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Depth of Boolean Algebras

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Abstract Suppose D is an ultrafilter on κ and $\lambda^\kappa = \lambda$. We prove that if $\mathbf{B}_i$ is a Boolean algebra for every $i < \kappa$ and λ bounds the depth of every $\mathbf{B}_i$, then the depth of the ultraproduct of the $\mathbf{B}_i$'s mod D is bounded by λ^+. We also show that for singular cardinals with small cofinality, there is no gap at all. This gives a partial answer to a previous problem raised by Monk.

1 Introduction

Let $\mathbf{B}$ be a Boolean Algebra. We define the Depth of it as the supremum of the cardinalities of well-ordered subsets in $\mathbf{B}$. Now suppose that $\langle \mathbf{B_i} : i < \kappa \rangle$ is a sequence of Boolean algebras, and D is an ultrafilter on κ. Define the ultraproduct algebra $\mathbf{B}$ as $\prod\limits_{i<\kappa} \mathbf{B}_i/D$. The question (raised also for other cardinal invariants by Monk in [5]) is about the relationship between $\mathrm{Depth}(\mathbf{B})$ and $\prod_{i<\kappa} \mathrm{Depth}(\mathbf{B}_i)/D$.

Let us try to draw the picture:

$$\begin{tikzcd}
 & & \langle \mathbf{B_i} : i < \kappa \rangle, D \arrow[dll] \arrow[drr] & & \\
\mathbf{B} = \prod\limits_{i<\kappa} \mathbf{B}_i/D \arrow[dr] & & & & \langle \mathrm{Depth}(\mathbf{B}_i) : i < \kappa \rangle \arrow[dl] \\
 & \mathrm{Depth}(\mathbf{B}) & & \prod\limits_{i<\kappa} \mathrm{Depth}\mathbf{B}_i/D &
\end{tikzcd}$$

As we can see from the picture, given a sequence of Boolean algebras (of length κ) and an ultrafilter on κ, we have two alternate ways to produce a cardinal value. The left course creates, first, a new Boolean algebra, namely, the ultraproduct algebra $\mathbf{B}$. Then we compute the depth of it. In the second way, we first get rid of the algebraic

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structure, producing a sequence of cardinals (namely, $\langle \text{Depth}(\mathbf{B}_i) : i < \kappa \rangle$). Then we compute the cardinality of its Cartesian product divided by D.

Shelah proved in [9], Section 5, under the assumption $\mathbf{V} = \mathbf{L}$, that if $\kappa = \text{cf}(\kappa) < \lambda$ and $\lambda = \lambda^\kappa$ (so $\kappa < \text{cf}(\lambda)$), then one can build a sequence of Boolean algebras $\langle \mathbf{B_i} : i < \kappa \rangle$ such that $\text{Depth}(\mathbf{B}_i) \leq \lambda$ for every $i < \kappa$, and $\text{Depth}(\mathbf{B}) > |\prod_{i<\kappa} \text{Depth}(\mathbf{B}_i)/D|$ for every uniform ultrafilter D. This result is based on the square principle, introduced and proved in $\mathbf{L}$ by Jensen.

A natural question is how far can this gap reach. We prove (in Section 2) that if $\mathbf{V} = \mathbf{L}$ then the gap is at most one cardinal. Summarizing, in $\mathbf{L}$, for every regular cardinal and for every singular cardinal with high cofinality, we can create a gap (having the square for every infinite cardinal in $\mathbf{L}$), but it is limited to one cardinal.

The assumption $\mathbf{V} = \mathbf{L}$ is just to make sure that every ultrafilter is regular, so the results in Section 2 apply also outside $\mathbf{L}$. On the other hand, if $\mathbf{V}$ is far from $\mathbf{L}$ we get a different picture. By [10] (see conclusion 2.2 there, page 94), under some reasonable assumptions, there is no gap at all above a compact cardinal.

We can ask further what happens if $\text{cf}(\lambda) < \lambda$, and $\kappa \geq \text{cf}(\lambda)$. We prove here that if λ is singular with small cofinality (i.e., the cases which are not covered in the previous paragraph), then $|\prod_{i<\kappa} \text{Depth}(\mathbf{B}_i)/D| \geq \text{Depth}(\mathbf{B})$. It is interesting to know (see [4], Section 2) that a similar result holds above a compact cardinal for singular cardinals with countable cofinality. We suspect that it holds (for such cardinals) in ZFC.

