Notre Dame Journal of Formal Logic
Volume IV, Number 1, January 1963

A RULE-COMPLETENESS THEOREM

NUEL D. BELNAP, JR. and RICHMOND H. THOMASON

From an intuitive standpoint it would seem that the connectives of conjunction and disjunction assume in intuitionistic logic the same role as in classical logic. We may lend precision to these intuitive ideas by considering Gentzen's formulation of intuitionistic logic, which separates the deductive roles of the various logical connectives, defining each connective by a pair of rules added to a structural system. Though it is possible to convert a Gentzen formulation of intuitionistic logic (with singular right sides) into a classical two-valued system by altering the rules for negation, implication, or equivalence,[1] Leblanc has conjectured[2] that no classically valid changes in structural rules (i.e., rules exhibiting no connectives) or rules for conjunction or disjunction can have this effect.

We here verify this conjecture by showing that the conjunction-disjunction fragment is "rule-complete" (in a sense to be specified) under the ordinary two-valued interpretation.

Notation. Let q_1, q_2, . . . range over propositional variables, and A, B, A_i, . . . over well-formed formulas (wffs) defined by the conditions (i) q_1, q_2, . . . are well-formed, and (ii) if A and B are well-formed, then so are $(A \wedge B)$ and $(A \vee B)$. Let S, S_1, . . . range over *statements* having the form

$$\text{(I)} \quad A_1, \ldots, A_n \vdash B \,.$$

Let α and β range over finite (possibly null) sequences of well-formed formulas separated by commas, and let Σ, Σ_1, Σ_2 range over finite (possibly null) sequences of statements having the form (I).

Definitions: Let LJ^* be the system defined by Gentzen's (intuitionistic) structural rules, together with his rules for $\wedge$ and $\vee$, as follows:

<u>Structural Rules</u>:

$$A \vdash A \qquad (Id)$$

$$\frac{\alpha \vdash A}{B, \alpha \vdash A} \qquad (K \vdash)$$

Received August 4, 1962

$$\frac{\alpha, A, B, \beta \vdash C}{\alpha, B, A, \beta \vdash C} \qquad (C \vdash)$$

$$\frac{A, A, \alpha \vdash B}{A, \alpha \vdash B} \qquad (W \vdash)$$

$$\frac{\alpha \vdash A \quad A, \beta \vdash B}{\alpha, \beta \vdash B} \qquad (Cut)$$

Rules for $\wedge$ and $\vee$:

$$\frac{\alpha \vdash A \quad \alpha \vdash B}{\alpha \vdash A \wedge B} \qquad (\vdash \wedge)$$

$$\frac{A, \alpha \vdash C}{A \wedge B, \alpha \vdash C} \qquad (\wedge \vdash)$$

$$\frac{A, \alpha \vdash C \quad B, \alpha \vdash C}{A \vee B, \alpha \vdash C} \qquad (\vee \vdash)$$

$$\frac{\alpha \vdash A}{\alpha \vdash A \vee B} \qquad \frac{\alpha \vdash B}{\alpha \vdash A \vee B} \qquad (\vdash \vee)$$

A *proof* (in LJ^*) *of* S *on the hypotheses* Σ is a finite sequence $S_1, \ldots, S_m$ of statements each having the form (I) such that S_m is S, and for all i $(1 \leq i \leq m)$ S_i either occurs in Σ, is an instance of *Id*, or follows from some S_j, where $j < i$, by $K \vdash$, $C \vdash$, $W \vdash$, $\wedge \vdash$, or $\vdash \vee$, or from some S_j and S_k, where $j, k < i$, by *Cut*, $\vdash \wedge$, or $\vee \vdash$.

A rule having the form

$$\text{(II)} \quad \frac{\Sigma}{\alpha \vdash A}$$

is said to be *derivable* (in LJ^*) iff there exists a proof (in LJ^*) of $\alpha \vdash A$ on the hypotheses Σ.

Where a classical truth-functional interpretation is given to statements of LJ^* (letting a statement (I) take the value t iff either some A_i $(1 \leq i \leq n)$ takes the value f or B takes the value t, under the usual truth-tables for $\wedge$ and $\vee$), we say a rule having the form (II) is *valid* if there is no assignment of values t and f to the variables occurring in Σ and $\alpha \vdash A$ such that every statement in Σ takes the value t and $\alpha \vdash A$ takes the value f.

