Notre Dame Journal of Formal Logic
Volume 48, Number 4, 2007

An Algebraic Approach to the Disjunction Property of Substructural Logics

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Abstract Some of the basic substructural logics are shown by Ono to have the disjunction property (DP) by using cut elimination of sequent calculi for these logics. On the other hand, this syntactic method works only for a limited number of substructural logics. Here we show that Maksimova's criterion on the DP of superintuitionistic logics can be naturally extended to one on the DP of substructural logics over **FL**. By using this, we show the DP for some of the substructural logics for which syntactic methods don't work well.

1 Algebraic Characterization of the Disjunction Property

We will give a brief explanation of substructural logics over **FL** and residuated lattices. For more information, see [2] and [7].

Let **FL** be the sequent system obtained from Gentzen's **LJ** for intuitionistic logic by deleting all structure rules and adding rules for *fusion*. Here we assume that the language for **FL** contains not only fusion, $\varphi \cdot \psi$, but also two types of *implication*, φ/ψ and $\psi\backslash\varphi$, and two types of *negation*, $-\varphi$ and $\sim\varphi$, for given formulas φ and ψ, because of the lack of exchange rule. When the exchage rule is added, they are written as $\varphi \to \psi$ and $\neg\varphi$, respectively. Sequent systems $\mathbf{FL_e}$ ($\mathbf{FL_{ew}}$, $\mathbf{FL_{ec}}$) are obtained from **FL** by adding the exchange rule (and also the weakening rules and the contraction rule, respectively). We use **FL** not only for the sequent system, but also for the set of formulas provable in it. By a *substructural logic over* **FL** we mean an axiomatic extension of **FL**. In the following, we call a substructural logic over **FL** simply a *logic* if no confusion occurs. For a given formula φ, if a logic **L** is obtained from **FL** by adding every initial sequent of the form $\Rightarrow \tilde{\varphi}$, where $\tilde{\varphi}$ is any substitution instance of φ, **L** is expressed as $\mathbf{FL}[\varphi]$. We use similar notation for axiomatic extensions of $\mathbf{FL_e}$ and $\mathbf{FL_{ew}}$.

Received March 28, 2006; accepted March 1, 2007; printed October 22, 2007
2000 Mathematics Subject Classification: Primary, 03B47, 03G25
Keywords: substructural logic, residuated lattice, disjunction property, well-connectedness

Now we consider the following axioms:

$\mathrm{E}^{\mathrm{m}}_{\mathrm{n}}$ $(p^m\backslash p^n)$ $(m \geq 0,\ n \geq 0)$,
DN $(\sim -p\backslash p) \wedge (- \sim p\backslash p)$ (*double negation*),
Dis $((p \wedge (q \vee r))\backslash((p \wedge q) \vee (p \wedge r)))$ (*distributive law*),

where p, q, r are propositional variables. Note that $\mathrm{E}^{\mathrm{m}}_{\mathrm{n}}$, when $m = 1$ and $n = 2$, is equivalent to the contraction rule, and when $m = 1$ and $n = 0$, it is equivalent to the weakening rule.

We say that a logic **L** has the *disjunction property* when, for any formulas φ and ψ, if $\varphi \vee \psi$ is provable in **L** then at least one of the formulas φ and ψ is provable in it. We know that even if a logic **L** is involutive, that is, the law of double negation holds in **L**, if **L** is cut-free and doesn't have right-contraction, then **L** has the disjunction property (see [6]). But having a cut-free sequent calculus is rather exceptional. For example, $\mathbf{FL_e}[\mathrm{E}^{\mathrm{n}}_{1}]$ and $\mathbf{FL}[\mathrm{E}^{0}_{1}]$ are cut-free for every n, but neither $\mathbf{FL}[\mathrm{E}^{\mathrm{n}}_{\mathrm{k}}]$ (where $n \neq 0$ or $k \neq 1$) nor $\mathbf{FL_w}[\mathrm{E}^{\mathrm{k+1}}_{\mathrm{k}}]$ are cut-free. See [3] and [1].

