M_i$ modulo $\sim_{\mathscr{U}}$ will be denoted by $[\![a]\!]_{\mathscr{U}}$.

Definition 5.6 An *ultraproduct* $\prod_{i\in I}\mathfrak{M}_i/\mathscr{U}$ is a model where

- $\prod_{i\in I} M_i/\mathscr{U} = \{[\![a]\!]_{\mathscr{U}} \mid a \in \prod_{i\in I} M_i\}$,
- $c^{\prod_{i\in I}\mathfrak{M}_i/\mathscr{U}} = [\![c]\!]_{\mathscr{U}}$,
- $f^{\prod_{i\in I}\mathfrak{M}_i/\mathscr{U}}([\![a_0]\!]_{\mathscr{U}}, \ldots, [\![a_{n-1}]\!]_{\mathscr{U}}) = [\![f^{\prod_{i\in I}\mathfrak{M}_i}(a_0, \ldots, a_{n-1})]\!]_{\mathscr{U}}$,
- $R^{\prod_{i\in I}\mathfrak{M}_i/\mathscr{U}}([\![a_0]\!]_{\mathscr{U}}, \ldots, [\![a_{n-1}]\!]_{\mathscr{U}}) = \imath\mathfrak{v}.\|R^{\mathfrak{M}_i}(a_0(i), \ldots, a_{n-1}(i)) = \mathfrak{v}\| \in \mathscr{U}$.

This final clause is the reason that we must restrict ourselves to finite logical systems. If the finite sequence $\mathfrak{v}_0, \ldots, \mathfrak{v}_{n-1}$ enumerates $\mathscr{V}_{\mathsf{L}}$ for some logical system L, then the family $\{\|\varphi\|_{\mathfrak{v}_0}, \ldots, \|\varphi\|_{\mathfrak{v}_{n-1}}\}$ is a finite partition of I. Properties of ultrafilters ensure that in this case, for precisely one $j < n$ is $\|\varphi\|_{\mathfrak{v}_j} \in \mathscr{U}$. We are thus safe in employing a definite description.

But if $\mathscr{V}_{\mathsf{L}}$ is infinite, one lacks this guarantee. For example, suppose that with respect to a sentence φ, for each element $\mathfrak{v}'$ of an infinite set $\mathscr{V}'$ of truth values there exists a model $\mathfrak{M}$ such that $v_{\mathfrak{M}}(\varphi) = \mathfrak{v}'$. For each $\mathfrak{v}' \in \mathscr{V}'$, let $\mathfrak{M}_{\mathfrak{v}'}$ be a model witnessing this, that is, such that $v_{\mathfrak{M}_{\mathfrak{v}'}}(\varphi) = \mathfrak{v}'$, and let $\mathscr{V}'$ itself index these models. Then let $\mathscr{U}$ be a *nonprincipal* ultrafilter on $\wp(\mathscr{V}')$, and consider the ultraproduct $\prod_{\mathfrak{v}' \in \mathscr{V}'} \mathfrak{M}_{\mathfrak{v}'}/\mathscr{U}$. For no $\mathfrak{v}'' \in \mathscr{V}'$ is $\|\varphi\|_{\mathfrak{v}''} \in \mathscr{U}$, whence the clause for relation symbols in Definition 5.6 would be ill defined.

We introduce an important type of quotient. Let $\{\mathfrak{M}_i\}$ be a family of L_0-models indexed by a set I, and for each $i \in I$, let $\sim_i$ be a congruence relation on $\mathfrak{M}_i$.

Then with respect to an ultrafilter $\mathscr{U} \subset \wp(I)$, we can define a further congruence relation $\sim_{I/\mathscr{U}}$ on $\prod_{i \in I} \mathfrak{M}_i/\mathscr{U}$ by the following definition:

$$\begin{aligned}&[\![a]\!]_{\mathscr{U}} \sim_{I/\mathscr{U}} [\![b]\!]_{\mathscr{U}} \text{ if there are } a' \in [\![a]\!]_{\mathscr{U}}, b' \in [\![b]\!]_{\mathscr{U}}\\ &\text{such that } \left\|a'(i) \sim_i b;(i)\right\| \in \mathscr{U}.\end{aligned}$$

That the definition induces a congruence relation is nontrivial, however. We prove this as follows.

