Lowness for Difference Tests

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Abstract   We show that being low for difference tests is the same as being computable and therefore lowness for difference tests is not the same as lowness for difference randomness. This is the first known example of a randomness notion where lowness for the randomness notion and lowness for the test notion do not coincide. Additionally, we show that for every incomputable set A, there is a difference test $\mathcal{T}^A$ relative to A which cannot even be covered by finitely many unrelativized difference tests.

Along with the study of random sequences comes the study of sequences that are far from random. There are many different ways in which an infinite binary sequence can be said to be far from random, such as triviality, lowness, lowness for tests, or being a base (see Downey and Hirschfeldt [3], Nies [10]). In this paper we will focus on the lowness notions.

The concept of lowness has been studied in many contexts. In general, a sequence is said to be low if it is indistinguishable from a computable sequence in the given context. For instance, a sequence A is low with respect to the Turing jump if $A' \equiv_T 0'$, that is, if its jump has the same Turing degree as that of a computable sequence. Lowness has also been studied in the contexts of learning theory (see Slaman and Solovay [11]) and recursive model theory (see Franklin and Solomon [7]), but it is most prominent in algorithmic randomness (see [5]).

The first notion is that of lowness for $\mathcal{R}$, and the second is that of lowness for tests.

Definition 1   A sequence A is *low* for a randomness notion $\mathcal{R}$ if $\mathcal{R}^A = \mathcal{R}$, that is, if it does not make any random sequence appear to be nonrandom when it is used as an oracle.

Definition 2   Let $\mathcal{R}$ be a randomness notion. A sequence A is *low for $\mathcal{R}$-tests* if for every $\mathcal{R}$-test relative to A $\langle V_i^A \rangle$ there is an unrelativized $\mathcal{R}$-test $\langle U_i \rangle$ such that $\bigcap_i V_i^A \subseteq \bigcap_i U_i$.

It is clear that for any notion $\mathcal{R}$, any sequence that is low for $\mathcal{R}$-tests must also be low for $\mathcal{R}$, but the converse is not obviously true since more than one unrelativized test may be required to cover a relativized test. Whenever a universal test exists, as for Martin-Löf randomness, these notions are clearly equivalent. If this is not the case, some work must be done to determine whether this equivalence holds. For instance, there is no universal Schnorr test. However, Kjos-Hanssen, Nies, and Stephan [8] showed that every sequence that is low for Schnorr randomness is low for Schnorr tests by characterizing them as the recursively traceable sets, the class which Terwijn and Zambella [13] had shown were precisely the sets that were low for Schnorr randomness.

Bienvenu and Miller [1] have recently characterized lowness in terms of open covers in a way that suggests that lowness for tests and lowness will coincide for any randomness notion where the tests are composed of Σ_1^0-sets. In this paper, we consider difference randomness, in which the test elements are differences of Σ_1^0-sets, and show that lowness for difference tests does not coincide with lowness for difference randomness. This is the first example of a randomness notion for which these classes are known to be different and suggests that there is no intrinsic reason for these classes to coincide. One could make the argument that difference randomness is not a natural randomness notion and that this example is therefore pathological; however, the difference random sets are precisely the Turing incomplete Martin-Löf random sets (see Franklin and Ng [6]) and as such form a very natural subclass of the Martin-Löf random sets (see Stephan [12]).

1 Introduction

In Section 2, we prove our main theorem.

Theorem 3   *If A is low for difference tests, then A is computable.*

Then, in Section 3, we show that incomputable sets are very far from being low for difference randomness.

Theorem 4   *If A is incomputable, then there is a difference test $\mathcal{T}^A$ relative to A which cannot be captured by finitely many unrelativized difference tests.*

Our notation is standard and will generally follow [10]. We will not define difference randomness as Franklin and Ng [6] originally did, but the following formulation is easily seen to be equivalent and is more appropriate for our purposes.