It is clear that our definitions of "derivable rule" and "valid rule" can be generalized to arbitrary systems and interpretations. We shall say that such a system is *rule-complete* if every valid rule is derivable. Obviously *rule-completeness* is a stronger property than *statement-completeness.* For example, the system considered by H. Hiż in "Extendible sentential calculus," *Journal of Symbolic Logic*, vol. 24 (1960), pp. 193-202, is statement-complete (every tautology is a theorem) without being rule-complete: modus ponens (though valid) is not derivable. Under this definition, the theorem stated immediately below asserts that LJ^* is rule-complete. (It is perhaps worth remarking that if the rule *Cut* were removed from LJ^*, the resulting system would by Gentzen's *Hauptsatz* remain statement-complete, but it would no longer be rule-complete, since the valid rule *Cut* itself would not be derivable.)

Theorem: Every valid rule is derivable.[3]

The theorem will follow directly from the following two lemmas:

Lemma 1: If a rule (II) is in *normal form* (i.e., if for every statement $\alpha \vdash B$ occurring therein, α is constituted only of propositional variables and B contains no occurrence of $\wedge$) then if it is valid, it is derivable.

Proof: Let k be the number of premisses of a rule (II) in normal form. We prove the lemma by induction on k. Suppose that (II) is valid. When $k = 0$ the validity of (II) guarantees that some propositional variable p of A occurs in α. Then $\alpha \vdash A$ is a theorem of LJ^*, by a number of applications of $K \vdash$, $C \vdash$ and $\vdash \vee$ to an instance $p \vdash p$ of *Id*, and so (II) is derivable. Suppose now as inductive hypothesis that all valid rules in normal form with n premisses are derivable, and let $k = n + 1$. If some propositional variable p is common to A and α, then $\alpha \vdash A$ is a theorem and (II) is derivable as before. Suppose therefore that α and A share no propositional variable. Then (II) must have the form

$$\text{(II*)} \quad \frac{\Sigma_1, \quad \alpha^* \vdash B, \quad \Sigma_2}{\alpha \vdash A},$$

where each variable occurring in α^* also occurs in α; for otherwise (i.e., if the antecedent of every statement in Σ contains some variable not occurring in α) we could invalidate (II) by the valuation v, where $v(p) = t$ iff p occurs in α.

Let $q_1, \ldots, q_m$ be the propositional variables of B. Then, for all i, $1 \leq i \leq m$, the rules

$$\text{(III.}i\text{)} \quad \frac{\Sigma_1, \Sigma_2}{q_i \quad \alpha \vdash A},$$

with only n hypotheses, must be valid, since any assignment v which would invalidate (III.i) would have to have $v(q_i) = t$, hence $v(B) = t$, and so would invalidate (II*). We may then show that (II) is derivable as follows:

1.	Σ_1	Hypotheses.
2.	$\alpha^* \vdash B$	Hypothesis.
3.	Σ_2	Hypotheses.
4_i.	$q_i, \alpha \vdash A$ (for all i, $1 \leq i \leq m$)	From 1 and 3, since by the hypothesis of induction the valid rules (III.i) are all derivable.

5. $B, \alpha^* \vdash A$	From 4, by m-1 applications of $\vee \vdash$.
6. $\alpha, \alpha^* \vdash A$	From 2 and 5, by *Cut*.
7. $\alpha \vdash A$	From 6, by a number of applications of $C \vdash$ and $W \vdash$ (since every variable occurring in α^* also occurs in α).

Lemma 1 now follows by induction.

Lemma 2: For every rule $\frac{\Sigma}{S}$ of LJ^* there exists a set R of rules such that (a) each rule in R is in normal form, (b) if $\frac{\Sigma}{S}$ is valid, so is every member of R, and (c) if every member of R is derivable, so is $\frac{\Sigma}{S}$.

Proof: We define inductively a sequence $R_1, \ldots, R_n$ of sets of rules, terminating in the desired set R. Let $R_1 = \left\{\frac{\Sigma}{S}\right\}$, and suppose R_k defined. If every rule in R_k is in normal form, the sequence terminates with R_k. Otherwise, we suppose R_k to be ordered in some way, and consider the first rule P (in the ordering) not in normal form.