Observation 5.7 *We have that $\sim_{I/\mathscr{U}}$ is a congruence relation.*

Proof Reflexivity and symmetry are nearly immediate. Every element $a \in [\![a]\!]_{\mathscr{U}}$ bears $\sim_i$ to itself in each $\mathfrak{M}_i$; hence, $\|a(i) \sim_i a(i)\| = I$ and $\|a(i) \sim_i a(i)\|$ is subsequently a member of $\mathscr{U}$. Likewise, inasmuch as $\|a(i) \sim_i b(i)\| = \|b(i) \sim_i a(i)\|$, any $a, b \in \prod_{i \in I} M_i$ witnessing that $[\![a]\!]_{\mathscr{U}} \sim_{I/\mathscr{U}} [\![b]\!]_{\mathscr{U}}$ will also serve to witness that $[\![b]\!]_{\mathscr{U}} \sim_{I/\mathscr{U}} [\![a]\!]_{\mathscr{U}}$.

For transitivity, let $[\![a]\!]_{\mathscr{U}} \sim_{I/\mathscr{U}} [\![b]\!]_{\mathscr{U}}$ and $[\![b]\!]_{\mathscr{U}} \sim_{I/\mathscr{U}} [\![c]\!]_{\mathscr{U}}$, with $a, b, c \in \Pi_{i \in I} M_i$ witnessing these facts. Then $\|a(i) \sim_i b(i)\|$ and $\|b(i) \sim_i c(i)\|$ are both members of $\mathscr{U}$. As $\mathscr{U}$ is closed under finite intersections, $\|a(i) \sim_i b(i)\| \cap \|b(i) \sim_i c(i)\| \in \mathscr{U}$. But at each $\mathfrak{M}_i$ whose index is counted as a member of this set, $a(i) \sim_i c(i)$ holds by the transitivity of $\sim_i$. Hence, by upward closure, $\|a(i) \sim_i c(i)\| \in \mathscr{U}$, whence $[\![a]\!]_{\mathscr{U}} \sim_{I/\mathscr{U}} [\![c]\!]_{\mathscr{U}}$.

Suppose that, for all $j < n$, $[\![a_j]\!]_{\mathscr{U}} \sim_{I/\mathscr{U}} [\![b_j]\!]_{\mathscr{U}}$, and let $a_0, \ldots, a_{n-1}, b_0, \ldots, b_{n-1}$ witness these facts. We now may show that $f^{\prod_{i \in I} \mathfrak{M}_i}([\![a_0]\!]_{\mathscr{U}}, \ldots, [\![a_{n-1}]\!]_{\mathscr{U}}) = f^{\prod_{i \in I} \mathfrak{M}_i}([\![b_0]\!]_{\mathscr{U}}, \ldots, [\![b_{n-1}]\!]_{\mathscr{U}})$.

By hypothesis, for all $j < n$, $\|a_j(i) \sim_i b_j(i)\| \in \mathscr{U}$. That $\mathscr{U}$ is closed under finite intersections entails that the set $\bigcap_{j<n}\|a_j(i) \sim_i b_j(i)\| \in \mathscr{U}$. Consider an arbitrary $i \in \bigcap_{j<n}\|a_j(i) \sim_i b_j(i)\|$; at $\mathfrak{M}_i$, by virtue of the fact that $\sim_i$ is a congruence relation, $\|f^{\mathfrak{M}_i}(a_0(i), \ldots, a_{n-1}(i)) = f^{\mathfrak{M}_i}(b_0(i), \ldots, b_{n-1}(i))\|$ extends this set, whence it is a member of $\mathscr{U}$. Let a and b denote some element of $\prod_{i \in I} M_i$ such that $a(i) = f^{\mathfrak{M}_i}(a_0(i), \ldots, a_{n-1}(i))$ for all $i \in I$, and mutatis mutandis for b.