Definition 5   A *difference test* is a pair $(\langle \mathcal{U}_n \rangle, \mathcal{C})$, where $\langle \mathcal{U}_n \rangle$ is a uniformly Σ_1^0-sequence and $\mathcal{C}$ is a Π_1^0-class such that for all n, the measure condition

$$\mu(\mathcal{U}_n \cap \mathcal{C}) < 2^{-n}$$

is satisfied.

We say that a difference test *captures* a real X if $X \in \bigcap_n \mathcal{U}_n \cap \mathcal{C}$. A real X is called *difference random* if it is not captured by any difference test.

Definition 6   A real X is *low for difference randomness* if every real which is difference random is difference random relative to X. A real X is *low for difference tests* if for every difference test $(\langle \mathcal{U}_n^X \rangle, \mathcal{C}^X)$ relative to X, there is an unrelativized difference test $(\langle \mathcal{V}_n \rangle, \mathcal{D})$ such that

$$\bigcap_n \mathcal{U}_n^X \cap \mathcal{C}^X \subseteq \bigcap_n \mathcal{V}_n \cap \mathcal{D}. \tag{1}$$

If equation (1) holds, then we say that $\mathcal{T}^X = (\langle \mathcal{U}_n^X \rangle, \mathcal{C}^X)$ *is covered by* $\mathcal{T}' = (\langle \mathcal{V}_n \rangle, \mathcal{D})$. This can be generalized to finitely many tests $\mathcal{T}_1 \cdots \mathcal{T}_n$ in the obvious way.

All of the proofs in this paper are done in the space $[0, 1]$, but all of the results hold identically for 2^ω as there is a computable measure isomorphism $X \mapsto 0.X$ from 2^ω to $[0, 1]$.

2 Lowness for Difference Tests

To show that no incomputable set is low for difference randomness, we begin with hyperimmune sets, that is, those sets that compute a function that cannot be dominated by any computable function. We show how to use such a function f to build an f-difference test $\mathcal{T}^f$ which is not covered by any single unrelativized difference test $\mathcal{T}'$. The first part of our strategy is to ensure that $\mathcal{T}^f$ captures Ω, which ensures that $\mathcal{T}^f$ is not covered by any difference test $\mathcal{T}'$ which does not capture Ω; we then need only concern ourselves with difference tests $\mathcal{T}'$ which capture Ω. Since Ω is Martin-Löf random, a difference test $\mathcal{T}'$ which captures Ω cannot be a Martin-Löf test. Moreover, this must hold locally in any interval around Ω. Such a $\mathcal{T}'$ must be missing many open intervals near Ω, which will be revealed as they show up in the Π_1^0-class that makes $\mathcal{T}'$ a difference test rather than a Martin-Löf test. When we see such an interval, we can ensure that some rational in the interval is captured by $\mathcal{T}^f$. We use places where f escapes domination to ensure that we see such an interval soon enough that we can use it to diagonalize, that is, before we have put it into the complement of the $\Pi_1^0[f]$-class that we are building as part of the test $\mathcal{T}^f$.

Theorem 7   *Let A have hyperimmune degree. Then A is not low for difference tests.*

Proof   Let $f \leq_T A$ escape domination by every computable function g. We will use f to define a difference test $(\langle \mathcal{U}_n^A \rangle, \mathcal{C}^A)$. Let $\langle \Omega_s \rangle$ be a monotone approximation of the halting probability Ω. We define the difference test by setting

$$\mathcal{U}_n^A = [0, \Omega + 2^{-n})$$

and constructing $\mathcal{C}^A$ in stages. Let $\langle \mathcal{D}_e \rangle$ be an effective enumeration of all Π_1^0-classes, where $\mathcal{D}_e[s]$ is the stage s approximation to $\mathcal{D}_e$. During the construction, we will enumerate a set of rationals Q; when a rational is enumerated into Q, it will be enumerated for some $\mathcal{D}_e$, and the requirement $\mathcal{R}_e$ will henceforth be marked as *satisfied*.