Where no statement in Σ_1 violates the normal form condition, where α contains no compound wffs, and where $A \wedge B$ is the leftmost disjunctive part of $\phi(A \wedge B)$,[4] P will have exactly one of the forms (1) - (6) below. R_{k+1} is defined as the result of replacing P in R_k by the matching primed rule or rules below.

$$\begin{cases} (1) & \dfrac{\Sigma_1, \quad \alpha, A \wedge B, \beta \vdash C, \quad \Sigma_2}{S_2} \\[2ex] (1)' & \dfrac{\Sigma_1, \quad \alpha, A, B, \beta \vdash C, \quad \Sigma_2}{S_2} \end{cases}$$

$$\begin{cases} (2) & \dfrac{\Sigma_1, \quad \alpha, A \vee B, \beta \vdash C, \quad \Sigma_2}{S_2} \\[2ex] (2)' & \dfrac{\Sigma_1, \quad \alpha, A, \beta \vdash C, \quad \alpha, B, \beta \vdash C, \quad \Sigma_2}{S_2} \end{cases}$$

$$\begin{cases} (3) & \dfrac{\Sigma_1}{\alpha, A \wedge B, \beta \vdash C} \\[2ex] (3)' & \dfrac{\Sigma_1}{\alpha, A, B, \beta \vdash C} \end{cases}$$

$$\left\{\begin{array}{l}(4)\ \dfrac{\Sigma_1}{\alpha,\ A \vee B,\ \beta \vdash C} \\[2ex] (4)'\ \dfrac{\Sigma_1}{\alpha,\ A,\ \beta \vdash C},\quad \dfrac{\Sigma_1}{\alpha,\ B,\ \beta \vdash C}\end{array}\right.$$

$$\left\{\begin{array}{l}(5)\ \dfrac{\Sigma_1,\quad \alpha \vdash \phi(A \wedge B),\quad \Sigma_2}{S_2} \\[2ex] (5)'\ \dfrac{\Sigma_1,\quad \alpha \vdash \phi(A),\quad \alpha \vdash \phi(B),\quad \Sigma_2}{S_2}\end{array}\right.$$

$$\left\{\begin{array}{l}(6)\ \dfrac{\Sigma_1}{\alpha \vdash \phi(A \wedge B)} \\[2ex] (6)'\ \dfrac{\Sigma_1}{\alpha \vdash \phi(A)},\quad \dfrac{\Sigma_1}{\alpha \vdash \phi(B)}\end{array}\right.$$

It is easily verified that the sequence thus defined will terminate in a set of rules, each in normal form; that if each member of R_{i+1} is derivable, so is each member of R_i; and that if each member of R_i is valid, so is each member of R_{i+1}. Lemma 2 follows immediately, and the theorem from lemmas 1 and 2.

The rule-completeness of Gentzen's intuitionistic structural rules follows as a corollary of lemma 1.[5] From the proof of this lemma it also follows that the single axiom $\alpha,\ A,\ \beta \vdash A$ and the rule $\dfrac{\alpha^* \vdash B \quad \alpha,\ B \vdash A}{\alpha \vdash A}$ (where each constituent of α^* occurs in α) generate the complete system of structural rules. We also remark that all these results easily generalize to Gentzen systems with multiple constituents on the right.

NOTES

1. See "Intuitionism reconsidered" by H. Leblanc and N. Belnap, *Notre Dame Journal of Formal Logic*, vol. 3 (1962), pp. 79-82.
2. *Ibid.* Note that the conjecture is due to Leblanc.
3. E. W. Beth has communicated to us an alternative proof of this theorem by D. H. J. de Jongh, contained in a report by de Jongh to Euratom. De Jongh uses an extension of semantic valuation notions due to Beth.
4. $\phi(A \wedge B)$ is a wff having $A \wedge B$ as a disjunctive part, and $\phi(A)$ [$\phi(B)$] is the result of replacing $A \wedge B$ in $\phi(A \wedge B)$ by A [B]. "Disjunctive part" is defined as follows: A is a disjunctive part of A, and if $B \vee C$ is a disjunctive part of A, then so are B and C.
5. This corollary is relevant to the thesis of "Tonk, Plonk, and Plink," by N. Belnap, *Analysis*, vol. 22 (1962), pp. 130-134, where philosophical use is made of the claim that Gentzen's structural rules are complete.

Yale University
New Haven, Connecticut