Notre Dame Journal of Formal Logic
Volume 50, Number 3, 2009

A Topological Approach to Yablo's Paradox

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Abstract Some years ago, Yablo gave a paradox concerning an infinite sequence of sentences: if each sentence of the sequence is 'every subsequent sentence in the sequence is false', a contradiction easily follows. In this paper we suggest a formalization of Yablo's paradox in algebraic and topological terms. Our main theorem states that, under a suitable condition, any continuous function from $\mathbf{2}^{\mathbf{N}}$ to $\mathbf{2}^{\mathbf{N}}$ has a fixed point. This can be translated in the original framework as follows. Consider an infinite sequence of sentences, where any sentence refers to the truth values of the subsequent sentences: if the corresponding function is continuous, no paradox arises.

1 Introduction

This paper deals with Yablo's paradox, which, according to its author, is a paradox without self-reference (see Yablo [11] or, for an earlier version, Yablo [10, p. 340]). There are several papers discussing Yablo's argument. In particular, there was much debate about whether Yablo's paradox involves self-reference or not: according to some criticisms, Yablo's construction *is*, in fact, self-referential (see, for instance, Priest [8]). Leitgeb [6] argues that the opponents in this debate refer to two different notions of self-reference, and, moreover, both of these notions seem to be inadequate. See also Leitgeb [7], where a definition of self-reference is given: if one assumes this definition, the statements in Yablo's paradox are not self-referential. According to other criticisms, it would be more appropriate to say that Yablo's argument leads to a situation that is ω-inconsistent rather than to a contradiction (see Cook [3], Ketland [5]). In my opinion, Yablo's paradox does not depend on a genuine self-referential argument, or, at least, its logical schema is quite different from the classical self-referential schemata. But this discussion is *not* the aim of this paper. Our purpose is to examine the paradox from a topological point of view: in a

Received November 14, 2008; accepted May 6, 2009; printed September 30, 2009
2000 Mathematics Subject Classification: Primary, 03A05; Secondary, 03F45, 54D30
Keywords: fixed point of a continuous function, ungrounded sentence
 10.1215/00294527-2009-014

more general context, we show that *a paradox arises when a certain function is not continuous* with respect to a suitable topology.

Let us point out that other known constructions similar to Yablo's paradox exist. In particular, we would like to mention the *paradox of the class of all grounded classes* in Shen-Yuting [13] and the *hypergame paradox* in Zwicker [14]. For a comparison between self-reference and these constructions, see Bernardi [1, Section 1]. See also Bernardi and D'Agostino [2], where a proof schema is extracted from Yablo-style arguments.

In Section 2 we recall Yablo's paradox and briefly comment on it. In Section 3 we introduce a formal framework, which allows us, in Section 4, to prove the main theorem of this paper. Finally, Section 5 is devoted to examples and further remarks.

2 Yablo's Paradox

Yablo's paradox can be stated as follows. Consider an infinite sequence of sentences $s_0, s_1, s_2, \ldots$, where each sentence s_n is 'every subsequent sentence in the sequence (i.e., every sentence s_m with $m > n$) is false'. A contradiction arises when we try to make a consistent assignment of truth values to all sentences. Let us suppose that there is a sentence s_i which is false; then for a suitable $j > i$ the sentence s_j must be true. So, in *any* case, we can assume that there is a true sentence s_j. We deduce that, for all n greater than j, all sentences s_n are false. However, as a consequence, the sentence s_{j+1} should be true, because what is stated is correct. In other words, every T-value must be followed only by F-values, while every F-value must be followed by at least one T-value, which is clearly impossible.

In the sequence of sentences, each of them refers only to the sentences that occur later in the sequence (i.e., sentences of higher index). In this way, circularity and self-reference seem to be avoided: if s_n refers to s_m, then s_m does not refer to s_n. One can object that the sequence refers to itself. On the other hand, it is quite obvious that, if each s_n refers to the preceding sentences, no paradox arises: starting from s_0 we can assign truth values to every sentence in one and only one way, whatever the sentences are.