2.1 Construction Stage 0: Let $\mathcal{C}^A[0] = [0, 1]$, and let $Q_0 = \emptyset$.
   Stage s: For each $e < s$ with $\mathcal{R}_e$ unsatisfied, search for the least rational q such that

   (1) $\Omega_{s-1} < q < \Omega_{f(s)}$,
   (2) $q \notin \mathcal{D}_e[f(s)]$, and
   (3) the denominator of q is at most $f(s)$.

For each e such that some q is found, enumerate that q into Q, and mark $\mathcal{R}_e$ as satisfied. Then remove the set $[0, \Omega_s] \setminus Q$ (which is a finite union of open[1] intervals) from $\mathcal{C}^A$.

2.2 Verification

Lemma 8   *We have $\mathcal{C}^A = Q \cup [\Omega, 1]$.*

Proof   We will show, by induction on s, that for all s,

$$\mathcal{C}^A[s] \supseteq Q_s \cup [\Omega_s, 1].$$

At stage 0, $\mathcal{C}^A[0] = [0, 1]$, so this is certainly true. Assume that

$$\mathcal{C}^A[s-1] \supseteq Q_{s-1} \cup [\Omega_{s-1}, 1].$$

If any rational numbers are enumerated into Q at stage s, they must be at least Ω_{s-1}, so

$$\mathcal{C}^A[s-1] \supseteq Q_s \cup [\Omega_s, 1].$$

At stage s, no element of Q_s is removed from $\mathcal{C}^A$, nor is anything larger than Ω_s removed, and therefore

$$\mathcal{C}^A[s] \supseteq Q_s \cup [\Omega_s, 1].$$

By induction, this holds for all s. Taking the lim inf on both sides, we have

$$\mathcal{C}^A = \liminf_s \mathcal{C}^A[s] \supseteq \liminf_s Q_s \cup [\Omega_s, 1].$$

Thus

$$\mathcal{C}^A \supseteq Q \cup [\Omega, 1].$$

On the other hand, if $x < \Omega$ and $x \notin Q$, then there is some s such that $x < \Omega_s$. Then x is removed from $\mathcal{C}^A$ by the end of stage s. Thus

$$\mathcal{C}^A = Q \cup [\Omega, 1],$$

as claimed.   □

From this, we can easily see that the measure condition is satisfied, that is, $(\langle \mathcal{U}_n^A \rangle, \mathcal{C}^A)$ is a difference test, and that

$$\bigcap_n \mathcal{U}_n^A \cap \mathcal{C}^A = [0, \Omega] \cap \left(Q \cup [\Omega, 1] \right) = Q \cup \{\Omega\}.$$

Lemma 9   *There is no difference test such that its intersection contains*

$$\bigcap_n \mathcal{U}_n^A \cap \mathcal{C}^A = Q \cup \{\Omega\}.$$

Proof   Consider a difference test $(\langle \mathcal{V}_n \rangle, \mathcal{D})$. There are two cases: when this difference test captures Ω, and when it does not. It the test fails to capture Ω, then certainly the intersection cannot contain $Q \cup \{\Omega\}$. Suppose that the difference test does capture Ω. Then for every s, the interval (Ω_s, Ω) cannot be a subset of $\mathcal{D}$, or else $\langle (\Omega_s, \Omega + 2^{-n-1}) \cap \mathcal{V}_{n+1} \rangle$ would satisfy the Martin-Löf measure condition and would therefore be a Martin-Löf test capturing Ω.

Let e be an index of $\mathcal{D}$. If $\mathcal{R}_e$ is ever marked as satisfied, there is some $q \notin \mathcal{D}_e$ which is enumerated into Q. By the above lemma, $q \in \bigcap_n U_n^A \cap \mathcal{C}^A$ but $q \notin \bigcap_n \mathcal{V}_n \cap \mathcal{D}$, so q witnesses that the difference test $(\langle U_n^A \rangle, \mathcal{C}^A)$ is not covered by the difference test $(\langle \mathcal{V}_n \rangle, \mathcal{D})$. So it suffices to show that $\mathcal{R}_e$ is eventually marked satisfied.