Then $a \in f^{\prod_{i \in I} \mathfrak{M}_i}([\![a_0]\!]_{\mathscr{U}}, \ldots, [\![a_{n-1}]\!]_{\mathscr{U}})$, $b \in f^{\prod_{i \in I} \mathfrak{M}_i}([\![b_0]\!]_{\mathscr{U}}, \ldots, [\![b_{n-1}]\!]_{\mathscr{U}})$, and $\|a(i) \sim_i b(i)\| \in \mathscr{U}$. But this is just to say that $f^{\prod_{i \in I} \mathfrak{M}_i}([\![a_0]\!]_{\mathscr{U}}, \ldots, [\![a_{n-1}]\!]_{\mathscr{U}}) = f^{\prod_{i \in I} \mathfrak{M}_i}([\![b_0]\!]_{\mathscr{U}}, \ldots, [\![b_{n-1}]\!]_{\mathscr{U}})$. □

Lemma 5.8 *Let a and b be elements of $\prod_{i \in I} M_i$. Then*

$$[\![a]\!]_I \sim_{\mathscr{U}} [\![b]\!]_I \quad \textit{iff } [\![a]\!]_{\mathscr{U}} \sim_{I/\mathscr{U}} [\![b]\!]_{\mathscr{U}}.$$

Proof For left-to-right, suppose that $[\![a]\!]_I \sim_{\mathscr{U}} [\![b]\!]_I$. This means that the set $\|[\![a]\!]_I(i) = [\![b]\!]_I(i)\| \in \mathscr{U}$. Now consider an arbitrary model $\mathfrak{M}_i^{\sim_i}$ at which $[\![a]\!]_I(i) = [\![b]\!]_I(i)$. This is just to say that, for all $a' \in M_i$, $a' \sim_i a(i)$ *iff* $a' \sim_i b(i)$, entailing that $\| [\![a]\!]_I(i) = [\![b]\!]_I(i) \| \subseteq \|a(i) \sim_i b(i)\|$. By upward closure of $\mathscr{U}$, it follows that $\|a(i) \sim_i b(i)\| \in \mathscr{U}$. But this is precisely to say that $[\![a]\!]_{\mathscr{U}} \sim_{I/\mathscr{U}} [\![b]\!]_{\mathscr{U}}$.

For right-to-left, we prove the contrapositive. Suppose that $[\![a]\!]_I \sim_{\mathscr{U}} [\![b]\!]_I$ fails. Then $\| [\![a]\!]_I(i) = [\![b]\!]_I(i) \| \notin \mathscr{U}$. By the maximality of $\mathscr{U}$, $\| [\![a]\!]_I(i) \neq [\![b]\!]_I(i) \| \in \mathscr{U}$. Now consider models $\mathfrak{M}_i$ at which $[\![a]\!]_I(i) \neq [\![b]\!]_I(i)$; either there exists an $a' \in M_i$ such that $a' \sim_i a(i)$ although $a' \nsim_i b(i)$ or there exists an a' such that $a' \sim_i b(i)$ although $a' \nsim_i a(i)$. The transitivity of $\sim_i$ implies that $a(i) \nsim_i b(i)$. Hence, $\| [\![a]\!]_I(i) \neq [\![b]\!]_I(i) \| \subseteq \|a(i) \nsim_i b(i)\|$. Hence, $\|a(i) \nsim_i b(i)\| \in \mathscr{U}$, $\|a(i) \sim_i b(i)\| \notin \mathscr{U}$, and, consequently, that $[\![a]\!]_{\mathscr{U}} \sim_{I/\mathscr{U}} [\![b]\!]_{\mathscr{U}}$ fails. □

In the sequel, let g be a map from $\prod_{i\in I} M_i^{\sim i}/\mathscr{U}$ to $[\prod_{i\in I} M_i/\mathscr{U}]^{\sim I/\mathscr{U}}$ defined by

$$g : [\![[\![a]\!]_I]\!]_{\mathscr{U}} \mapsto [\![[\![a]\!]_{\mathscr{U}}]\!]_{I/\mathscr{U}}.$$

We are able to prove some essential features of g.

Lemma 5.9 *The function g is a bijection.*

Proof We show that g is both an injection and a surjection. That g is a surjection is immediate.

