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A Simple Embedding of T into Double S5

Steven Kuhn

Abstract The system obtained by adding full propositional quantification to **S5** is known to be decidable, while that obtained by doing so for **T** is known to be recursively intertranslatable with full second-order logic. Recently it was shown that the system with two **S5** operators and full propositional quantification is also recursively intertranslatable with second-order logic. This note establishes that the map assigning $[1][2]p$ to $\Box p$ provides a validity and satisfaction preserving translation between the **T** system and the double **S5** system, thus providing an easier proof of the recent result.

1 Introduction

For a natural number n, an *n-modal system* is a language with operators $[1], \ldots, [n]$ interpreted by world-world relations $R_1, \ldots, R_n$ according to the familiar Kripke semantics. **Double S5** is the 2-modal system determined by all frames (W, R_1, R_2) such that R_1 and R_2 are equivalence relations. (The nomenclature system envisioned here would, for example, take **S5S4K** to be the 3-modal system determined by frames (W, R_1, R_2, R_3) where R_1 is an equivalence relation and R_2 is symmetric and transitive and it would take **Double S5** to be **S5S5**.) In this paper we give a simple embedding of **T** into **Double S5** that extends to the case where both systems are supplemented by propositional quantifiers ranging over all subsets of worlds. This provides a quick proof that **Double S5** with such quantifiers is recursively intertranslatable with full second-order logic, a result that was recently obtained by more arduous methods in Antonelli and Thomason [1]. The result is noteworthy because ordinary **S5** with full propositional quantifiers is known to be decidable. (See Fine [2].)

2 Languages, Interpretations, and Systems

The *formulas* of $\mathcal{L}_\Box$ are built up in the usual way from a countable set $p_1, p_2, \ldots$ of propositional variables by the classical connectives $\neg$ and $\vee$ and the unary modal

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operator $\Box$. The formulas of $\mathcal{L}_{[1][2]}$ are defined similarly using the unary modal operators [1] and [2] in place of $\Box$. The formulas of $\mathcal{L}^{\pi}_{\Box}$ and $\mathcal{L}^{\pi}_{[1][2]}$ are defined by adding to the definitions of the formulas of $\mathcal{L}_{\Box}$ and $\mathcal{L}_{[1][2]}$ the clause: if p is a propositional variable and A is a formula then $\forall pA$ is a formula.

A *frame* for $\mathcal{L}_{\Box}$ or $\mathcal{L}^{\pi}_{\Box}$ is a pair $F = (W, R)$ where W is a nonempty set (the *worlds* of F) and R is a binary relation (*accessibility*) on W. A frame for $\mathcal{L}_{[1][2]}$ or $\mathcal{L}^{\pi}_{[1][2]}$ is a triple $F = (W, R_1, R_2)$ where W is a nonempty set, $R_1 \subseteq W \times W$, and $R_2 \subseteq W \times W$. A *model* for $\mathcal{L}_{\Box}$ or $\mathcal{L}^{\pi}_{\Box}$ is a triple $M = (W, R, V)$ where (W, R) is a frame for that language and $V : N \to 2^W$. V is the *valuation function* of M. If $V(i) = U$ we say that U is the *proposition* expressed by p_i in M. A model (W, R, V) is said to be a model *on the frame* (W, R). Similarly, a model for $\mathcal{L}_{[1][2]}$ or $\mathcal{L}^{\pi}_{[1][2]}$ is a quadruple (W, R_1, R_2, V) where (W, R_1, R_2) is a frame for that language and V is a valuation function as above.

Definition 2.1 Suppose $M = (W, R, V)$ is a model for $\mathcal{L}_{\Box}$ and $w \in W$. The notion that A is true at w in M (written $M, w \models A$) is defined by the following clauses:

1. $M, w \models p_i$ iff $w \in V(i)$;
2. $M, w \models (B \vee C)$ iff $M, w \models B$ or $M, w \models C$ (or both);
3. $M, w \models \neg B$ iff it is not the case that $M, w \models B$;
4. $M, w \models \Box B$ iff, for all v such that wRv, $M, v \models B$.

To define truth for formulas of $\mathcal{L}^{\pi}_{\Box}$ we add an additional clause.

5. $M, w \models \forall p_j B$ iff, for every $X \subseteq W$, $M^X_j, w \models B$ where M^X_j is the model (W, R, V^*) such that $V^*(i) = V(i)$ for $i \neq j$ and $V(j) = X$.