In any interval of the form (Ω_{s-1}, Ω), there must be some rational subinterval removed from $\mathcal{D} = \mathcal{D}_e$. We will define a computable function $g : \omega \to \omega$. To define $g(s)$, search for some natural number n and rational number q with denominator

at most n such that $\Omega_{s-1} < q < \Omega_n$ and some interval containing q is removed from $\mathcal{D}_e$ by stage n; let $g(s)$ be the least such n. The above argument shows that such an n will always be found, and so g is a total computable function. Since f escapes domination by g, there is a stage $s > e$ with $f(s) > g(s)$. During stage s of the construction, the search for a rational number q meeting conditions (1)–(3) will succeed, so $\mathcal{R}_e$ will be marked as satisfied during stage s if it is not already so marked. $\qquad\square$

Therefore, A is not low for difference tests. Theorem 7 is proved. $\qquad\square$

Corollary 10   *If A is low for difference tests, then A is computable.*

Proof   If A is Δ_2^0 and incomputable, then A is hyperimmune, whence A is not low for difference tests.

Downey, Nies, Weber, and Yu [4] showed that the K-trivial sets are the sets that are low for weak 2-randomness versus Martin-Löf randomness, and since difference randomness is intermediate between these notions (see [6]), we know that every set that is low for difference randomness is K-trivial.

If A is not Δ_2^0, then A is not K-trivial and hence not low for difference randomness. A fortiori, A is not low for difference tests. $\qquad\square$

In fact, the sets which are low for difference randomness are precisely the K-trivials. Franklin and Ng [6] observed that for computably enumerable (c.e.) sets A, lowness for difference randomness is equivalent to another property, being *weakly ML-noncuppable*, meaning that there is no Martin-Löf random set $B \not\geq_T \emptyset'$ such that $\emptyset' \leq_T A \oplus B$. Weak ML-noncuppability was recently shown by Day and Miller [2] to be the same as K-triviality. Since every K-trivial set is below a c.e. K-trivial (see Nies [9]), every K-trivial set is low for difference randomness.

This equivalence gives us the following additional corollary.

Corollary 11   *Lowness for difference tests and lowness for difference randomness do not coincide.*

Remark 12   It is worth pointing out that in order to satisfy requirement $\mathcal{R}_e$— diagonalizing against the eth difference test—the construction completely ignores the sequence of Σ_1^0-classes which forms the first half of the test and focuses entirely on the Π_1^0-class $\mathcal{D}_e$ which forms the second half of the test. As such, it does not matter whether we diagonalize against an enumeration of all difference tests or merely an enumeration of all Π_1^0-classes.

The verification, however, does depend on the entire difference test, because whether we succeed at diagonalizing against a difference test using some element of Q or using Ω depends on whether the difference test captures Ω.

3   How Far from Low?

Theorem 3 tells us that no incomputable set is low for difference tests but does so in a very weak way. Examining the test $\mathcal{T}^A$ built by the proof, one sees that apart from Ω, every real captured by $\mathcal{T}^A$ is rational. Hence each real captured by $\mathcal{T}^A$ is either captured by a single difference test capturing Ω or by a universal Martin-Löf test. One is therefore led to ask whether this is an essential feature of the proof. We know that incomputable K-trivials are low for difference randomness but not difference tests, but might they be somehow very close to being low for difference tests? In this

section we show that this is not the case and that in fact every incomputable set is as far as possible from being low for difference tests in the following way.