To show that g is injective, suppose for contradiction that g is not an injection, and let $[\![[\![a]\!]_I]\!]_{\mathscr{U}}$ and $[\![[\![b]\!]_I]\!]_{\mathscr{U}}$ be distinct elements of $\prod_{i\in I} M_i^{\sim i}/\mathscr{U}$ such that $g([\![[\![a]\!]_I]\!]_{\mathscr{U}}) = g([\![[\![b]\!]_I]\!]_{\mathscr{U}})$. But that g maps each to the same element implies that $[\![a]\!]_{\mathscr{U}} \sim_{I/\mathscr{U}} [\![b]\!]_{\mathscr{U}}$. By Lemma 5.8, this entails that $[\![a]\!]_I \sim_{\mathscr{U}} [\![b]\!]_I$. This, however, is just to say that $[\![[\![a]\!]_I]\!]_{\mathscr{U}} = [\![[\![b]\!]_I]\!]_{\mathscr{U}}$, contradicting the assumed distinctness of these elements. □

Lemma 5.10 *For all n-ary f and elements $a_0, \ldots, a_{n-1} \in \prod_{i\in I} M_i$, $f^{[\prod_{i\in I}\mathfrak{M}_i/\mathscr{U}]^{\sim I/\mathscr{U}}}(g([\![[\![a_0]\!]_I]\!]_{\mathscr{U}}), \ldots, g([\![[\![a_{n-1}]\!]_I]\!]_{\mathscr{U}})) = g(f^{\prod_{i\in I}\mathfrak{M}_i^{\sim i}/\mathscr{U}} \times ([\![[\![a_0]\!]_I]\!]_{\mathscr{U}}, \ldots, [\![[\![a_{n-1}]\!]_I]\!]_{\mathscr{U}}))$.*

Proof First, note that $f^{[\prod_{i\in I}\mathfrak{M}_i/\mathscr{U}]^{\sim I/\mathscr{U}}}([\![[\![a_0]\!]_{\mathscr{U}}]\!]_{I/\mathscr{U}}, \ldots, [\![[\![a_{n-1}]\!]_{\mathscr{U}}]\!]_{I/\mathscr{U}})$ is, by the definition of a Dunn–Priest quotient, $[\![f^{\prod_{i\in I}\mathfrak{M}_i/\mathscr{U}}([\![a_0]\!]_{\mathscr{U}}, \ldots, [\![a_{n-1}]\!]_{\mathscr{U}})]\!]_{I/\mathscr{U}}$. By the interpretation of f in an ultraproduct, this is equal to $[\![[\![f^{\prod_{i\in I}\mathfrak{M}_i}(a_0, \ldots, a_{n-1})]\!]_{\mathscr{U}}]\!]_{I/\mathscr{U}}$.

Likewise, $f^{\prod_{i\in I}\mathfrak{M}_i^{\sim i}/\mathscr{U}}([\![[\![a_0]\!]_I]\!]_{\mathscr{U}}, \ldots, [\![[\![a_{n-1}]\!]_I]\!]_{\mathscr{U}})$ is by definition $[\![f^{\prod_{i\in I}\mathfrak{M}_i^{\sim i}}([\![a_0]\!]_I, \ldots, [\![a_{n-1}]\!]_I)]\!]_{\mathscr{U}}$. As each $\sim_i$ is a congruence relation, this is just the object $[\![[\![f^{\prod_{i\in I}\mathfrak{M}_i}(a_0, \ldots, a_{n-1})]\!]_I]\!]_{\mathscr{U}}$.

Then, for each a_j, $g([\![[\![a_j]\!]_I]\!]_{\mathscr{U}}) = [\![[\![a_j]\!]_{\mathscr{U}}]\!]_{I/\mathscr{U}}$. So we infer that $f^{[\prod_{i\in I}\mathfrak{M}_i/\mathscr{U}]^{\sim I/\mathscr{U}}}(g([\![[\![a_0]\!]_I]\!]_{\mathscr{U}}), \ldots, g([\![[\![a_{n-1}]\!]_I]\!]_{\mathscr{U}}))$ is identical to the element $f^{[\prod_{i\in I}\mathfrak{M}_i/\mathscr{U}]^{\sim I/\mathscr{U}}}([\![[\![a_0]\!]_{\mathscr{U}}]\!]_{I/\mathscr{U}}, \ldots, [\![[\![a_{n-1}]\!]_{\mathscr{U}}]\!]_{I/\mathscr{U}})$, which by the above observation is just $[\![[\![f^{\prod_{i\in I}\mathfrak{M}_i}(a_0, \ldots, a_{n-1})]\!]_{\mathscr{U}}]\!]_{I/\mathscr{U}}$.