Algebras for logics over **FL** are defined by using *residuated lattices* (RLs). An algebra $\mathbf{A} = \langle \mathrm{A}, \wedge, \vee, \cdot, /, \backslash, 1\rangle$ is an RL if it satisfies the following three conditions.

(R1) $\langle \mathrm{A}, \wedge, \vee\rangle$ is a lattice,
(R2) $\langle \mathrm{A}, \cdot, 1\rangle$ is a monoid with the unit 1,
(R3) for $x, y, z \in \mathrm{A}$, $x \cdot y \leq z \Leftrightarrow x \leq z/y \Leftrightarrow y \leq x\backslash z$.

(R3) is called the residuation law. An **FL**-*algebra* is an algebra $\langle \mathrm{A}, \wedge, \vee, \cdot, /, \backslash, 0, 1\rangle$ where $\langle \mathrm{A}, \wedge, \vee, \cdot, /, \backslash, 1\rangle$ is an RL and 0 is an arbitrary element of A. Formulas of the language for **FL** are interpreted in a given **FL**-algebra **A** in the usual way by using valuations on it. We remark that negations are defined by $-x = 0/x$ and $\sim x = x\backslash 0$. Since 1 is not always the greatest element of A, we need to modify the validity of formulas as follows. A formula φ is *valid* in **A** if and only if $v(\varphi) \geq 1$ for every valuation v on **A**. An **FL**-algebra in which $\cdot$ satisfies commutativity, that is, $x \cdot y = y \cdot x$, is called a commutative **FL**-algebra, or an $\mathbf{FL_e}$-*algebra*. In any $\mathbf{FL_e}$-algebra, $x\backslash y$ is equal to y/x for all x, y.

For each logic **L**, let $V(\mathbf{L})$ be the class of all **FL**-algebras in which every formula provable in **L** is valid. In fact, $V(\mathbf{L})$ is a subvariety of the variety $\mathcal{FL}$ of all **FL**-algebras, and, conversely, any subvariety of $\mathcal{FL}$ is equal to $V(\mathbf{L})$ for some logic **L**. See [2] for further details.

A logic **L** is said to be *complete* with respect to a subclass $\mathcal{K}$ of $V(\mathbf{L})$ when $\mathcal{K}$ generates $V(\mathbf{L})$. In other words, if a formula φ is not valid in an algebra in $V(\mathbf{L})$ it is not valid in an algebra in $\mathcal{K}$ either.

In [4], Maksimova gave an algebraic characterization of the disjunction property for *superintuitionistic logics*, that is, logics over intuitionistic logic, by using *well-connected algebras*. More precisely, a Heyting algebra **A** is well-connected if and only if, for all $x, y \in A$, $x \vee y = 1$ implies either $x = 1$ or $y = 1$. Since the unit 1 is not always the greatest element of A in the present case, again we need some obvious modifications. An **FL**-algebra **A** is *well-connected* if and only if for all $x, y \in A$, $x \vee y \geq 1$ implies either $x \geq 1$ or $y \geq 1$. Then we have the following. The proof is given almost in the same way as that in [4].

Theorem 1.1 *Suppose that a logic* **L** *over* **FL** *is complete with respect to a class* $\mathcal{K}$ *of* **FL**-*algebras. Then the following are equivalent:*

1. **L** *has the disjunction property;*

2. *For all* $\mathbf{A}, \mathbf{B} \in \mathcal{K}$ *there exist a well-connected* $\mathbf{FL}$*-algebra* $\mathbf{C} \in V(\mathbf{L})$ *and a surjective homomorphism from* $\mathbf{C}$ *onto the direct product* $\mathbf{A} \times \mathbf{B}$ *of* $\mathbf{A}$ *and* $\mathbf{B}$*.*

2 Disjunction Property of Some Substructural Logics

We show the disjunction property of some logics over $\mathbf{FL}$ by using Theorem 1.1.