Theorem 13   *If A is incomputable, there is a difference test $\mathcal{T}^A$ relative to A which cannot be covered by finitely many unrelativized difference tests.*

To prove this theorem, we can again focus on sets of hyperimmune degree, since incomputable sets not of hyperimmune degree cannot even be low for difference randomness. We use a variation on our strategy from Section 2. Again we build a test $\mathcal{T}^f$ and ensure that it captures Ω, but now we have to diagonalize against finite unions of difference tests of the form $\mathcal{T}_1 \cdots \mathcal{T}_m$. Again, in order to capture $\mathcal{T}^f$ at least one of these tests, say, $\mathcal{T}_i$, must contain Ω. We can use the same strategy from before to find some interval I in the complement of $\mathcal{T}_i$. We then repeat the same strategy inside I but now only have to worry about $m-1$ of the tests from the union. Continuing like this for up to m steps if necessary, we diagonalize against the entire ensemble $\mathcal{T}_1 \cdots \mathcal{T}_m$. To prevent strategies for separate requirements from interfering with each other, we will work with infinitely many disjoint open subintervals of $[0, 1]$,

$$\{I_E : E \subset \omega \text{ is finite}\}.$$

(We could have done this in the previous construction as well, but it was unnecessary there.)

Theorem 14   *Let A have hyperimmune degree. Then there is a difference test relative to A which cannot be covered by any finite collection of unrelativized difference tests.*

Proof   As before, we will build an A-difference test $\mathcal{T}^A = (\langle \mathcal{U}_n^A \rangle, \mathcal{C}^A)$, defined using $f \leq_T A$, which escapes every computable function. Let $\langle \mathcal{T}_e \rangle$ be an enumeration of all unrelativized difference tests, where $\mathcal{T}_e = (\langle V_n^e \rangle, \mathcal{D}_e)$. We need to satisfy the requirements

$$\mathcal{R}_E : \mathcal{T}^A \text{ is not covered by } \{\mathcal{T}_e : e \in E\},$$

where E ranges over all finite subsets of the natural numbers. We will work for requirement $\mathcal{R}_E$ on the interval I_E, so in our construction we may fix a specific requirement and define $\langle U_n^A \rangle$ and $\mathcal{C}^A$ restricted to that interval for that requirement, ignoring all other requirements. To ensure that we can put all the separate pieces together and get a valid difference test, we need only ensure that

$$\mu(U_n^A \cap \mathcal{C}^A \cap I_E) \leq \epsilon_E 2^{-n},$$

where ϵ_E is a computable real-valued function of E (a finite set of natural numbers) with $\sum_E \epsilon_E = 1$.

Once again, we fix a monotone approximation $\langle \Omega_s \rangle$ of Ω. For an interval I with rational endpoints a, b, let $\Omega_I = a + (b-a)\Omega$; we see immediately that Ω_I has the following properties:

(1) Ω_I is in the interior of I,
(2) $\Omega_I[s] := a + (b-a)\Omega_s$ is a monotone approximation to Ω_I, with range contained in I,
(3) Ω_I is Martin-Löf random, and
(4) Ω_I is not difference random.

We define a procedure $\mathcal{M}(I, E, \epsilon)$ that works for a requirement $\mathcal{R}_E$ inside the interval I and adds measure at most $\epsilon 2^{-n}$ to the nth level of the test we build. We need to succeed at requirement $\mathcal{R}_E$ inside the interval I_E; the reason we define things in terms of procedures $\mathcal{M}(I, E, \epsilon)$ is that if $E = E' \cup \{e\}$ and we succeed at requirement $\mathcal{R}_{E'}$ inside an interval I in the complement of $\mathcal{D}_e$, then we also succeed at requirement $\mathcal{R}_E$. Therefore, our strategy for $\mathcal{R}_E$ involves finding such a pair e, E' and interval I and recursing to perform the strategy for $\mathcal{R}_{E'}$ inside that interval.

The definition of the procedure $\mathcal{M}(I, E, \epsilon)$ depends on the stage s_0 when it is called and on whether the set E is empty. If E is nonempty, then it looks for a place to recurse and call a subprocedure for $E \setminus \{e\}$, but if E is the empty set, then it merely has to ensure that $\mathcal{T}^A$ captures some element of the interval I.

3.1 Procedure $\mathcal{M}(I, E, \epsilon)$, for E nonempty Stage s_0: We assume that $\mathcal{U}_n^A[s_0] \cap I = \emptyset$ for all n and that $\mathcal{C}^A[s_0] \cap I = I$. Procedure $\mathcal{M}(I, E, \epsilon)$ has two states, "searching" and "found," and begins in the "searching" state.