The proof of these results is based on an improvement to the main theorem in [3]. It says that under some assumptions we can dominate the gap between $\text{Depth}(\mathbf{B})$ and $|\prod_{i<\kappa} \text{Depth}(\mathbf{B}_i)/D|$. In this paper we use weaker assumptions. We give here the full proof, so the paper is self-contained. We intend to shed light on the other side of the coin (i.e., under large cardinal assumptions) in a subsequent paper.

2 The Main Theorem

Definition 2.1 (Depth) Let $\mathbf{B}$ be a Boolean Algebra.

$$\text{Depth}(\mathbf{B}) := \sup\{\theta : \exists \bar{b} = (b_\gamma : \gamma < \theta),\ \text{increasing sequence in } \mathbf{B}\}.$$

Remark 2.2 Clearly, we can use *decreasing* instead of *increasing* in the definition of Depth. We prefer the *increasing* version, since it is coherent with the terminology of [6].

Discussion 2.3 $\text{Depth}(\mathbf{B})$ is always a cardinal, but it does not have to be a regular cardinal. It is achieved in the case of a successor cardinal (i.e., $\text{Depth}(\mathbf{B}) = \lambda^+$ for some infinite cardinal λ), and in the case of a singular cardinal with countable cofinality (i.e., $\text{Depth}(\mathbf{B}) = \lambda > \text{cf}(\lambda) = \aleph_0$). In all other cases, one can create an example of a Boolean Algebra $\mathbf{B}$, whose Depth is not attained. A detailed survey of these facts appears in [6].

We use also an important variant of the Depth.

Definition 2.4 (Depth^+) Let $\mathbf{B}$ be a Boolean Algebra.

$$\text{Depth}^+(\mathbf{B}) := \sup\{\theta^+ : \exists \bar{b} = (b_\gamma : \gamma < \theta),\ \text{increasing sequence in } \mathbf{B}\}.$$

Discussion 2.5 Assume λ is a limit cardinal. The question of achieving the Depth (for a Boolean Algebra $\mathbf{B}$ such that $\text{Depth}(\mathbf{B}) = \lambda$) demonstrates the difference between Depth and Depth^+. If $\text{cf}(\lambda)$ is uncountable, we can imagine two situations.

In the first one the Depth is achieved, and in that case we have $\mathrm{Depth}^+(\mathbf{B}) = \lambda^+$. In the second, the Depth is not achieved. Consequently, $\mathrm{Depth}^+(\mathbf{B}) = \lambda$. Notice that $\mathrm{Depth}(\mathbf{B}) = \lambda$ in both cases, so Depth^+ is more delicate and using it (as a scaffold) helps us to prove our results.

Throughout the paper, we use the following notation.

Notation 2.6

(a) κ, λ are infinite cardinals.
(b) D is a uniform ultrafilter on κ.
(c) $\mathbf{B}_i$ is a Boolean Algebra, for any $i < \kappa$.
(d) $\mathbf{B} = \prod\limits_{i<\kappa} \mathbf{B}_i / D$.
(e) For $\kappa = \mathrm{cf}(\kappa) < \lambda$, $S^\lambda_\kappa = \{\alpha < \lambda : \mathrm{cf}(\alpha) = \kappa\}$.

We state our main result.

Theorem 2.7 *Assume*

(a) $\lambda \geq \mathrm{cf}(\lambda) > \kappa$,
(b) $\lambda = \lambda^\kappa$,
(c) $\mathrm{Depth}^+(\mathbf{B}_i) \leq \lambda$, *for every* $i < \kappa$.

Then $\mathrm{Depth}^+(\mathbf{B}) \leq \lambda^+$.

Proof Assume toward a contradiction that $\langle a_\alpha : \alpha < \lambda^+ \rangle$ is an increasing sequence in $\mathbf{B}$. Let us write a_α as $\langle a^\alpha_i : i < \kappa \rangle / D$ for every $\alpha < \lambda^+$. We shall apply Claim 2.8, so λ, κ, D are given and we define R_i for every $i < \kappa$ as the set $\{(\alpha, \beta) : \alpha < \beta < \lambda^+$ and $a^\alpha_i < a^\beta_i\}$. As $\alpha < \beta \Rightarrow a_\alpha <_D a_\beta \Rightarrow \{i < \kappa : \mathbf{B}_i \models a^\alpha_i < a^\beta_i\} \in D$, all the assumptions of Claim 2.8 hold; hence the conclusion also holds. So there are $i_* < \kappa$ and $Z \subseteq \lambda^+$ of order type λ as there.

