Notre Dame Journal of Formal Logic
Volume VIII, Numbers 1 and 2, April 1967

THE DEDUCTION THEOREM IN S4, S4.2, AND S5

J. JAY ZEMAN

In a certain sense, there is no trick to merely stating the deduction theorem for a given system (on the assumption, of course, that it holds for that system). The general statement of the theorem might be, "*If there is a proof from the hypotheses* $A_1, \ldots, A_n$ *for the formula* B, *then there is a proof from the hypotheses* $A_1, \ldots, A_{(n-1)}$ *for the formula* $A_n \supset B$." The problem in formulating the deduction theorem lies not in simply stating it as above, but in defining just what we mean by "proof from hypotheses" for the system in question. Once we have such a definition, the statement and proof of the theorem will ordinarily present no real problem.

The three Lewis-modal systems with which we are concerned will be considered to be formulated on a **CPC** base, following, in general, Lemmon [2]. They will contain, first of all, any basis sufficient for the complete **CPC**, including the rules of substitution and detachment. Each of these systems will also contain the rule **RL**: "*If* α *is a theorem, so too is* $L\alpha$." The additional axioms are, for S4:

1. *CLCpqLCLpLq*
2. *CLpp*.

For S4.2, axioms 1. and 2. and also (see [3], p. 313):

3. *CMLpLMp*.

For S5, axioms 1. and 2. and also:

4. *CNLpLNLp*.

Since these systems are formulated on a **PC** base, we might suspect that a good part of the definition of "proof from hypotheses" for these systems will be exactly as for the **CPC**. This is the case; here we shall make use of Church's definition of "proof from hypotheses" for the **CPC** in [1], p. 87. The clauses of the definition as he states it are easily extended to our modal systems; we may thus present what will amount to most of our definitions:

A finite sequence of wffs $B_1, B_2, \ldots, B_m$ *is called a "proof from the hypotheses* $A_1, A_2, \ldots, A_n$*" if for each* i, $i \leq m$, *either*

Received August 29, 1965

1. *B_i is one of the $A_1, A_2, \ldots, A_n$, or*
2. *B_i is a varient of an axiom* (this understood as in [1]), *or*
3. *B_i is inferred by the rule of detachment from B_j and B_k, where $j, k < i$ and B_j is of form $B_k \supset B_i$, or*
4. *B_i is inferred by the rule of substitution from B_j, where $j < i$, and the variable substituted for does not occur in the $A_1, A_2, \ldots, A_n$.*

Note that there will be no difficulty in extending clause 2. above to include the axioms of the modal systems with which we are concerned.

One thing, however, is missing from the above definition, so far as S4, S4.2, and S5 are concerned. This is a consideration of the role of the rule **RL** in a proof from hypotheses. It is obvious that this rule is analogous to the rule of "universal generalization" in predicate calculi; we might, then, expect to get a hint of how to account for **RL** by an examination of the way that universal generalization is handled in statements of the deduction theorem for predicate calculi.

In the definition of "proof from hypotheses" in the predicate calculus, as in [1], p. 196, the following move is permitted in the inference of a B_i from a B_j by universal generalization: The inferred B_i will be of form $(a)B_j$, where $j < i$ and the variable a does not occur free in any of the hypotheses $A_1, A_2, \ldots, A_n$.

Our problem is now to find, for the systems S4, S4.2, and S5, an appropriate analog of the statement, "The variable a does not occur free in any of the hypotheses $A_1, A_2, \ldots, A_n$."

Such an analog is available. Prior, in [3], p. 312, has shown that S5 is derivable, and I have shown that S4 and S4.2 are derivable [4] by subjoining to the **CPC** the following rules:

R1: *If $C\alpha\beta$ is a theorem, so too is $CL\alpha\beta$.*
R2: *If $C\alpha\beta$ is a theorem, so too is $C\alpha L\beta$, provided α is completely modalized.*

The definition of "completely modalized" varies among these systems, and is the factor which distinguishes them. In S4, a wff α is completely modalized iff either

1. *It is a law of the system, every propositional variable of which is in the scope of the modal operator belonging to α, or*
2. *It is of the form $KL\delta KL\gamma \ldots L\nu$ with $L\delta$ as a limiting case.*

For S4.2 we have—in addition to the above—that α is completely modalized if:

3. *It is of the form $NLNL\gamma$.*

For S5, any wff α is completely modalized provided every propositional variable of α is in the scope of a modal operator belonging to α.

Now note that the complete quantification theory is formulable by subjoining to a complete **CPC** base the following:

RΠ1: *If $C\alpha\beta$ is a theorem, so too is $C\Pi x\alpha\beta$.*

RΠ**2:** *If* $C\alpha\beta$ *is a theorem, so too is* $C\alpha\Pi x\beta$, *provided* x *is not free in* α.

The similarity of the above rules to **R1** and **R2** for 'L' is obvious. And this similarity indicates to us what the analog for S4, S4.2, and S5 for the statement "The variable a does not occur free in any of the hypotheses" will be. Let us now move to a statement of the final clause in our definition of "proof from hypotheses" for S4, S4.2, and S5.