Stage $s > s_0$: We have the following.

- *Step* 1: If this procedure is in the "found" state, then go to step (2) below. If this procedure is in the "searching" state, then search for the least pair (a, b) of rationals according to the lexicographic order on $\mathbb{Q}^2$ such that
  (1) $\Omega_I[s - 1] < a < b < \Omega_I[f(s)]$,
  (2) there is some $e \in E$ such that the open interval (a, b) is a subset of the complement of $\mathcal{D}_e[f(s)]$, and
  (3) the denominators of a and b are at most $f(s)$.
  If such a pair is found, then switch to the "found" state and begin the procedure $\mathcal{M}((a, b), E \setminus \{e\}, \epsilon/2)$.
- *Step* 2: If this procedure is in the "searching" state, then let $\hat{I} = I$; if this procedure is in the "found" state and (a, b) is the interval it found, then let $\hat{I} = I \setminus [a, b]$. Enumerate $\hat{I} \cap [0, \Omega_I[s] + \epsilon 2^{-n-1})$ into $\mathcal{U}_n^A$ for each n, and enumerate $\hat{I} \cap [0, \Omega_I[s])$ into the complement of $\mathcal{C}^A$.

3.2 Procedure $\mathcal{M}(I, E, \epsilon)$, for E empty Stage s_0: We assume that $\mathcal{U}_n^A[s_0] \cap I = \emptyset$ for all n, and that $\mathcal{C}^A[s_0] \cap I = I$. Let q be the midpoint of the interval I.

Stage $s_0 + 1$: Enumerate I into $\mathcal{U}_n$, and enumerate $I \setminus \{q\}$ into the complement of $\mathcal{C}^A$.

No action will be taken by this procedure at stages after $s_0 + 1$.

3.3 Construction Let E_s denote the finite set of natural numbers with canonical index s.

At stage s, call the procedure $\mathcal{M}(I_{E_s}, E_s, \epsilon_{E_s})$, and take one step in each procedure already called.

3.4 Verification When one procedure $\mathcal{M}(I, E, \epsilon)$ calls another procedure $\mathcal{M}(I', E', \epsilon')$, call the latter procedure a *subprocedure* of the former. We inductively define $\mathcal{M}(I', E', \epsilon')$ to be a *child* of $\mathcal{M}(I, E, \epsilon)$ (and the latter a *parent* of the former) if $\mathcal{M}(I', E', \epsilon')$ is a subprocedure of $\mathcal{M}(I, E, \epsilon)$ or a child of a subprocedure of $\mathcal{M}(I, E, \epsilon)$. Looking at the procedure and construction definitions, one can verify the following properties:

- Each procedure has at most one subprocedure.

- No action taken by $\mathcal{M}(I, E, \epsilon)$ can affect any part of $\mathcal{U}_n^A$ or $\mathcal{C}^A$ outside of I.
- If $\mathcal{M}(I, E, \epsilon)$ calls $\mathcal{M}(I', E', \epsilon')$ at a stage s, then I' is a proper subinterval of I, and the procedure $\mathcal{M}(I, E, \epsilon)$ has not affected any part of $\mathcal{U}_n^A$ or $\mathcal{C}^A$ inside I' before the end of stage s and will never subsequently do so.
- The only procedures called by the construction are the procedures $\mathcal{M}(I_E, E, \epsilon_E)$ and their children.

By the last two properties, if two procedures $\mathcal{M}(I, E, \epsilon)$ and $\mathcal{M}(I', E', \epsilon')$ are called during the construction, then either I and I' are disjoint or one is a proper subinterval of the other and the corresponding procedure is a child of the other procedure. For each procedure $\mathcal{M}(I, E, \epsilon)$ called during the construction, define $\hat{I}$ to be $I \setminus I'$, where $\mathcal{M}(I', E', \epsilon')$ is the unique subprocedure of $\mathcal{M}(I, E, \epsilon)$ (or $\hat{I} = I$ if $\mathcal{M}(I, E, \epsilon)$ has no subprocedure). Since at most one procedure is called with interval I as the first argument and each procedure calls at most one subprocedure, this is well defined. Finally, the four properties together imply that the action of procedure $\mathcal{M}(I, E, \epsilon)$ is the only thing that can affect any part of $\mathcal{U}_n^A$ or $\mathcal{C}^A$ inside $\hat{I}$ during the construction.