Now, if $\alpha < \beta$ are from Z we have $\iota \in (\alpha, \beta)$ which satisfies $\alpha R_{i_*} \iota$ and $\iota R_{i_*} \beta$. It means that $a^\alpha_{i_*} <_{\mathbf{B}_{i_*}} a^\iota_{i_*} <_{\mathbf{B}_{i_*}} a^\beta_{i_*}$. By the transitivity of $<_{\mathbf{B}_{i_*}}$ (see Corollary 2.9), we have $a^\alpha_{i_*} <_{\mathbf{B}_{i_*}} a^\beta_{i_*}$ for every $\alpha < \beta$ from Z. Since $|Z| = \lambda$, we have an increasing sequence of length λ in $\mathbf{B}_{i_*}$, so $\mathrm{Depth}^+(\mathbf{B}_{i_*}) \geq \lambda^+$, contradicting the assumptions of the theorem. $\square$

Claim 2.8 *Assume*

(a) $\lambda = \lambda^\kappa$,
(b) *D is an ultrafilter on κ,*
(c) *$R_i \subseteq \{(\alpha, \beta) : \alpha < \beta < \lambda^+\}$ is a two place relation on λ^+ for every $i < \kappa$,*
(d) $\alpha < \beta \Rightarrow \{i < \kappa : (\alpha, \beta) \in R_i\} \in D$.

Then there exists $i_* < \kappa$ and $Z \subseteq \lambda^+$ of order type λ such that for every $\alpha < \beta$ from Z, for some $\iota \in (\alpha, \beta)$, we have $(\alpha, \iota), (\iota, \beta) \in R_{i_*}$.

Proof Let $\bar{M} = \langle M_\alpha : \alpha < \lambda^+ \rangle$ be a continuous and increasing sequence of elementary submodels of $(\mathcal{H}(\chi), \in)$ for sufficiently large χ, with the following properties for every $\alpha < \lambda^+$:

(a) $\|M_\alpha\| = \lambda$,
(b) $\lambda + 1 \subseteq M_\alpha$,
(c) $\langle M_\beta : \beta \leq \alpha \rangle \in M_{\alpha+1}$,
(d) $[M_{\alpha+1}]^\kappa \subseteq M_{\alpha+1}$.

For every $\alpha < \beta < \lambda^+$, define

$$A_{\alpha,\beta} = \{i < \kappa : \alpha R_i \beta\}.$$

By the assumption, $A_{\alpha,\beta} \in D$ for all $\alpha < \beta < \lambda^+$. Define

$$C := \{\gamma < \lambda^+ : \gamma = M_\gamma \cap \lambda^+\} \text{ and } S := C \cap S^{\lambda^+}_{\mathrm{cf}(\lambda)}.$$

Since C is a club subset of λ^+, S is a stationary subset of λ^+. Choose δ^* as the λth member of S. For every $\alpha < \delta^*$, let A_α denote the set A_{α,δ^*}.

Let $u \subseteq \delta^*$, $|u| \leq \kappa$. Notice that $u \in M_{\delta^*}$, by the assumptions on $\bar{M}$. Define

$$S_u = \{\beta < \lambda^+ : \beta > \sup(u), \mathrm{cf}(\beta) = \mathrm{cf}(\lambda), \text{ and } (\forall \alpha \in u)(A_{\alpha,\beta} = A_\alpha)\}.$$

Notice that $S_u \neq \varnothing$ as $\delta^* \in S_u$; hence if $u \subseteq \delta^*$ and $|u| \leq \kappa$ then $S_u \cap \delta^* \neq \varnothing$. Let $\langle \delta_\epsilon : \epsilon < \lambda \rangle$ be the increasing enumeration of $C \cap S \cap \delta^*$. Define, for every $\epsilon < \lambda$, the following family:

$$\mathcal{A}_\epsilon = \{S_u \cap \delta_{\epsilon+1} \setminus \delta_\epsilon : u \in [\delta_{\epsilon+1}]^{\leq\kappa}\}.$$