To define truth for formulas of $\mathcal{L}_{[1][2]}$ and $\mathcal{L}^{\pi}_{[1][2]}$ we replace clause (4) with two similar clauses with R_1 and R_2 playing the role of R and [1] and [2] playing the role of $\Box$.

If M is a model with worlds W for any of these systems then A *is valid in* M (written $M \models A$) if $M, w \models A$ for all $w \in W$. If F is a frame then A *is valid in* F ($F \models A$) if A is valid in every model on F.

Definition 2.2

1. **T** is the set of formulas of $\mathcal{L}_{\Box}$ valid on all frames (W, R) such that R is reflexive.
2. **Double S5** (or **S5S5** or **2S5**) is the set of all formulas of $\mathcal{L}_{[1][2]}$ valid on all frames (W, R_1, R_2) such that R_1 and R_2 are equivalence relations.
3. $\mathbf{T}^{\pi}$ is the set of all formulas of $\mathcal{L}^{\pi}_{\Box}$ valid on all frames with reflexive accessibility relations.
4. $\mathbf{2S5}^{\pi}$ is the set of all formulas of $\mathcal{L}^{\pi}_{[1][2]}$ valid on all frames (W, R_1, R_2) such that R_1 and R_2 are equivalence relations.

3 Generated Models

For R a binary relation, let xR^0y if and only if $x = y$ and $xR^{n+1}y$ if and only if, for some z, xRz and zR^ny. The *ancestral of* R (written R^*) is the relation that holds between x and y if and only if xR^ky for some k.

Let $M = (W, R, V)$ be a model for $\mathcal{L}_{\Box}$ or $\mathcal{L}^{\pi}_{\Box}$ and let $w \in W$. The *model generated by* M *from* w (written M^w) is the model (W^w, R^w, V^w) where

$W^w = \{x \in W : wR^*x\}$, $R^w = R \cap (W^w \times W^w)$, and, for every natural number i, $V^w(i) = V(i) \cap W^w$

The following result is well known in $\mathcal{L}_\Box$ and extends easily to $\mathcal{L}_\Box^\pi$.

Theorem 3.1 *For every formula A of $\mathcal{L}_\Box^\pi$, $M, w \models A$ if and only if $M^w, w \models A$.*

Proof By induction on A. We do the quantifier case.

$$\begin{aligned} M, w \models \forall p_j A &\text{ iff } M_j^X, w \models A \text{ for all } X \subseteq W \quad \text{(by truth definition)} \\ &\text{ iff } (M_j^X)^w, w \models A \text{ for all } X \subseteq W \quad \text{(by induction hypothesis)} \\ &\text{ iff } (M_j^{X \cap W^w})^w, w \models A \text{ for all } X \subseteq W \\ &\qquad\qquad \text{(by definition of model generated from w)} \\ &\text{ iff } (M_j^Y)^w, w \models A \text{ for all } Y \subseteq W^w \quad \text{(because } X \cap W^w \subseteq W^w\text{)} \\ &\text{ iff } M^w \models \forall p_j A \quad \text{(by truth definition).} \end{aligned}$$

□

4 Mappings

Definition 4.1 The translation t from $\mathcal{L}_\Box$ to $\mathcal{L}_{[1][2]}$ is defined by the following clauses:

1. $t(p_i) = p_i$,
2. $t(B \vee C) = t(B) \vee t(C)$,
3. $t(\neg B) = \neg t(B)$,
4. $t(\Box B) = [1][2]t(B)$.

t extends to a map from $\mathcal{L}_\Box^\pi$ to $\mathcal{L}_{[1][2]}^\pi$ with the addition of the clause,

5. $t(\forall pA) = \forall pt(a)$.

For any model $M = (W, R_1, R_2, V)$ for $\mathcal{L}_{[1][2]}$ (or $\mathcal{L}_{[1][2]}^\pi$), the *product of* M (written M^p) is the model (W, R, V) for $\mathcal{L}_\Box$ (or $\mathcal{L}_\Box^\pi$) where W and V are as in M and $R = R_1R_2$, that is, wRv if and only if, for some x in W, wR_1x and xR_2v.

Notice that if the accessibility relations in M are reflexive, the accessibility relation in M^p is also reflexive.