If E is nonempty, then the procedure $\mathcal{M}(I, E, \epsilon)$ enumerates all of the interval $I \cap [0, \Omega_I + \epsilon 2^{-n-1})$ into $\mathcal{U}_n^A$, except for the interval I' if a subprocedure $\mathcal{M}(I', E', \epsilon')$ is called. Therefore,

$$\hat{I} \cap \mathcal{U}_n^A = \hat{I} \cap [0, \Omega_I + \epsilon 2^{-n-1}).$$

If E is nonempty, then $\mathcal{M}(I, E, \epsilon)$ also enumerates all of the interval $I \cap [0, \Omega_I)$ into the complement of $\mathcal{C}^A$, excluding the interval I' if a subprocedure $\mathcal{M}(I', E', \epsilon')$ is called. Hence if a procedure $\mathcal{M}(I, E, \epsilon)$ is called during the construction with E nonempty, then $\hat{I} \cap \mathcal{C}^A = \hat{I} \cap [0, \Omega_I)$. If E is empty, then $\hat{I} = I$, $I \subset U_n^A$ for all n, and $I \cap \mathcal{C}^A = \{q\}$, where q is the midpoint of I.

Lemma 15   *If the procedure $\mathcal{M}(I, E, \epsilon)$ is called during the construction, then*

$$\mu(I \cap \mathcal{U}_n^A \cap \mathcal{C}^A) \leq \epsilon 2^{-n}.$$

Proof   We prove the lemma by induction on the size of E. If E is empty, then by the above we have $I = \hat{I}$ and $I \cap \mathcal{U}_n^A \cap \mathcal{C}^A$ is the singleton $\{q\}$, where q is the midpoint of I, and hence I has measure 0. If E is nonempty and no subprocedure is called by $\mathcal{M}(I, E, \epsilon)$, then we again have $I = \hat{I}$, and

$$I \cap \mathcal{U}_n^A \cap \mathcal{C}^A = I \cap [\Omega_I, \Omega_I + \epsilon 2^{-n-1}),$$

so the measure is at most $\epsilon 2^{-n-1}$. If E is nonempty and $\mathcal{M}(I, E, \epsilon)$ calls a subprocedure $\mathcal{M}(I', E', \epsilon')$, then $|E'| = |E| - 1$ and $\epsilon' = \epsilon/2$, so by the inductive hypothesis,

$$\mu(I' \cap \mathcal{U}_n^A \cap \mathcal{C}^A) \leq \epsilon 2^{-n-1}.$$

Also,

$$\hat{I} \cap \mathcal{U}_n^A \cap \mathcal{C}^A = \hat{I} \cap [\Omega_I, \Omega_I + \epsilon 2^{-n-1}),$$

which has measure at most $\epsilon 2^{-n-1}$. Hence

$$I \cap \mathcal{U}_n^A \cap \mathcal{C}^A = (\hat{I} \cap \mathcal{U}_n^A \cap \mathcal{C}^A) \cup (I' \cap \mathcal{U}_n^A \cap \mathcal{C}^A)$$

has measure at most $\epsilon 2^{-n}$.   □

Thus

$$\mu(I_E \cap \mathcal{U}_n^A \cap \mathcal{C}^A) \leq 2^{-n} \epsilon_E$$

and

$$\mu(\mathcal{U}_n^A \cap \mathcal{C}^A) = \mu\left(\bigcup_E I_E \cap \mathcal{U}_n^A \cap \mathcal{C}^A\right) \le \sum_E 2^{-n}\epsilon_E = 2^{-n},$$

so $(\langle \mathcal{U}_n^A \rangle, \mathcal{C}^A)$ is a difference test relative to A.