The crucial point is that $\mathcal{A}_\epsilon$ is not empty for each ϵ. We shall prove this in Lemma 2.10 below. Observe also that $\mathcal{A}_\epsilon$ is downward κ^+-directed since if $u_\alpha \in [\delta_{\epsilon+1}]^{\leq\kappa}$ for each $\alpha < \kappa$ then

$$\bigcap \{S_{u_\alpha} \cap [\delta_\epsilon, \delta_{\epsilon+1}) : \alpha < \kappa\} = S_{\bigcup_{\alpha<\kappa} u_\alpha} \cap [\delta_\epsilon, \delta_{\epsilon+1}) \in \mathcal{A}_\epsilon.$$

So we have a family of nonempty sets, which is downward κ^+-directed. Hence, there is a κ^+-complete filter E_ϵ on $[\delta_\epsilon, \delta_{\epsilon+1})$, with $\mathcal{A}_\epsilon \subseteq E_\epsilon$, for every $\epsilon < \lambda$.

Define, for any $i < \kappa$ and $\epsilon < \lambda$, the sets $W_{\epsilon,i} \subseteq [\delta_\epsilon, \delta_{\epsilon+1})$ and $B_\epsilon \subseteq \kappa$ by

$$W_{\epsilon,i} := \{\beta : \delta_\epsilon \leq \beta < \delta_{\epsilon+1} \text{ and } i \in A_{\beta,\delta_{\epsilon+1}}\},$$
$$B_\epsilon := \{i < \kappa : W_{\epsilon,i} \in E^+_\epsilon\}.$$

Finally, set

$$W_\epsilon = [\delta_\epsilon, \delta_{\epsilon+1}) \setminus \bigcup_{i \in \kappa \setminus B_\epsilon} W_{\epsilon,i}\,.$$

Notice that $W_\epsilon \in E_\epsilon$, as it results out of throwing κ-many small sets from $[\delta_\epsilon, \delta_{\epsilon+1})$, and E_ϵ is κ^+-complete, so clearly $W_\epsilon \neq \varnothing$. We shall prove below that $B_\epsilon \in D$ for every $\epsilon < \lambda$. Clearly, this implies $B_\epsilon \cap A_{\delta_{\epsilon+1}} \in D$ as well.

Choose $\beta = \beta_\epsilon \in W_\epsilon$. If $i \in A_{\beta,\delta_{\epsilon+1}}$, then $W_{\epsilon,i} \in E^+_\epsilon$, so $A_{\beta,\delta_{\epsilon+1}} \subseteq B_\epsilon$ (by the definition of B_ϵ). But, $A_{\beta,\delta_{\epsilon+1}} \in D$, so $B_\epsilon \in D$. For every $\epsilon < \lambda$, $A_{\delta_{\epsilon+1}}$ (which equals to $A_{\delta_{\epsilon+1},\delta^*}$) belongs to D, so $B_\epsilon \cap A_{\delta_{\epsilon+1}} \in D$.

Choose $i_\epsilon \in B_\epsilon \cap A_{\delta_{\epsilon+1}}$, for every $\epsilon < \lambda$. We choose, in this process, λ i_ϵ-s from κ, so as $\mathrm{cf}(\delta^*) = \mathrm{cf}(\lambda) > \kappa$, there is an ordinal $i_* \in \kappa$ such that the set $Y = \{\epsilon < \lambda : \epsilon \text{ is an even ordinal, and } i_\epsilon = i_*\}$ has cardinality λ.

The last step will be as follows. Define $Z = \{\delta_{\epsilon+1} : \epsilon \in Y\}$. Clearly, $Z \in [\delta^*]^\lambda \subseteq [\lambda^+]^\lambda$. We will show that for $\alpha < \beta$ from Z we can find $\imath \in (\alpha, \beta)$ so that $(\alpha R_{i_*} \imath)$ and $(\imath R_{i_*} \beta)$. The idea is that if $\alpha < \beta$ and $\alpha, \beta \in Z$, then $i_* \in A_{\alpha,\beta}$.