Theorem 4.2 *Let $M = (W, R_1, R_2, V)$ be a model for $\mathcal{L}_{[1][2]}$ or $\mathcal{L}_{[1][2]}^\pi$, and $w \in W$. Then $M, w \models tA$ if and only if $M^p, w \models A$.*

Proof By induction on A. We do the □ case.

$$\begin{aligned} M, w \models t(\Box B) &\text{ iff } M, w \models [1][2]t(B) \quad \text{(by definition of t)} \\ &\text{ iff, for all } x, wR_1x \text{ implies } M, x \models [2]t(B) \quad \text{(by truth definition)} \\ &\text{ iff, for all } x, wR_1x \text{ implies, for all } y, xR_2y \text{ implies } M, y \models t(B) \\ &\qquad\qquad \text{(by truth definition)} \\ &\text{ iff, for all } y, wR^py \text{ implies } M, y \models t(B) \quad \text{(by definition of } R^p\text{)} \\ &\text{ iff, for all } y, wR^py \text{ implies } M^p, y \models B \text{ (by induction hypothesis)} \\ &\text{ iff } M^p, w \models \Box B \quad \text{(by truth definition definition).} \end{aligned}$$

□

The product provides a mapping from $\mathcal{L}_{[1][2]}$ or $\mathcal{L}^{\pi}_{[1][2]}$ models to $\mathcal{L}_{\Box}$ or $\mathcal{L}^{\pi}_{\Box}$ models. Now we define a kind of inverse mapping. The idea is that whenever uRv in a frame for the 1-modal system we insert a world x so that uR_1x and xR_2v in the corresponding frame of the 2-modal system. More precisely, suppose $F = (W, R)$ is a frame for $\mathcal{L}_{\Box}$ or $\mathcal{L}^{\pi}_{\Box}$ and $M = (W, R, V)$ is a frame on F. Let W^i be the result of adding to W, a new world $i(u, v)$ for each pair of distinct worlds u and v in W such that uRv. (We call these *infill* worlds and the remaining worlds of W^i the *original* worlds.) For any original world u, let *right*$(u) = \{u\} \cup \{i(u, x) : uRx\}$ and let *left*$(u) = \{u\} \cup \{i(x, u) : xRu\}$. For all x and y in W^i, let xR_1y if and only if $x = y$ or, for some original world w, x and y are both in *right*(w). Similarly, let xR_2y if and only if $x = y$ or, for some original world w, x and y are both in *left*(w). The *infill of* F is the frame $F^i = (W^i, R_1, R_2)$ (unique up to isomorphism), where W^i, R_1, R_2 are as defined above. An *infill of* M is a model $M^i = (W^i, R_1, R_2, V^i)$ on F^i in which, for all natural numbers i, $V^i(i) \cap W = V(i)$ (so the truth value of propositional variables in M^i on the original worlds agrees with their truth value in M).

Theorem 4.3 *Suppose $M = (W, R, V)$ is a model for $\mathcal{L}_{\Box}$ or $\mathcal{L}^{\pi}_{\Box}$ with R reflexive and $M^i = (W^i, R_1, R_2, V^i)$ is an infill of M.*

1. *R_1 and R_2 are equivalence relations.*
2. *For all u and v in W, uRv if and only if uR_1R_2v.*
3. *For all $w \in W$, $M^w = (((M^w)^i)^p)^w$.*

Proof (1) Observe first that if $u \neq v$ then *right*(u) and *right*(v) are disjoint. For suppose they had a world w in common. Since the only original world in *right*(u) is u and the only original world in *right*(v) is v, w cannot be an original world. But if w were an infill world it would have to be $i(u, x)$ for some x and $i(v, y)$ for some y which is not possible when $u \neq v$. Since each original world u is in *right*(u) and each infill world $i(x, y)$ is in *right*(x), the sets *right*(u) partition W^i into disjoint sets containing u. It follows that R_1 is an equivalence relation. A similar argument establishes that R_2 is an equivalence relation.

(2) Suppose uRv. By the definition of R_1, $uR_1i(u, v)$. By the definition of R_2, $i(u, v)R_2v$. Hence uR_1R_2v. Conversely, suppose uR_1R_2v for u and v in W. Then, for some x, uR_1x and xR_2v. If x is an original world then $u = x$ and $x = v$ and so $u = v$. By the reflexivity of R, uRv as was to be shown. If x is an infill world, then $x = i(u, y)$ for some y and $x = i(z, v)$ for some z. Hence $x = i(u, v)$ and uRv as was to be shown.