Lemma 16   *The A-difference test $\mathcal{T}^A = (\langle \mathcal{U}_n^A \rangle, \mathcal{C}^A)$ is not covered by any finite ensemble of unrelativized difference tests.*

Proof   We say that $\mathcal{R}_E$ is *satisfied inside I* if there is some real $X \in I$ which is captured by the test $\mathcal{T}^A$ but is not captured by any of the tests $\mathcal{T}_e$ for $e \in E$. (In the case when E is empty, this just says that $\mathcal{T}^A$ captures some real in the interval I.) We will show that for every set E, the requirement $\mathcal{R}_E$ is satisfied inside I_E. Notice that if $e \in E$ and $E' = E \setminus \{e\}$, then if $\mathcal{R}_{E'}$ is satisfied on some interval $I' \subset I$ in the complement of $\mathcal{D}_e$, there is some real $X \in I'$ not captured by any of the tests $\mathcal{T}_{e'}$ for $e' \in E'$. Since $X \notin \mathcal{D}_e$, it is also not captured by $\mathcal{T}_e$, and hence $\mathcal{R}_E$ is satisfied inside I.

We will prove by induction on $|E|$ that if a procedure $\mathcal{M}(I, E, \epsilon)$ is called during the construction, then $\mathcal{R}_E$ is satisfied inside I.

Suppose that E is empty. We showed above that in this case, $\mathcal{T}^A$ captures the midpoint of I. Hence $\mathcal{R}_E$ is satisfied inside I.

Now suppose that E is nonempty. We showed above that $\Omega_I \in I$ and Ω_I is captured by the A-difference test $\mathcal{T}^A$, so if Ω_I is not also captured by one of the difference tests $\mathcal{T}_e$ where $e \in E$, then $\mathcal{R}_E$ is satisfied inside I. Suppose that Ω_I is captured by at least one of these tests. If the procedure $\mathcal{M}(I, E, \epsilon)$ eventually goes from the "searching" state to the "found" state after finding some subinterval $I' = (a, b)$ in the complement of some $\mathcal{D}_e$ (with $e \in E$) and calling the subprocedure $\mathcal{M}(I', E', \epsilon/2)$ (where $E' = E \setminus \{e\}$), then $|E'| = |E| - 1$, so by the inductive hypothesis we may assume that $\mathcal{R}_{E'}$ is satisfied inside I'. But as shown above, this implies that $\mathcal{R}_E$ is satisfied inside I. So it suffices to show that if Ω_I is captured by at least one of the tests $\mathcal{T}_e$ where $e \in E$, then the procedure $\mathcal{M}(I, E, \epsilon)$ eventually goes from the "searching" state to the "found" state.

Let s_0 be the stage at which the procedure $\mathcal{M}(I, E, \epsilon)$ is called. Fix $e \in E$ such that $\mathcal{T}_e$ captures Ω_I. As before, in any interval of the form $(\Omega_I[s-1], \Omega_I)$, there must be some rational subinterval removed from $\mathcal{D}_e$. We will define a computable function $g : \omega \to \omega$. To define $g(s)$, search for some natural number n and pair of rational numbers a, b with denominator at most n, such that $\Omega_I[s - 1] < a < b < \Omega_I[n]$ and the interval (a, b) is in the complement of $\mathcal{D}_e[n]$; let $g(s)$ be the least such n. Such an n will always be found, and so g is a total computable function. Since f escapes domination by g, there is a stage $s > s_0$ with $f(s) > g(s)$. During stage s of the construction, the search for a rational interval (a, b) meeting conditions (1)–(3) must succeed, so the procedure $\mathcal{M}(I, E, \epsilon)$ will move to the "found" state at stage s, provided that it is still searching. □

This completes the proof. □

Note

1. In the space $[0, 1]$, intervals of the form $[0, x)$ are indeed open.

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