This is equal to $g([\![[\![f^{\prod_{i\in I}\mathfrak{M}_i}(a_0, \ldots, a_{n-1})]\!]_I]\!]_{\mathscr{U}})$, which by the above reasoning is precisely $g(f^{\prod_{i\in I}\mathfrak{M}_i^{\sim i}/\mathscr{U}}([\![[\![a_0]\!]_I]\!]_{\mathscr{U}}, \ldots, [\![[\![a_{n-1}]\!]_I]\!]_{\mathscr{U}}))$. □

Lemma 5.11 *For all n-ary R and elements $a_0, \ldots, a_{n-1} \in \prod_{i\in I} M_i$, $R^{[\prod_{i\in I}\mathfrak{M}_i/\mathscr{U}]^{\sim I/\mathscr{U}}}(g([\![[\![a_0]\!]_I]\!]_{\mathscr{U}}), \ldots, g([\![[\![a_{n-1}]\!]_I]\!]_{\mathscr{U}})) = R^{\prod_{i\in I}\mathfrak{M}_i^{\sim i}/\mathscr{U}} \times ([\![[\![a_0]\!]_I]\!]_{\mathscr{U}}, \ldots, [\![[\![a_{n-1}]\!]_I]\!]_{\mathscr{U}})$.*

Proof $R^{[\prod_{i\in I}\mathfrak{M}_i/\mathscr{U}]^{\sim I/\mathscr{U}}}(g(\llbracket\llbracket a_0\rrbracket_I\rrbracket_{\mathscr{U}}),\ldots,g(\llbracket\llbracket a_{n-1}\rrbracket_I\rrbracket_{\mathscr{U}}))$ is defined as $R^{[\prod_{i\in I}\mathfrak{M}_i/\mathscr{U}]^{\sim I/\mathscr{U}}}(\llbracket\llbracket a_0\rrbracket_{\mathscr{U}}\rrbracket_{I/\mathscr{U}},\ldots,\llbracket\llbracket a_{n-1}\rrbracket_{\mathscr{U}}\rrbracket_{I/\mathscr{U}})$. This is by definition of $\sim_{I/\mathscr{U}}$ $h(v^{\mathsf{Out}}_{\prod_{i\in I}\mathfrak{M}_i/\mathscr{U}(\prod_{i\in I}M_i/\mathscr{U})}(R(\underline{\llbracket a_0\rrbracket}_{\mathscr{U}},\ldots,\underline{\llbracket a_{n-1}\rrbracket}_{\mathscr{U}})))$.

Now, as $v^{\mathsf{Out}}_{\prod_{i\in I}\mathfrak{M}_i/\mathscr{U}(\prod_{i\in I}M_i/\mathscr{U})}(R(\underline{\llbracket a_0\rrbracket}_{\mathscr{U}},\ldots,\underline{\llbracket a_{n-1}\rrbracket}_{\mathscr{U}}))$ is the set $\{\mathfrak{v}\mid \|\mathfrak{v}\in v^{\mathsf{Out}}_{\mathfrak{M}_i(M_i)}(R(\underline{a_0(i)},\ldots,\underline{a_{n-1}(i)}))\|\in\mathscr{U}\}$, this reduces to the unique element $h(\{\mathfrak{v}\mid \|\mathfrak{v}\in v^{\mathsf{Out}}_{\mathfrak{M}_i(M_i)}(R(\underline{a_0(i)},\ldots,\underline{a_{n-1}(i)}))\|\in\mathscr{U}\})$.

But this is just the element $\iota\mathfrak{v}.\|h(v^{\mathsf{Out}}_{\mathfrak{M}_i(M_i)}(R(\underline{a_0(i)},\ldots,\underline{a_{n-1}(i)})))=\mathfrak{v}\|\in\mathscr{U}$, which by definition is $\iota\mathfrak{v}.\|R^{\mathfrak{M}_i^{\sim i}}(\llbracket a_0\rrbracket_I(i),\ldots,\llbracket a_{n-1}\rrbracket_I(i))=\mathfrak{v}\|\in\mathscr{U}$. By the definition of an ultraproduct, this is $R^{\prod_{i\in I}\mathfrak{M}_i^{\sim i}/\mathscr{U}}(\llbracket\llbracket a_0\rrbracket_I\rrbracket_{\mathscr{U}},\ldots,\llbracket\llbracket a_{n-1}\rrbracket_I\rrbracket_{\mathscr{U}})$, as expected. □

We now proceed to the main result of the section.