So why does a suitable $\imath$ exist? Recall that $\alpha = \delta_{\epsilon+1}$ and $\beta = \delta_{\zeta+1}$, for some $\epsilon < \zeta < \lambda$ (that's the form of the members of Z). Define

$$U_1 := S_{\{\delta_{\epsilon+1}\}} \cap [\delta_\zeta, \delta_{\zeta+1}).$$

Observe that $U_1 \in \mathcal{A}_\zeta$ and $\mathcal{A}_\zeta \subseteq E_\zeta$; hence $U_1 \in E_\zeta$. Observe also that $\alpha R_{i_*} \imath$ for any $\imath \in U_1$. Now define

$$U_2 := \{\gamma : \delta_\zeta \leq \gamma < \delta_{\zeta+1}, i_* \in A_{\gamma,\delta_{\zeta+1}}\}.$$

Notice that $U_2 \in E_\zeta^+$, since $U_2 \equiv W_{\zeta,i_*}$ and $i_* = i_\zeta \in B_\zeta$. It follows that $\imath R_{i_*} \beta$ for every $\imath \in U_2$. So $U_1 \cap U_2 \neq \varnothing$, and we can choose $\imath \in U_1 \cap U_2$.

Now the following statements hold:

(a) $\alpha R_{i_*} \imath$
[Why? Well, $\imath \in U_1$, so $A_{\delta_{\epsilon+1,\imath}} = A_{\delta_{\epsilon+1}}$. But, $i_* \in B_\epsilon \cap A_{\delta_{\epsilon+1}} \subseteq A_{\delta_{\epsilon+1}}$, so $i_* \in A_{\delta_{\epsilon+1,\imath}}$, which means that $\delta_{\epsilon+1} R_{i_*} \imath$].

(b) $\imath R_{i_*} \beta$
[Why? Well, $\imath \in U_2$, so $i_* \in A_{\imath,\delta_{\zeta+1}}$, which means that $\imath R_{i_*} \delta_{\zeta+1}$].

So, we are done. □

Corollary 2.9 *Suppose each R_i in the above claim is transitive. Then $(\alpha, \beta) \in R_{i_*}$ as well.*

Lemma 2.10 *Let $\mathcal{A}_\epsilon = \{S_u \cap \delta_{\epsilon+1} \setminus \delta_\epsilon : u \in [\delta_{\epsilon+1}]^{\leq\kappa}\}$.*

(a) *$\mathcal{A}_\epsilon$ is not empty, for every $\epsilon < \lambda$.*
(b) *Moreover, $u \in [\delta_{\epsilon+1}]^{\leq\kappa} \Rightarrow S_u \cap \delta_{\epsilon+1} \setminus \delta_\epsilon$ is unbounded in $\delta_{\epsilon+1}$.*

Proof Clearly, (b) implies (a). Let us prove part (b). First we observe that if $u \in [\delta_{\epsilon+1}]^{\leq\kappa}$ then $\sup(u) < \delta_{\epsilon+1}$ (since $\delta_{\epsilon+1} \in S \subseteq S^{\lambda^+}_{\mathrm{cf}(\lambda)}$, and $\kappa < \mathrm{cf}(\lambda)$). Second, $M_{\delta_{\epsilon+1}} = \bigcup\{M_\alpha : \alpha < \delta_{\epsilon+1}\}$ (since $\delta_{\epsilon+1}$ is a limit ordinal and $\bar{M}$ is continuous).

Consequently, there exists $\alpha < \delta_{\epsilon+1}$ so that $u \subseteq M_\alpha$. Choose such α, and observe that $u \in M_{\alpha+1}$ (again, this follows from the properties of $\bar{M}$). We derive $S_u \in M_{\alpha+1}$ as well (since it is definable from parameters in $M_{\alpha+1}$). By the definition of S_u, $\delta^* \in S_u$. We conclude

$$M_{\alpha+1} \cap \lambda^+ \subseteq M_{\delta_{\epsilon+1}} \cap \lambda^+ = \delta_{\epsilon+1} < \delta^* \in S_u.$$

We can infer that $\sup(S_u) = \lambda^+$, so $M_{\delta_{\epsilon+1}} \models S_u \subseteq \lambda^+$, unbounded in λ^+. Since $M_{\delta_{\epsilon+1}} \cap \lambda^+ = \delta_{\epsilon+1}$ and by virtue of elementarity, $S_u \cap \delta_{\epsilon+1}$ is unbounded in $\delta_{\epsilon+1}$. Recall that $\delta_\epsilon < \delta_{\epsilon+1}$, so $S_u \cap \delta_{\epsilon+1} \setminus \delta_\epsilon$ is also unbounded, and we are done. □

Corollary 2.11 (GCH) *Assume*

(a) $\kappa < \mu$,
(b) $\mathrm{Depth}(\mathbf{B}_i) \leq \mu$, *for every* $i < \kappa$.