Remark 2.1 In the original version, the sentence s_n is 'for all $m > n$, s_m is untrue'. Let us briefly explain the difference between *false* and *untrue*. A spontaneous way to try to overcome the Liar paradox is to say that the sentence 'this sentence is false' is meaningless; so, it is neither true nor false. However, this is not a solution: it is sufficient to consider the so-called Strengthened Liar sentence 'this sentence is untrue' (in the sense that it is either meaningless or false), and we get again a contradiction. However, in our approach, we will always refer to a two-valued logic and make no difference between the terms *false* and *untrue*.

In Yablo's paradox, every sentence must be considered as a non-well-founded or an *ungrounded* sentence in the sense of Kripke: the process of determining its truth value, by examining the sentences it refers to, does not terminate. Note that we cannot define a rank, or a level for ungrounded sentences—see also Yi [12, p. 561] for a distinction between grounded and rooted. Are we allowed to consider ungrounded sentences? There is no doubt that there are "serious difficulties in dealing with sentences that form an infinite chain of semantic attributions" [12]. Similarly, in [8, footnote 5], discussing a set-theoretic analogue of the paradox given in Goldstein [4], the author says, "Given the non-well-foundedness of the situation, it is not at all clear that there is such a sequence, even in naïve set theory." Our main result (Theorem 4.2 in Section 4) states a condition under which ungrounded situations can be safely considered.

In Section 3, in order to approach a general situation, we will make use of a denumerable sequence of sentences $(s_n)_{n\in\mathbf{N}}$ indexed by the set $\mathbf{N}$. Instead of $\mathbf{N}$, we could also consider the set $\mathbf{Z}^-$ of negative integers: in fact, from an intuitive point of view, one might prefer the set $\mathbf{Z}^-$ because the concept of an ungrounded sentence corresponds to an infinite regression (it can be useful to think of a *descending* chain —see [12] and [1]). Without any substantial difference, we could also consider the set $\mathbf{Z}$ of all integers.

3 Formalizing the Paradox

Let us translate the situation in algebraic terms. Instead of false and true, consider 0 and 1, and let $\mathbf{2}$ be the set $\{0, 1\}$. Let us suppose that every sentence s_n in a sequence $(s_n)_{n\in\mathbf{N}}$ refers to the truth values of sentences in the sequence. Assume also that each sentence is truth functional; this means that its truth value depends only on the truth values of the involved sentences. Under these assumptions, any sequence $(s_n)_{n\in\mathbf{N}}$ induces a function f from $\mathbf{2}^\mathbf{N}$ to $\mathbf{2}^\mathbf{N}$. Indeed, what is stated in the sentence s_n (for every n) induces a function f_n whose value can be either 0 or 1 depending only on the truth values of the sentences involved in s_n. We can express all the functions f_n by means of a single function f from $\mathbf{2}^\mathbf{N}$ to $\mathbf{2}^\mathbf{N}$.

Definition 3.1 An *assignment* of truth values to all sentences is a function from $\mathbf{N}$ to $\mathbf{2}$, that is, an element a of the set $\mathbf{2}^\mathbf{N}$. (Of course, $\mathbf{2}^\mathbf{N}$ can be identified with the power set $\mathcal{P}(\mathbf{N})$.) We shall refer to a_n as the nth component of a.

Given any *assignment* of truth values (that is, an element a of $\mathbf{2}^\mathbf{N}$), let us consider it as expressing the truth values of sentences of the given sequence. Starting from this assignment, the function f_n gives the truth value that the nth sentence assumes as a consequence. For instance, let us assume that the second sentence is 'the next two sentences are false'. Then the second component of $f(a)$ is 1 if and only if $a_3 = 0$ and $a_4 = 0$.

Now, the key remark is the following. Given a sequence of sentences, assigning truth values to all sentences in a consistent way corresponds to finding a *fixed point* for the corresponding function f (see also [8]). Therefore, we look for an element a of $\mathbf{2}^\mathbf{N}$ such that $a = f(a)$.

Definition 3.2 An *initial segment* of an element a of $\mathbf{2}^\mathbf{N}$ is a finite family $a|_m = a|_{\{0,\ldots,m-1\}} = (a_j)|_{j<m} = (a_0, a_1, \ldots, a_{m-1})$. A *final segment* of an element a of $\mathbf{2}^\mathbf{N}$ is an infinite family $a|_{\mathbf{N}-\{0,\ldots,m\}} = (a_j)|_{j>m} = (a_{m+1}, \ldots)$.