Theorem 5.12 *Let $\prod_{i\in I}\mathfrak{M}_i$ be a product of L_0-models, for each $i\in I$, let $\sim_i$ be a congruence relation inducing a Dunn–Priest quotient $\mathfrak{M}_i^{\sim i}$, and let $\mathscr{U}\subset\wp(I)$ be an ultrafilter. Then*

$$\prod_{i\in I}\mathfrak{M}_i^{\sim i}/\mathscr{U}\cong\Big[\prod_{i\in I}\mathfrak{M}_i/\mathscr{U}\Big]^{\sim I/\mathscr{U}}.$$

Proof The candidate isomorphism is the function g previously defined. By Lemma 5.9, g is bijective. By Lemma 5.10, g preserves the behavior of functions. By Lemma 5.11, g also preserves the behavior of relation symbols.

Definition 2.21 suggests that this is precisely what it means to be an isomorphism between $\prod_{i\in I}\mathfrak{M}_i^{\sim i}/\mathscr{U}$ and $[\prod_{i\in I}\mathfrak{M}_i/\mathscr{U}]^{\sim I/\mathscr{U}}$. Hence, g witnesses that $\prod_{i\in I}\mathfrak{M}_i^{\sim i}/\mathscr{U}\cong[\prod_{i\in I}\mathfrak{M}_i/\mathscr{U}]^{\sim I/\mathscr{U}}$. □

By this observation, we are free to consider Dunn–Priest quotients of ultraproducts or ultraproducts of such quotients as equivalent.

6 Conclusions

The theorems of Dunn and Priest have been very versatile tools. Generalizing this construction to arbitrary many-valued logics increases the utility of the construction, and its well-behavior with respect to ultraproducts, in particular, ensures that it can be deployed in nonclassical mathematics unproblematically.

An interesting question is what further applications can be found for Dunn–Priest quotients and when suitable maps exist. Clearly, there is no a priori prohibition on a Dunn–Priest map running from a logical system to itself. In, for example, a fuzzy logic for which the interval $[0,1]$ serves as set of truth values, it seems that a map from $\wp^+([0,1])\to[0,1]$ determined by some topological properties of subsets of the interval might give something like a "derivative" of a model. Are such constructions possible and, if so, useful? When can a Dunn–Priest map successfully map a logical system to itself?

There is also an interpretative matter. In particular cases, for example, inconsistent models of arithmetic, there are some independent interpretations concerning a particular quotient; Priest [14] uses a "greatest number" theme to motivate certain Dunn–Priest quotients of the standard model of arithmetic. But in general, there is no a priori unifying interpretation of Dunn–Priest quotients in general.

The set $\mathscr{V}_{\mathsf{FDE}}$ of four truth values in the Dunn–Belnap logic FDE are interpreted in, for example, Belnap [1] as epistemic accounts of the data—possibly contradictory—received by a computer from multiple sources. In [18], Shramko and Wansing considered complications of this picture, by considering *networks* of such computers. A network in which a number of "Belnap computers" report to a central system is captured by a 16-valued logic corresponding to the logic of the power set of $\mathscr{V}_{\mathsf{FDE}}$. This appears to give rise to a reading of Dunn–Priest quotients such that the "collapsed" model represents collective disagreement between agents concerning some atomic formulae. Can anything more be made of this interpretation?

Finally, Priest [17] describes "general plurivalence" as a further generalization of the notion of plurivalent semantics by considering not only the *positive* power set of a set of truth values $\mathscr{V}$, but the entire power set, so that $\varnothing$ must be interpreted as a distinct value. As our treatment of many-valued model theory has treated all interpretations of n-ary predicates as *total* functions from M^n to $\mathscr{V}$, this value will never be yielded in a quotient. Given the natural relationship between plurivalence and Dunn–Priest quotients, it is a reasonable question to ask whether such a value can be represented or generated in a Dunn–Priest quotient, perhaps by allowing partial functions as interpretations of predicate symbols.

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Notes

1. Dunn's proof of the nontriviality of relevant arithmetic relies on a similar observation.

2. We borrow the notation of [9], Lloyd Humberstone's companion piece to [17]. Priest uses the decoration "$\dot{M}$" to denote the plurivalent system corresponding to a matrix M.

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