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COMBINATORIAL DESIGNS ON INFINITE SETS

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I. INTRODUCTION

§1. *Generalities*[1]. Roughly speaking, the field of combinatorial mathematics can be said to deal with those problems of arranging objects according to some fixed pattern and in determining how many distinct ways this can be accomplished[2]. Observe that no restriction is placed on the set to which the objects can be considered to belong. Consequently, a combinatorics of the infinite naturally evolves when investigations are concerned with arrangements of sets which are not finite. Frequently it happens that a meaningful question of a combinatorial nature, initially posed with reference to a finite collection, retains its interest when one allows the collection to be infinite. The generalization is usually realized by permitting one or more of those symbols, which represents a natural number in the finite formulation of the problem, to now stand for an arbitrary cardinal number. Very often, however, such a simplistic generalization trivializes a very interesting finite problem. In some cases, therefore, to recapture the spirit of a finite combinatorial problem in the infinite case, it is necessary to effect more sophisticated alterations in the hypotheses of the original problem.

In the course of the present report this method of generalization will be exhibited. Our interests will converge upon a single, yet important, area of combinatorial research: the existence and construction of designs. Design problems in combinatorics, which are both intriguing and difficult, almost always deal with arrangements of finite sets. The aim, herewith, is to develop a theory of combinatorial designs on infinite sets which bears a

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1. The present researches constitute a part of the author's doctoral dissertation, *Block Designs On Infinite Sets*, written under the direction of Professor B. Sobociński and accepted by the University of Notre Dame in partial fulfullment of the requirements of the degree of Ph.D. in Mathematics, February, 1966.
2. M. Hall [5] asserts this as a working definition of combinatorial analysis.

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strong resemblance to the existing theory in the finite case. To accomplish this we follow the procedure that was outlined above. It will become more than evident, however, that our results are far from complete and that many open questions remain. On the other hand, we do hope to offer an adequate formulation of the notion of a combinatorial design on an infinite set and to establish the existence, on every infinite set, of a wide class of these designs. Hopefully this will provide a sufficient foundation upon which a more elaborate theory can rest.

§2. *Designs.* Generally, when one speaks of a mathematical design it is taken to mean a certain arrangement which can be imposed on some given set. In a very abstract manner we may define a mathematical design to be an ordered quadruple $< S, P, L, R >$ where S is a set, P and L are families of subsets of S, and R is a relation of 'incidence' between these two families. Clearly this concept, under one form or another, is found in almost every branch of mathematics.

Combinatorics, however, is the place where the notion of a design reveals itself with particular clarity. The official appearance of the design concept in combinatorial analysis occurred in 1853. In that year the well-known geometer Jacob Steiner, while investigating a problem in algebraic geometry, posed in [13] his now famous problem of triples. He asked which finite sets could be decomposed into a collection of unordered triples such that any two distinct elements of the original set be contained in exactly one triple of the collection. Six years later the problem was completely solved by M. Reiss [9] who showed that a finite set could be so decomposed if and only if its cardinality were congruent to *1* or *3* modulo *6*. Steiner triple systems, as such decompositions were to be called, are the first of the many combinatorial designs that were to follow.

As with most problems in combinatorics, the only significant parameter in the Steiner problem is the cardinality of the finite set. In fact, instead of asking which finite sets possess Steiner triple systems we could unambiguously inquire as to which natural numbers possess Steiner triple systems. All this follows from the fact that if one set possesses a Steiner triple system we can, in the obvious way, construct a Steiner triple system on every other set having the same cardinality. The notion of a Steiner triple system can easily be extended to the non-finite case by asking which infinite sets (or, equivalently, which non-finite cardinal numbers) can be decomposed into unordered triples such that every two distinct elements of the set are contained in exactly one triple. In 1945 W. Sierpiński [11] settled this question by proving every infinite set possesses a Steiner triple system. This appears to be the first time the designs of finite combinatorics were formally considered to be imposed on infinite sets.

§3. *Tactical configurations.* At the turn of the century, due largely to the effort of E. H. Moore [8], the design concept in combinatorics was considerably widened. This enlargement is called a tactical configuration. Although always considered to be an arrangement of a finite set, we may regard a tactical configuration to be defined more generally as follows.

Definition I.1. Let n, k and p be non-zero cardinal numbers. A set S, of cardinality v, (or, equivalently, a cardinal number v) is said to possess a (k, n, p)-tactical configuration if there exists a family F of subsets of S such that 1) each member of F is of cardinality n and 2) every subset of S, having cardinality k, is contained in exactly p members of the family F.