Then $\mathrm{Depth}(\mathbf{B}) \leq \mu^+$.

Proof For every successor cardinal μ^+, and every $\kappa < \mu$, we have (under the GCH) $(\mu^+)^\kappa = \mu^+$. By assumption (b), we know that $\mathrm{Depth}^+(\mathbf{B}_i) \leq \mu^+$ for every $i < \kappa$. Now apply Theorem 2.7 (upon noticing that μ^+ here is standing for λ there), and conclude that $\mathrm{Depth}^+(\mathbf{B}) \leq \mu^{+2}$, so $\mathrm{Depth}(\mathbf{B}) \leq \mu^+$ as required. □

Remark 2.12 Notice that the corollary holds even if almost every $\mathbf{B}_i$ has μ as its Depth. Recall that for a sequence of cardinals $\bar{\lambda} = \langle \lambda_i : i < \kappa \rangle$ and an ultrafilter D on κ we define $\mu = \lim_D(\bar{\lambda})$ when μ is the unique cardinal such that $\{i < \kappa : \beta < \lambda_i \leq \mu\} \in D$ for every $\beta < \mu$. Hence we may replace assumption (b) in Corollary 2.11 by the weaker assumption that $\mu = \lim_D(\langle \mathrm{Depth}(\mathbf{B}_i) : i < \kappa \rangle)$.

This assumption becomes important if we try to phrase an equality (not just $\leq$), as in the theorem of the next section.

3 Depth in $\mathbf{L}$

Monk's problem No. 12 from [6] is whether there exists a ZFC example of $\mathrm{Depth}(\prod_{i<\kappa} \mathbf{B}_i/D) > |\prod_{i<\kappa} \mathrm{Depth}(\mathbf{B}_i)/D|$. We shall work in $\mathbf{L}$, showing that $\mathbf{V} = \mathbf{L}$ rules out a ZFC example in the case of singular cardinals with countable cofinality. Essentially, we do not use the fundamental structure of $\mathbf{L}$ but rather the fact that every ultrafilter is regular in $\mathbf{L}$. We start with a short discussion on regular ultrafilters. A good source for the subject is [1], Section 4.3. Recall the following definition.

Definition 3.1 (Regular ultrafilters) Let D be an ultrafilter on an infinite cardinal κ, and $\theta \leq \kappa$.

(a) D is θ-regular if there exists $E \subseteq D$, $|E| = \theta$, so that

$$\alpha < \kappa \Rightarrow |\{e \in E : \alpha \in e\}| < \aleph_0,$$

(b) D is called regular when D is κ-regular.

Remark 3.2 (Measurability and $\aleph_0$-regular ultrafilters) An ultrafilter D on κ is $\aleph_0$-regular if and only if D is $\aleph_1$-incomplete (The proof appears, for instance, in [1], Proposition 4.3.4, page 249). If κ is below the first measurable cardinal, then every nonprincipal ultrafilter on κ is $\aleph_1$-incomplete, hence $\aleph_0$-regular.

The following is a fundamental result of Donder from [2].

Theorem 3.3 (Regular ultrafilters in the constructible universe) *Assume* $\mathbf{V} = \mathbf{L}$. *Let D be a nonprincipal ultrafilter on an infinite cardinal κ. Then D is regular.*

It is proved (see [1], Proposition 4.3.5, page 249) that for every infinite cardinal κ there exists a regular ultrafilter D over κ. Having a regular ultrafilter D, one can estimate the cardinality of an ultraproduct divided by D. A proof of the following claim can be found in [1], Proposition 4.3.7, page 250.

Claim 3.4 *Suppose D is a regular ultrafilter on κ. Then* $|\prod_{i<\kappa} \lambda/D| = \lambda^\kappa$.