As we have already noted, in order to avoid both direct and indirect self-reference, the truth value of every sentence must depend only on the truth value of the subsequent sentences. The following definition translates this property in our context.

Definition 3.3 A function f from $\mathbf{2}^\mathbf{N}$ to $\mathbf{2}^\mathbf{N}$ is said to be *ordered* if, for every n, the nth component of $f(a)$ depends only on the final segment $a|_{\mathbf{N}-\{0,\ldots,n\}} = (a_j)|_{j>n}$. This means that, given two elements a and b, if their components having indices greater than n are equal, then the nth components of $f(a)$ and $f(b)$ are equal too. Note that, if f is ordered, the value a_0 has no influence on $f(a)$.

In the following sections, we will consider $\mathbf{2^N}$ as a *topological space*, that is, as the product of denumerably many copies of the discrete space {0, 1}. According to the definition of a product space, $\mathbf{2^N}$ is not a discrete space, but it is compact by the Tychonoff Theorem. In this space, a sequence $(a^h)_{h\in\mathbf{N}}$ of elements of $\mathbf{2^N}$ converges to an element a of $\mathbf{2^N}$ if and only if, for every n, the nth components of a^h eventually equal the nth component of a. As is well known, a function $f : \mathbf{2^N} \to \mathbf{2^N}$ is continuous if and only if all its component functions $f_n : \mathbf{2^N} \to \mathbf{2}$ are continuous. The space $\mathbf{2^N}$ is metrizable; in fact, it is homeomorphic to Cantor set. It follows that a function $f : \mathbf{2^N} \to \mathbf{2^N}$ is continuous if and only if it is sequentially continuous (i.e., it preserves the limit of any given sequence).

4 If the Function Is Continuous, a Fixed Point Does Exist

First of all, we have to point out that not all ordered functions from $\mathbf{2^N}$ to $\mathbf{2^N}$ have a fixed point. For example, it is enough to express Yablo's paradox in this context. The function g that corresponds to the sentences 'every subsequent sentence is false' can be described as follows:

$$\text{the } n\text{th component of } g(a) \text{ is } \begin{cases} 1, & \text{if } a_m = 0 \text{ for all } m > n \\ 0, & \text{otherwise.} \end{cases}$$

A fixed point, that is, a solution of the equation $g(a) = a$, should be an element of $\mathbf{2^N}$ where every digit 1 is followed only by 0s, whereas every digit 0 is followed by at least one 1. This is clearly impossible. It is important to note that the function g is not continuous with respect to the topology on $\mathbf{2^N}$. This can be proved using the characterization of continuous functions given in Lemma 4.1, but we prefer to verify that g does not preserve the limit of a sequence. Let a^h be the element of $\mathbf{2^N}$ with all components equal to 0 with the only exception of the hth component that equals 1. It is easy to see that

1. the limit of the sequence $(a^h)_{h\in\mathbf{N}}$ is $\mathbf{0}$ (where $\mathbf{0}$ is the element of $\mathbf{2^N}$ with all components 0),
2. $g(a^h)$ is the element $00\dots0111\dots$ where the first h digits are 0,
3. also the limit of the sequence $(g(a^h))_{h\in\mathbf{N}}$ is $\mathbf{0}$.

However, $g(\mathbf{0}) = \mathbf{1}$ (where $\mathbf{1}$ is the element of $\mathbf{2^N}$ with all components 1): the image of the limit is different from the limit of images.

Lemma 4.1

(i) *A function f_n from $\mathbf{2^N}$ in $\mathbf{2}$ is continuous if and only if $f_n(a)$ depends only on finitely many components of a. More precisely, f_n is continuous if and only if there exists a number m such that, for every element a, the value $f_n(a)$ depends only on the initial segment $a|_m = (a_j)|_{j<m}$.*

(ii) *A function f from $\mathbf{2^N}$ in $\mathbf{2^N}$ is continuous if and only if, for every n, the nth component of $f(a)$ depends only on finitely many components of a.*

Proof (i) ($\Rightarrow$) The proof follows by contradiction. Assume that $f_n(a)$ does not depend on finitely many components of a. Then we can find, for every m, two distinct elements b^m and c^m of $\mathbf{2^N}$ which share the first m components (in the sense that $b^m|_m = c^m|_m$) but are such that $f_n(b^m) = 0$ and $f_n(c^m) = 1$. Consider a convergent subsequence of the sequence $(b^m)_{m\in\mathbf{N}}$ (remember that $\mathbf{2^N}$ is a compact space) and let b be its limit. Note that b is also the limit of the corresponding sequence of c^m. But this contradicts the continuity of f_n, because this continuity would imply that $f_n(b)$ equals both 0 and 1.