(3) Let $M' = (W', R', V')$ be the model $(((M^w)^i)^p)^w$. We must prove that each component of M' is identical to the corresponding component of M^w. Let Q_1 and Q_2 be the accessibility relations of $(M^w)^i$ and let Q be the accessibility relation of $((M^w)^i)^p$. Then

(i)	$x \in W^w$	iff wR^*x	(by definition of W^w)
		iff $w(R^w)^*x$	(by definition of R^w)
		iff $w(Q_1Q_2)^*x$	(by 2 above)
		iff wQ^*x	(by definition of the product of a model)
		iff $x \in W'$	(by the definition of W').

(ii) $uR^w v$ iff uQ_1Q_2v (by 2 above)
iff uQv (by the definition of the product of a model)
iff $uR'v$ since u and v are both in W^w and hence in W' by i.

(iii) Since the generation, product, and infill constructions never change the valuation function on any world, it is clear that $x \in V^w(i)$ if and only if $x \in V'(i)$ for all natural numbers i and all $x \in W^w$.

□

5 The Embedding Result

Theorem 5.1 *(1) $A \in$ **T** if and only if $t(A) \in$ **2S5**; (2) $A \in T^\pi$ if and only if $t(A) \in$ **2S5**$^\pi$.*

Proof Suppose $A \notin$ **2S5**. Then there is a model $M = (W, R_1, R_2, V)$ with R_1 and R_2 equivalence relations and some $w \in W$ such that $M^w \not\models t(A)$. By Theorem 4.2, $M^p, w \not\models A$. By an earlier observation, M^p is reflexive. Hence $A \notin$ **T**. Conversely, suppose $A \notin$ **T**. Then there is some model $M = (W, R, V)$ with reflexive R and some $w \in W$ such that $M, w \not\models A$. By Theorem 3.1, $M^w, w \not\models A$. By part 3 of Theorem 4.3, $(((M^w)^i)^p)^w, w \not\models A$. By Theorem 3.1 again, $((M^w)^i)^p, w \not\models A$. By Theorem 4.2, $(M^w)^i \not\models t(A)$. By part 1 of Theorem 4.3 the relations in this model are equivalence relations. It follows that $t(A) \notin$ **2S5**. This proves (1). Since all the results appealed to carry over in the presence of full propositional quantifiers, this proof also suffices for (2). □

Corollary 5.2 **2S5**$^\pi$ *is recursively intertranslatable with full second-order logic.*

Proof Well-known methods assure that any simple n-modal system with propositional quantifiers can be recursively embedded in second-order logic. More particularly, take any such system S determined by the class of all frames $(W, R_1, \ldots, R_n)$ meeting some first- or second-order condition $\Phi(R_1, \ldots, R_n)$. Then we first define a base function s from $\mathcal{L}_{[1],\ldots,[n]}$ to the formulas of second-order logic with x as the only individual variable by a simple induction:

1. $s(p_i) = P_i x$ (where P_i is the ith one-place predicate symbol),
2. $s(B \vee C) = s(B) \vee s(C)$,
3. $s(\neg B) = \neg s(B)$,
4. $s([i]B) = \forall y(xR_i y \to (s(B))^y_x)$ where y is the first individual variable that does not occur in $s(B)$, and D^y_x is the result of replacing x in D by y,
5. $s(\forall p_j B) = \forall P_j s(B)$.

Now, for any formula A of $\mathcal{L}^\pi_{[1],\ldots,[n]}$, let $t(A)$ be the formula

$$\forall R_1 \ldots \forall R_n(\Phi(R_1, \ldots, R_n) \to \forall P_{i_1} \ldots \forall P_{i_m} \forall x s(A)),$$

where $p_{i_1}, \ldots, p_{i_m}$ are all the propositional variables that occur free in A and then, by the truth definition for $\mathcal{L}^\pi_{[1],\ldots,[n]}$, $\models A$ if and only if $\models t(A)$. So to show that such systems are recursively intertranslatable with second-order logic it is sufficient to find a recursive embedding in the other direction, that is, from second-order logic to the modal system. This is done for **T**$^\pi$, (among other systems) in [2]. Since Theorem 5.1 provides a recursive embedding of **T**$^\pi$ into **2S5**$^\pi$, it follows that the same can be done for **2S5**$^\pi$. □

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Department of Philosophy
Georgetown University
Washington DC 20015
kuhns@georgetown.edu