Theorem 2.1 (Disjunction property for $\mathbf{FL}[\mathrm{E}^{\mathrm{m}}_{\mathrm{n}}]$) *Both* $\mathbf{FL}[\mathrm{E}^{\mathrm{m}}_{\mathrm{n}}]$ *and* $\mathbf{FL_e}[\mathrm{E}^{\mathrm{m}}_{\mathrm{n}}]$ *have the disjunction property for every* m, n*.*

Proof To prove this theorem we construct a suitable RL $\mathbf{C}$ for RLs $\mathbf{A}$ and $\mathbf{B}$ which are given as follows.

1. $\mathbf{A} = \langle \mathrm{A}, \wedge_{\mathbf{A}}, \vee_{\mathbf{A}}, \cdot_{\mathbf{A}}, /_{\mathbf{A}}, \backslash_{\mathbf{A}}, 0_{\mathbf{A}}, 1_{\mathbf{A}} \rangle$.
2. $\mathbf{B} = \langle \mathrm{B}, \wedge_{\mathbf{B}}, \vee_{\mathbf{B}}, \cdot_{\mathbf{B}}, /_{\mathbf{B}}, \backslash_{\mathbf{B}}, 0_{\mathbf{B}}, 1_{\mathbf{B}} \rangle$.

Define an RL $\mathbf{C} = \langle \mathrm{C}, \wedge, \vee, \cdot, /, \backslash, \mathbf{0}, \mathbf{1} \rangle$ as follows: Let $\mathbf{2}$ be the two-element Boolean algebra with the universe $\{0, 1\}$. Take the direct product $\mathbf{A} \times \mathbf{B} \times \mathbf{2}$. Consider a subset $\mathrm{C} = \{(a, b, 0) | a \in \mathrm{A}, b \in \mathrm{B}\} \cup \{(a, b, 1) | a \in \mathrm{A}, b \in \mathrm{B}, a \geq_{\mathbf{A}} 1_{\mathbf{A}}, b \geq_{\mathbf{B}} 1_{\mathbf{B}}\}$ of $\mathrm{A} \times \mathrm{B} \times 2$. Observe that C is closed under lattice operations but not under multiplication or residuals. Define multiplication on C as follows:

$$\langle a, b, i \rangle \cdot \langle a', b', j \rangle = \langle a \cdot_{\mathbf{A}} a', b \cdot_{\mathbf{B}} b', i \cdot j \rangle.$$

Now observe that the unit element for multiplication is $(1_{\mathbf{A}}, 1_{\mathbf{B}}, 1)$ and that it is join irreducible in the lattice ordering of C since it has a unique subcover in C, namely, $(1_{\mathbf{A}}, 1_{\mathbf{B}}, 0)$. Then we define residuals on C as follows:

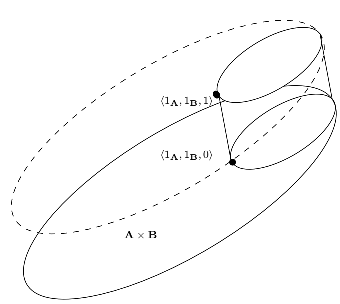
$$\langle a, b, i \rangle \backslash \langle a', b', j \rangle = \begin{cases} \langle a \backslash_{\mathbf{A}} a', b \backslash_{\mathbf{B}} b', i \backslash j \rangle & \text{if } \langle a \backslash_{\mathbf{A}} a', b \backslash_{\mathbf{B}} b', i \backslash j \rangle \in \mathrm{C} \\ \langle a \backslash_{\mathbf{A}} a', b \backslash_{\mathbf{B}} b', 0 \rangle & \text{if } \langle a \backslash_{\mathbf{A}} a', b \backslash_{\mathbf{B}} b', i \backslash j \rangle \notin \mathrm{C}. \end{cases}$$

Similarly for left residuals.