($\Leftarrow$) Conversely, let us prove that, given a sequence $(a^h)_{h\in\mathbf{N}}$, when $(a^h) \to a$ we have $(f_n(a^h)) \to f_n(a)$. Assume that $f_n(a)$ depends only on $a|_m$. Since $(a^h) \to a$, there is a k such that, for any $h > k$, we have $a^h|_m = a|_m$. Therefore, under our hypotheses, $f_n(a) = f_n(a^h)$ for every $h > k$. So we conclude that $(f_n(a^h))_{h\in\mathbf{N}} \to f_n(a)$.

(ii) Obvious from part (i). □

Note that, if f is a continuous ordered function, then the nth component of $f(a)$ depends only on a finite family $(a_h/n < h < m) = (a_{n+1}, \dots, a_{m-1})$.

Theorem 4.2 *Every continuous ordered function f from $\mathbf{2^N}$ in $\mathbf{2^N}$ has a fixed point.*

Proof Let us start with a remark. Any final segment $a|_{\mathbf{N}-\{0,\dots,m\}} = (a_j)|_{j>m}$ can be "completed" to an element of $\mathbf{2^N}$ by means of the function f. More precisely, since f is ordered, the given segment allows us to compute the mth component of $f(a)$; if we put this element in front of the given segment, we obtain another final segment, whose domain is $\{j \in \mathbf{N}/j > m-1\}$. Iterating this procedure m times, we get a final segment whose domain is the whole set $\mathbf{N}$. Now, consider any element a of $\mathbf{2^N}$. Let b^q be the element of $\mathbf{2^N}$ obtained with the procedure we have just described, starting from the *final segment* $a|_{j>q}$. Consider a convergent subsequence of the sequence $(b^q)_{q\in\mathbf{N}}$ and let b be its limit. We claim that b is a fixed point for the function f.

Indeed, from Lemma 4.1, we know that the nth component of $f(b)$ depends only on an initial segment $b|_m$ of b. Considering a suitable index q_0 (greater than m and n), we have that $b|_m = b^q|_m$ for all $q > q_0$ in the considered subsequence. This equality guarantees that b_n is the nth component of $f(b^q)$ and, as a consequence, it is also the nth component of $f(b)$. In other words, $f(b) = b$. □

Corollary 4.3 *Let f be an ordered function from $\mathbf{2^N}$ in $\mathbf{2^N}$ with only finitely many discontinuous components f_n. Then f has a fixed point.*

Proof Let n_0 be the maximum index such that the n_0th component of f is discontinuous. Let us consider the function f^* restricted to the set $\{n/n > n_0\}$: just applying the theorem we have a final segment which is a fixed point for f^*. By completing this final segment as before, we have a fixed point for f. □

Combining the results of Lemma 4.1 and Theorem 4.2, we have the following application. Consider any infinite sequence of sentences $s_0, s_1, s_2, \dots$, where each sentence s_n refers to the truth values of sentences in the sequence. If each sentence refers only to finitely many subsequent sentences, then one can assign truth values to all sentences in a consistent way, and no paradox arises. (A similar result is quoted in Schlenker [9, footnote 3] and is credited to Martin and Visser.) This result holds also in case finitely many sentences refer to infinitely many subsequent sentences.

5 Some Examples

5.1 Examples of continuous functions We start from two trivial cases. Consider the sequence of sentences where every sentence is 'the next sentence is true'. The corresponding function f is a translation: $f(a)(n) = a_{n+1}$. This function has two fixed points: the sequence constituted by all 1s and the sequence constituted by all 0s. Likewise, there are just two fixed points when every sentence is 'the next sentence is false'. In this case, the fixed points are 010101 ... and 101010 ....