A finite sequence of wffs $B_1, B_2, \ldots, B_m$ is called a "proof from the hypotheses $A_1, A_2, \ldots, A_n$" if for each i, $i \leqslant m$, either one of the four previously mentioned clauses (as stated for the **PC** in [1]) holds, or,

5. *B_i is inferred from B_j by* **RL**, *where $j < i$ and each of the hypotheses $A_1, A_2, \ldots, A_n$ is completely modalized in the sense of the system in which we are working.*

With these five clauses, then, defining "proof from hypotheses" in S4, S4.2, and S5, we shall write

$$A_1, A_2, \ldots, A_n \vdash B$$

for "there is a proof from the hypotheses $A_1, A_2, \ldots, A_n$ for the wff B." The statement of the deduction theorem for these systems is now:

If it is the case that

$$A_1, A_2, \ldots, A_n \vdash B,$$

it is also the case that

$$A_1, A_2, \ldots, A_{(n-1)} \vdash A_n \supset B.$$

The proof of this theorem for the first four clauses of our definition of proof from hypotheses will be just as in [1], pp. 88-89. The only extension of the proof needed is to cover our clause 5; this is easily accomplished.

Let each of the $A_1, A_2, \ldots, A_n$ be completely modalized. And let B be B_i, such that if $k < i$, then

$$A_1, A_2, \ldots, A_{(n-1)} \vdash A_n \supset B_k, \tag{1}$$

whenever

$$A_1, A_2, \ldots, A_n \vdash B_k. \tag{2}$$

Now let B_i be inferred from B_j, $j < i$, by **RL**. This means that, by our definition of proof from hypotheses and the fact that B is B_i, whenever (2) holds, then

$$A_1, A_2, \ldots, A_n \vdash B. \tag{3}$$

With (2) and (3) holding, note that since $j < i$, then also $j \leqslant k$, and by (1),

$$A_1, A_2, \ldots, A_{(n-1)} \vdash A_n \supset B_j. \tag{4}$$

But then, since each of the hypotheses is completely modalized, we have also, by **RL** and our definition of proof from hypotheses:

$$A_1, A_2, \ldots, A_{(n-1)} \vdash L(A_n \supset B_j). \tag{5}$$

It is easily provable as a theorem of S4, S4.2, or S5 that, where α is completely modalized,

$$CLC\alpha pC\alpha Lp. \tag{6}$$

(This schema is, of course, analogous to the predicate calculus theorem

$C\Pi xC\phi\psi C\phi\Pi x\psi$, *where x is not free in* ϕ.)

But this means that we may move to

$$A_1, A_2, \ldots, A_{(n-1)} \vdash A_n \supset LB_j \tag{7}$$

from (5), since A_n, along with all the other hypotheses, is completely modalized. But B_i was inferred from B_j by **RL**, and so is of the form 'LB_j', This means that whenever (2) and (3) hold, then also

$$A_1, A_2, \ldots, A_{(n-1)} \vdash A_n \supset B \tag{8}$$

is true, B being B_i, From here it is a simple matter of mathematical induction to complete the proof of the deduction theorem for S4, S4.2, and S5, given our definition of "proof from hypotheses."

As an example of a proof in these systems employing the deduction theorem, we may quickly show that

$$CLCpqCLCqpC\delta p\delta q \tag{9}$$

is a theorem schema of S4, S4.2, and S5. (Note that we do not, strictly speaking, employ 'δ' as a "functor variable," as is commonly done; rather, we employ this sign as a symbol of the metalanguage, letting 'δp' be a *schema* representing any wf function of p, including, in this case, modal functions.)

By the rule of substitutivity of strict equivalence and our definition of proof from hypotheses, we may write

$$LCpq, LCqp \vdash C\delta p\delta q. \tag{10}$$

Note that the hypotheses for this case are completely modalized in any of the three systems in question. But by the deduction theorem, the schema (9) stands proven.

Note that we could *not* in the general case for these systems have stated

$$Cpq, Cqp \vdash C\delta p\delta q.$$

This in spite of the fact that—as a rule of inference—the substitutivity of material equivalence holds in these systems. For there is no guarantee that the rule **RL** would not have to be applied in order to get the desired results, and by our definition of proof from hypotheses this application would not be allowed in the last case, since neither of the "hypotheses" is completely modalized in any of the three systems in question.

BIBLIOGRAPHY

[1] Church, Alonzo. *Introduction to Mathematical Logic*. Princeton: Princeton, 1956, I.

[2] Lemmon, E. J. "New Foundations for the Lewis Modal Systems," *The Journal of Symbolic Logic*, XXII (1956), 176-86.

[3] Prior, A. N. *Formal Logic*. 2nd ed. Oxford, 1962.

[4] Zeman, J. Jay. "Bases for S4 and S4.2 without added axioms," *Notre Dame Journal of Formal Logic*, IV (1963), 227-30.

Data Systems Division
Edgewood Arsenal, Maryland