We can easily show that $\mathbf{C}$ is a well-connected RL. The mapping h defined by $h(\langle a, b, i \rangle) = \langle a, b \rangle$ for $a \in \mathrm{A}, b \in \mathrm{B}, i \in \{0, 1\}$ is a surjective homomorphism from $\mathbf{C}$ to $\mathbf{A} \times \mathbf{B}$. Now we show that $\mathbf{C}$ satisfies the condition $\mathrm{E}^{\mathrm{m}}_{\mathrm{n}}$, assuming that both $\mathbf{A}$ and $\mathbf{B}$ $\mathrm{E}^{\mathrm{m}}_{\mathrm{n}}$. Then $1_{\mathbf{A}} \leq a^m \backslash a^n$, $1_{\mathbf{B}} \leq b^m \backslash b^n$. For all $\langle a, b, i \rangle \in \mathrm{C}$,

$$\begin{aligned} \langle a, b, i \rangle^m \backslash \langle a, b, i \rangle^n &= \langle a^m, b^m, i^m \rangle \backslash \langle a^n, b^n, i^n \rangle \\ &= \langle a^m \backslash_{\mathbf{A}} a^n, b^m \backslash_{\mathbf{B}} b^n, i^m \backslash i^n \rangle \\ &\geq \langle 1_{\mathbf{A}}, 1_{\mathbf{B}}, 1 \rangle. \end{aligned}$$

Hence $\langle 1_{\mathbf{A}}, 1_{\mathbf{B}}, u \rangle \leq \langle a, b, i \rangle^m \backslash \langle a, b, i \rangle^n$. Thus the algebra $\mathbf{C}$ satisfies $\mathrm{E}^{\mathrm{m}}_{\mathrm{n}}$.



It is clear that if both $\mathbf{A}$ and $\mathbf{B}$ are commutative then so is $\mathbf{C}$. From this observation, the DP of $\mathbf{FL_e}[\mathrm{E_n^m}]$ follows. □

As special cases, we have the DP of $\mathbf{FL}$, $\mathbf{FL_e}$, $\mathbf{FL}[\mathrm{E_k}]$, and $\mathbf{FL_e}[\mathrm{E_k}]$, where $\mathrm{E_k}$ is the *weak k-potency* (i.e., $\mathrm{E_k^{k+1}}$). See [9] and [10]. Since $\mathrm{E_2^1}$ is the axiom of contraction, the DP of $\mathbf{FL_{ec}}$ follows also.

We note that in this proof if we assume, moreover, that both $\mathbf{A}$ and $\mathbf{B}$ satisfy Dis, that is, their lattice reducts are distributive, then we can show that also $\mathbf{C}$ satisfies Dis. Hence we have also the following theorem.

Corollary 2.2 *Both* $\mathbf{FL}[\mathrm{E_n^m}, \mathrm{Dis}]$ *and* $\mathbf{FL_e}[\mathrm{E_n^m}, \mathrm{Dis}]$ *have the disjunction property. In particular,* $\mathbf{FL}[\mathrm{Dis}]$ *has the disjunction property.*

As the existence of the zero element $\mathbf{0}$ of $\mathbf{C}$ doesn't play any particular role in the proof of Theorem 2.1, we can derive that each positive fragment of these logics has also the disjunction property. Since the positive relevant logic $\mathbf{R}^+$ is equal to the positive fragment of $\mathbf{FL_e}[\mathrm{E_2^1}, \mathrm{Dis}]$, we have an alternative proof of the disjunction property of $\mathbf{R}^+$, which was first proved by Meyer in [5].