In the next example there is *only one* fixed point. Consider the following sentences:

$$s_0: \quad \text{the sentences } s_1 \text{ and } s_2 \text{ are false,}$$
$$s_1: \quad \text{the sentence } s_2 \text{ is false;}$$

similarly, for every n,

$$s_{2n}: \quad \text{the sentences } s_{2n+1} \text{ and } s_{2n+2} \text{ are false,}$$
$$s_{2n+1}: \quad \text{the sentence } s_{2n+2} \text{ is false.}$$

It is easy to see that every s_{2n} must be false; therefore, every s_{2n+1} must be true. The only fixed point is 010101 .... We get a similar situation considering the sentences,

$$s_{2n}: \quad \text{the sentences } s_{2n+1} \text{ and } s_{2n+2} \text{ have the same truth value,}$$
$$s_{2n+1}: \quad \text{the sentence } s_{2n+2} \text{ is false.}$$

The only fixed point is the same as in the previous example.

Now, for every n, consider the sentence s_n : 'the sentence s_{n+2} is true'. Since sentences with even indices and sentences with odd indices are independent from each other, there are 4 fixed points: $0000\ldots, 1111\ldots, 0101\ldots, 1010\ldots.$ In a similar way, we can construct examples of continuous ordered functions from $\mathbf{2}^{\mathbf{N}}$ in $\mathbf{2}^{\mathbf{N}}$ with 8 or more fixed points. Moreover, partition the set $\mathbf{N}$ in infinitely many infinite sets $(A_i)_{i\in\mathbf{N}}$ and list any A_i in increasing order: $A_i = \{x_{i,0}, x_{i,1}, x_{i,2}, \ldots\}$. If, for every $x_{i,n}$, the sentence with index $x_{i,n}$ is 'the sentence with index $x_{i,n+1}$ is true', the corresponding function f has *infinitely many* fixed points (more precisely, nondenumerably many fixed points).

Remark 5.1 Let f be a continuous ordered function from $\mathbf{2}^{\mathbf{N}}$ in $\mathbf{2}^{\mathbf{N}}$. Two distinct fixed points of f cannot eventually coincide: indeed, if a is a fixed point, the final segment $\{a_m/m > n\}$ determines the initial segment $a|_{m+1}$ of a in a unique way. The set of fixed points of f is a closed set with empty interior. Indeed, the set of fixed points is closed for every continuous function. If the set of all sequences sharing the same initial segment would be included in the set of fixed points, then there would be two distinct fixed points with the same final segment.

5.2 Examples of noncontinuous functions (variants of Yablo's paradox) We have a contradiction also in case we consider the sequence in which all sentences are 'at least one of the subsequent sentences is false'. Indeed, every 1 must be followed by at least one 0, while every 0 must be followed only by 1s. Therefore, it is impossible to make any consistent assignment of truth values to all sentences. Of course, continuity is not a necessary condition for the existence of a fixed point. If all sentences are 'at least one of the subsequent sentences is true', there are two fixed points (all sentences may be true, and all sentences may be false). Similarly, there are the same two solutions when all sentences are 'all the subsequent sentences are true'.

Slightly modifying the original situation, we can construct another couple of paradoxes. The first one is a "Curry-style" or "Löb-style" version, which can be regarded as a negation-free paradox. Given any statement H, we can prove H by considering the sequence where each sentence is 'if at least one of the subsequent sentences is true, then H'. The statement H can easily be deduced (assume $\neg H$: each sentence becomes equivalent to the negation of 'at least one of the subsequent sentences is true', and so we get a contraction as in Yablo's paradox).

Finally, assume that each sentence is 'among the subsequent sentences, only finitely many are true'. Also in this case we quickly reach a paradox. Indeed, after a true sentence there are only finitely many true sentences, while after a false sentence there are infinitely many true sentences, which is clearly impossible. One could say that the last paradox is without self-reference, without negation, and without implication, but I understand that somebody may disagree with this...

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Acknowledgments

Study realized within the project PRIN 2007B2M4EK, jointly funded by MIUR and by Sapienza, Università di Roma. I would like to thank Laura Mazzoli and Andrea Sellaroli for their helpful comments and for supplying me with bibliographical references.

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