By [9], Section 5, if λ is regular and $\kappa < \lambda$, or even $\lambda > \mathrm{cf}(\lambda) > \kappa$, we can build in $\mathbf{L}$ an example for $\mathrm{Depth}(\mathbf{B}) > |\prod_{i<\kappa} \mathrm{Depth}(\mathbf{B}_i)/D|$, but the discrepancy is just one cardinal as shown in Corollary 2.11. We can ask what happens if λ is singular with small cofinality. The theorem below says that equality holds. The theorem answers problem No. 12 from [6], for the case of singular cardinals with countable cofinality (since then $\mathrm{cf}(\lambda) \leq \kappa$ for every infinite cardinal κ).

Theorem 3.5 *Assume*

(a) $\lambda > \kappa \geq \mathrm{cf}(\lambda)$,
(b) $\mathrm{Depth}(\mathbf{B}_i) \leq \lambda$, *for every* $i < \kappa$,
(c) $\lambda = \lim_D(\langle \mathrm{Depth}(\mathbf{B}_i) : i < \kappa \rangle)$.

Then

($\aleph$) $\mathbf{V} = \mathbf{L}$ *implies* $\mathrm{Depth}(\mathbf{B}) = |\prod_{i<\kappa} \mathrm{Depth}(\mathbf{B}_i)/D|$;
($\beth$) *instead of* $\mathbf{V} = \mathbf{L}$ *it suffices that D is a κ-regular ultrafilter, and* $\lambda^\kappa = \lambda^+$.

Proof ($\aleph$) First we claim that $|\prod_{i<\kappa} \mathrm{Depth}(\mathbf{B}_i)/D| = \lambda^+$. It follows from the fact that in $\mathbf{L}$ we know that D is regular (by Theorem 3.3 of Donder, taken from [2]), so (using assumption (c), and Claim 3.4),

$$\Big|\prod_{i<\kappa}\mathrm{Depth}(\mathbf{B}_i)/D\Big| = \lambda^\kappa = \lambda^+$$

(recall that $\mathrm{cf}(\lambda) \leq \kappa$).

Now $\mathrm{Depth}(\mathbf{B}) \geq |\prod_{i<\kappa} \mathrm{Depth}(\mathbf{B}_i)/D| = \lambda^+$, by Theorem 4.14 from [6] (since $\mathbf{L} \models \mathrm{GCH}$). On the other hand, Corollary 2.11 makes sure that $\mathrm{Depth}(\mathbf{B}) \leq \lambda^+$ (by (b) of the present theorem). So $|\prod_{i<\kappa} \mathrm{Depth}(\mathbf{B}_i)/D| = \lambda^+ = \mathrm{Depth}(\mathbf{B})$, and we are done.

($\beth$) Notice that in the proof of ($\aleph$) we use just the regularity of D (and κ-regularity suffices), and the assumption that $\lambda^\kappa = \lambda$. □

We know that if κ is less than the first measurable cardinal, then every uniform ultrafilter on κ is $\aleph_0$-regular, as noted in Remark 3.2. It gives us the result of Theorem 3.5 for singular cardinals with countable cofinality, if $\mathbf{V} = \mathbf{L}$ or a similar weaker assumption.

On the other hand, we have good evidence that something similar holds for singular cardinals with countable cofinality above a compact cardinal (as shown in [4], Theorem 2.5 there). Moreover, if $\mathrm{cf}(\lambda) = \aleph_0$ then $\kappa \geq \mathrm{cf}(\lambda)$ for every infinite cardinal κ. It means that it is consistent with ZFC not to have a counterexample in this case. So the following conjecture does make sense.

Conjecture 3.6 (ZFC) *Assume*

(a) $\aleph_0 = \mathrm{cf}(\lambda) < \lambda$,
(b) $\kappa < \lambda$, *and* $2^\kappa < \lambda$,
(c) $\mathrm{Depth}(\mathbf{B}_i) \leq \lambda$, *for every* $i < \kappa$,
(d) $\lambda = \lim_D(\langle \mathrm{Depth}(\mathbf{B}_i) : i < \kappa\rangle)$,
(e) *D is not $\aleph_1$-complete.*

Then $\mathrm{Depth}(\mathbf{B}) \leq |\prod_{i<\kappa} \mathrm{Depth}(\mathbf{B}_i)/D|$.

Notice that by [7] we know that this question is independent when $2^{\aleph_0} > \lambda$, as follows from Theorem 3.2 there. More independent results (under some *pcf* assumptions) can be derived from [8], Section 3.

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