In general, the RL $\mathbf{C}$ in the proof of Theorem 2.1 doesn't satisfy DN, even if both $\mathbf{A}$ and $\mathbf{B}$ satisfy DN. For example, if $1_\mathbf{A} \leq_\mathbf{A} a$, $1_\mathbf{B} \leq_\mathbf{B} b$ then

$$\begin{aligned} \sim -\langle a, b, 0\rangle \backslash \langle a, b, 0\rangle &= \langle \sim_\mathbf{A} -_\mathbf{A} a, \sim_\mathbf{B} -_\mathbf{B} b, 1\rangle \backslash \langle a, b, 0\rangle \\ &= \langle \sim_\mathbf{A} -_\mathbf{A} a \backslash_\mathbf{A} a, \sim_\mathbf{B} -_\mathbf{B} b \backslash_\mathbf{B} b, 0\rangle \\ &\not\geq \langle 1_\mathbf{A}, 1_\mathbf{B}, 1\rangle. \end{aligned}$$

So we need some modification of the definition of $\mathbf{C}$ in proving the disjunction property of $\mathbf{FL}[\mathrm{DN}]$.

Note that $\mathbf{FL_e}[\mathrm{DN}]$ is nothing but the multiplicative additive linear logic MALL. As mentioned in [6], $\mathbf{FL_e}[\mathrm{DN}]$ has the DP since it is formulated by a cut-free sequent system without the right contraction rule. Here we give an algebraic proof of it.

Theorem 2.3 (Disjunction property for FL[DN]) *Both* $\mathbf{FL}[\mathrm{DN}]$ *and* $\mathbf{FL_e}[\mathrm{DN}]$ *have the disjunction property.*

Proof We prove this theorem in the same way as Theorem 2.1. Suppose that $\mathbf{A}$ and $\mathbf{B}$ are given. Define an RL $\mathbf{D} = \langle \mathrm{D}, \wedge, \vee, \cdot, \backslash, /, \mathbf{0}, \mathbf{1}\rangle$ as follows: Let $\mathbf{C_3}$ be the three element MV-algebra with the universe $\{0, \frac{1}{2}, 1\}$. Take the direct product $\mathbf{A} \times \mathbf{B} \times \mathbf{C_3}$. Consider a subset $\mathrm{D} = \{\langle a, b, \frac{1}{2}\rangle \mid a \in \mathrm{A}, b \in \mathrm{B}\} \cup \{\langle a, b, 1\rangle \mid a \in \mathrm{A}, b \in \mathrm{B}, a \geq_\mathbf{A} 1_\mathbf{A}, b \geq_\mathbf{B} 1_\mathbf{B}\} \cup \{\langle a, b, 0\rangle \mid a \in \mathrm{A}, b \in \mathrm{B}, a \leq_\mathbf{A} 0_\mathbf{A}, b \leq_\mathbf{B} 0_\mathbf{B}\}$ of $\mathrm{A} \times \mathrm{B} \times \mathrm{C}_3$. Similarly to the previous construction, we can show that D is closed under lattice operations but not under multiplication or residuals. Define multiplication on D as follows:

1. If $a \cdot_\mathbf{A} a' \leq 0_\mathbf{A}$, $b \cdot_\mathbf{B} b' \leq_\mathbf{B} 0_\mathbf{B}$ and $i, j \notin \{0, \frac{1}{2}\}$, then
$$\langle a, b, i\rangle \cdot \langle a', b', j\rangle = \langle a \cdot_\mathbf{A} a', b \cdot_\mathbf{B} b', 0\rangle.$$
2. Otherwise,
$$\langle a, b, i\rangle \cdot \langle a', b', j\rangle = \begin{cases} \langle a \cdot_\mathbf{A} a', b \cdot_\mathbf{B} b', i \cdot j\rangle & \text{if } \langle a \cdot_\mathbf{A} a', b \cdot_\mathbf{B} b', i \cdot j\rangle \in \mathrm{D} \\ \langle a \cdot_\mathbf{A} a', b \cdot_\mathbf{B} b', \frac{1}{2}\rangle & \text{if } \langle a \cdot_\mathbf{A} a', b \cdot_\mathbf{B} b', i \cdot j\rangle \notin \mathrm{D}. \end{cases}$$

The unit element for multiplication is $\langle 1_\mathbf{A}, 1_\mathbf{B}, 1\rangle$ and is join irreducible in the lattice ordering of D since it has a unique subcover $\langle 1_\mathbf{A}, 1_\mathbf{B}, \frac{1}{2}\rangle$ in D. Then we define residuals on D as follows:

1. If $i = \frac{1}{2}$ and $\langle a', b', j\rangle \leq \langle 0_{\mathbf{A}}, 0_{\mathbf{B}}, 0\rangle$, then
$$\langle a, b, i\rangle\backslash\langle a', b', j\rangle = \langle a\backslash_{\mathbf{A}}a', b\backslash_{\mathbf{B}}b', \tfrac{1}{2}\rangle.$$
2. Otherwise,
$$\langle a, b, i\rangle\backslash\langle a', b', j\rangle = \begin{cases} \langle a\backslash_{\mathbf{A}}a', b\backslash_{\mathbf{B}}b', i\backslash j\rangle & \text{if } \langle a\backslash_{\mathbf{A}}a', b\backslash_{\mathbf{B}}b', i\backslash j\rangle \in \mathrm{D} \\ \langle a\backslash_{\mathbf{A}}a', b\backslash_{\mathbf{B}}b', \tfrac{1}{2}\rangle & \text{if } \langle a\backslash_{\mathbf{A}}a', b\backslash_{\mathbf{B}}b', i\backslash j\rangle \notin \mathrm{D}. \end{cases}$$

Similarly for left residuals.

Then we can show that $\sim\langle a, b, 0\rangle = \langle\sim_{\mathbf{A}} a, \sim_{\mathbf{B}} b, 1\rangle$, $\sim\langle a, b, \frac{1}{2}\rangle = \langle\sim_{\mathbf{A}} a, \sim_{\mathbf{B}} b, \frac{1}{2}\rangle$, $\sim\langle a, b, 1\rangle = \langle\sim_{\mathbf{A}} a, \sim_{\mathbf{B}} b, 0\rangle$, $-\langle a, b, 0\rangle = \langle -_{\mathbf{A}}a, -_{\mathbf{B}}b, 1\rangle$, $-\langle a, b, \frac{1}{2}\rangle = \langle -_{\mathbf{A}}a, -_{\mathbf{B}}b, \frac{1}{2}\rangle$, and $-\langle a, b, 1\rangle = \langle -_{\mathbf{A}}a, -_{\mathbf{B}}b, 0\rangle$.

We can show that the algebra $\mathbf{D}$ is a well-connected RL and that the mapping h defined by $h(\langle a, b, i\rangle) = \langle a, b\rangle$ for $a \in \mathrm{A}, b \in \mathrm{B}, i \in \{0, \frac{1}{2}, 1\}$ is a surjective homomorphism from $\mathbf{D}$ to $\mathbf{A} \times \mathbf{B}$. We show now that $\mathbf{D}$ satisfies DN, assuming that both $\mathbf{A}$ and $\mathbf{B}$ satisfy the condition DN, that is, $1_{\mathbf{A}} \leq_{\mathbf{A}} (\sim_{\mathbf{A}} -_{\mathbf{A}}a\backslash_{\mathbf{A}}a) \wedge_{\mathbf{A}} (a/-_{\mathbf{A}} \sim_{\mathbf{A}} a)$, $1_{\mathbf{B}} \leq_{\mathbf{B}} (\sim_{\mathbf{B}} -_{\mathbf{B}}b\backslash_{\mathbf{B}}b) \wedge_{\mathbf{B}} (b/-_{\mathbf{B}} \sim_{\mathbf{B}} b)$. For all $\langle a, b, i\rangle \in \mathrm{D}$, if $i = 0$, then from $a \leq_{\mathbf{A}} 0_{\mathbf{A}}$ and $b \leq_{\mathbf{B}} 0_{\mathbf{B}}$,
$$\begin{aligned} \sim -\langle a, b, i\rangle\backslash\langle a, b, i\rangle &= \sim\langle -_{\mathbf{A}}a, -_{\mathbf{B}}b, 1\rangle\backslash\langle a, b, i\rangle \\ &= \langle\sim_{\mathbf{A}} -_{\mathbf{A}}a, \sim_{\mathbf{B}} -_{\mathbf{B}}b, 0\rangle\backslash\langle a, b, i\rangle \\ &= \langle\sim_{\mathbf{A}} -_{\mathbf{A}}a\backslash_{\mathbf{A}}a, \sim_{\mathbf{B}} -_{\mathbf{B}}b\backslash b, 0\backslash_{\mathbf{B}}i\rangle \\ &\geq \langle 1_{\mathbf{A}}, 1_{\mathbf{B}}, 1\rangle. \end{aligned}$$
If $i = \frac{1}{2}$, then
$$\begin{aligned} \sim -\langle a, b, i\rangle\backslash\langle a, b, i\rangle &= \sim\langle -_{\mathbf{A}}a, -_{\mathbf{B}}b, \tfrac{1}{2}\rangle\backslash\langle a, b, i\rangle \\ &= \langle\sim_{\mathbf{A}} -_{\mathbf{A}}a, \sim_{\mathbf{B}} -_{\mathbf{B}}b, \tfrac{1}{2}\rangle\backslash\langle a, b, i\rangle \\ &= \langle\sim_{\mathbf{A}} -_{\mathbf{A}}a\backslash_{\mathbf{A}}a, \sim_{\mathbf{B}} -_{\mathbf{B}}b\backslash_{\mathbf{B}}b, \tfrac{1}{2}\backslash i\rangle \\ &\geq \langle 1_{\mathbf{A}}, 1_{\mathbf{B}}, 1\rangle. \end{aligned}$$
If $i = 1$, then
$$\begin{aligned} \sim -\langle a, b, i\rangle\backslash\langle a, b, i\rangle &= \sim\langle -_{\mathbf{A}}a, -_{\mathbf{B}}b, 0\rangle\backslash\langle a, b, i\rangle \\ &= \langle\sim_{\mathbf{A}} -_{\mathbf{A}}a, \sim_{\mathbf{B}} -_{\mathbf{B}}b, 1\rangle\backslash\langle a, b, i\rangle \\ &= \langle\sim_{\mathbf{A}} -_{\mathbf{A}}a\backslash_{\mathbf{A}}a, \sim_{\mathbf{B}} -_{\mathbf{B}}b\backslash_{\mathbf{B}}b, 1\backslash i\rangle \\ &\geq \langle 1_{\mathbf{A}}, 1_{\mathbf{B}}, 1\rangle. \end{aligned}$$
Similarly, we can easily show $-\sim\langle a, b, i\rangle\backslash\langle a, b, i\rangle \geq \langle 1_{\mathbf{A}}, 1_{\mathbf{B}}, 1\rangle$. So $(\sim-\langle a, b, i\rangle\backslash\langle a, b, i\rangle) \wedge (- \sim\langle a, b, i\rangle\backslash\langle a, b, i\rangle) \geq \langle 1_{\mathbf{A}}, 1_{\mathbf{B}}, 1\rangle$. Thus the algebra $\mathbf{D}$ satisfies DN.

It is clear that if both $\mathbf{A}$ and $\mathbf{B}$ are commutative then so is $\mathbf{D}$. From this observation, the DP of $\mathbf{FL_e}$[DN] follows. Thus, by Theorem 1.1, $\mathbf{FL}$[DN] has the DP. □

In this proof, suppose, moreover, that both $\mathbf{A}$ and $\mathbf{B}$ satisfy Dis. $\langle \mathrm{D}, \wedge, \vee\rangle$ is a sublattice of the direct product $\mathbf{A} \times \mathbf{B} \times \mathbf{C_3}$. Since the lattice reduct of any of $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{3}$ is distributive, $\mathbf{D}$ satisfies also Dis. We can show the following corollary.

Corollary 2.4 *Both* $\mathbf{FL}$[Dis, DN] *and* $\mathbf{FL_e}$[Dis, DN] *have the disjunction property.*

Note that $\mathbf{FL_e}$[Dis, DN] is equivalent to the contractionless relevant logic $\mathbf{RW}$, whose disjunction property is shown in [8] by using *metavaluations*.

We can show the following by extending Theorems 2.1 and 2.3 when we have weakening rules, that is, when we assume $x \le 1$ and $0 \le x$ for any x in algebras.

Corollary 2.5 *$\mathbf{FL_{ew}}[\mathrm{E}_n^m]$ and $\mathbf{FL_{ew}}[\mathrm{DN}]$ have the disjunction property.*

Since $\mathbf{FL_{ew}}[\mathrm{E}_2^1]$ is equal to intuitionistic logic **Int**, this leads to an algebraic proof of the DP of **Int**.

On the other hand, these proofs cannot be combined together. That is, the argument doesn't work well for $\mathbf{FL_x}[\mathrm{E}_n^m, \mathrm{DN}]$, where $\mathbf{x}$ is either empty or $\mathbf{e}$ or $\mathbf{ew}$. In fact, the DP doesn't hold for cases like $\mathbf{FL_{ew}}[\mathrm{E}_n^m, \mathrm{DN}]$, since the latter is equal to classical logic. Note that in the proof of Theorem 2.3, $\mathbf{D}$ is not always a Boolean algebra even if both $\mathbf{A}$ and $\mathbf{B}$ are Boolean algebras.

References

[1] Bayu Surarso, and H. Ono, "Cut elimination in noncommutative substructural logics," *Reports on Mathematical Logic*, no. 30 (1996), pp. 13–29. Zbl 0896.03048. MR 1642157. 490

[2] Galatos, N., and H. Ono, "Algebraization, parametrized local deduction theorem and interpolation for substructural logics over **FL**," *Studia Logica*, vol. 83 (2006), pp. 279–308. Zbl 1105.03021. MR 2250112. 489, 490

[3] Hori, R., H. Ono, and H. Schellinx, "Extending intuitionistic linear logic with knotted structural rules," *Notre Dame Journal of Formal Logic*, vol. 35 (1994), pp. 219–42. Zbl 0812.03008. MR 1295560. 490

[4] Maksimova, L. L., "On maximal intermediate logics with the disjunction property," *Studia Logica*, vol. 45 (1986), pp. 69–75. Zbl 0635.03019. MR 877302. 490

[5] Meyer, R. K., "Metacompleteness," *Notre Dame Journal of Formal Logic*, vol. 17 (1976), pp. 501–16. Zbl 0232.02015. MR 0439593. 492

[6] Ono, H., "Proof-theoretic methods in nonclassical logic—An introduction," pp. 207–54 in *Theories of Types and Proofs (Tokyo, 1997)*, vol. 2 of *MSJ Memoirs*, Mathmatical Society of Japan, Tokyo, 1998. Zbl 0947.03073. MR 1728763. 490, 492

[7] Ono, H., "Substructural logics and residuated lattices—An introduction," pp. 193–228 in *Trends in Logic. 50 Years of Studia Logica*, vol. 21 of *Trends in Logic Studia Logica Library*, Kluwer Academic Publications, Dordrecht, 2003. Zbl 1048.03018. MR 2045284. 489

[8] Slaney, J. K., "A metacompleteness theorem for contraction-free relevant logics," *Studia Logica*, vol. 43 (1984), pp. 159–68. Zbl 0576.03014. MR 782856. 493

[9] Souma, D., "Algebraic approach to disjunction property of substructural logics," pp. 26–28 in *Proceedings of 38th MLG Meeting at Gamagori, Japan 2004*, 2004. 492

[10] Souma, D., "An algebraic approach to the disjunction property of substructural logics," Master's thesis, Japan Advanced Institute of Science and Technology, Ishikawa, 2005. 492

Acknowledgments

The author thanks the anonymous referee for her/his suggestions, and also Prof. H. Ono and Dr. T. Seki for valuable advice and guidance in this work.

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