Definition I.2. The symbol $\mathsf{C}[v, k, n, p]$ will be used to assert the fact that the cardinal number v possesses a (k, n, p)-tactical configuration.

The notion of a tactical configuration subsumes, as special cases, most designs of interest to the combinatorialist. The major question raised here is one of existence. More precisely, given natural numbers k, n and p, for what natural numbers v do we have $\mathsf{C}[v, k, n, p]$? As mentioned above, when $k = 2$, $n = 3$ and $p = 1$, Reiss [9] determined the precise range of v. On the other hand, the general existence question for finite tactical configurations remains unanswered. In recent times, however, numerous partial results have been obtained by methods ranging from strictly counting arguments to those which employ the Hasse-Minkowski theory of algebraic numbers. In the way of examples, the range of v has been determined by very nice arithmetical conditions when *i)* $k = 2$, $n = 3$, $p = 2$, (Bose [1]); *ii)* $k = 2$, $n = 3$ and 4, p arbitrary (Hanani [6]); *iii)* $n = 3$, $p = 4$, p arbitrary (Hanani [7]). Yet no satisfactory theory which would unify the study of tactical configurations on finite sets has appeared. These designs remain one of the genuine mysteries in finite combinatorics.

§4. *Tactical configurations on infinite sets.* As previously mentioned, Sierpiński [11] showed $\mathsf{C}[v, 2, 3, 1]$ for every non-finite cardinal number v. It is interesting to note that Sierpiński employs the axiom of choice to achieve this result and, in fact, remarks that it seems the axiom's role is essential since, without its use, he was unable to prove $\mathsf{C}[\mathfrak{c}, 2, 3, 1]$ where $\mathfrak{c}$ represents the cardinality of the set of all real numbers[3]. This remark led B. Sobociński [12] to prove that the statement, "$\mathsf{C}[v, 2, 3, 1]$ for every non-finite cardinal number v", is equivalent to the axiom of choice. In Frascella [2, 3] both these results are extended where it is shown for every natural number $n > 2$, "$\mathsf{C}[v, n-1, n, 1]$ for every non-finite cardinal number v," and, furthermore, each such statement is equivalent to the axiom of choice. At this point we mention that H. Rubin and J. Rubin [10], working independently from the present author, have duplicated some results which will appear here. In particular, they have shown if k and n are natural numbers such that $1 < k < n$, then the following statement is equivalent to the axiom of choice: "$\mathsf{C}[v, k, n, 1]$ for all non-finite cardinal numbers v". That this statement follows from the axiom of choice can be deduced from more general results given in Section III of the present report. We intend to publish the converse statement along with other

3. That $\mathsf{C}[\mathfrak{c}, 2, 3, 1]$ is provable without the axiom of choice was shown by the present author in [4].

metamathematical consequences of block designs on infinite sets in a forthcoming paper.

§5. *A new design-concept for infinite sets.* A natural extension to the existence results for tactical configurations on infinite sets would be the following.

> **(A)** *Let k, n and p be natural numbers such that $k < n$. Then, for every non-finite cardinal number v, $\mathbf{C}[v, k, n, p]$.*

In fact, this statement is true and will be verified in III, (*cf.* a remark following the proof of Theorem III.12). Continuing in a desire for greater generality we would like to let, not only v, but the symbols k, n and p represent arbitrary cardinal numbers. However, the following result thwarts our hope.

> **(B)** *Let k, n and v be any cardinal numbers such that 1) $k < n < v$ and 2) k is non-finite. Then $\mathbf{C}[v, k, n, p]$ is not true for each non-zero cardinal number $p < v$.*

The proof of **(B)** will also be offered in III, (*cf.* Theorem III.1). It is clear from **(B)** that the straightforward generalization of **(A)** to the higher cardinals is not realizable. To circumvent this difficulty we will structure a new notion of a combinatorial design on an infinite set. It will be clear that such a definition will subsume, as a very special case, the notion of a tactical configuration. Then we shall prove that every infinite set possesses a wide class of such designs. This class of designs will include those mentioned in **(A)** together with a collection of designs which can be considered "generalized tactical configurations". The structuring of this definition will be the chief concern of section II.

§6. *Conventions and notations.* The subject matter of the present researches are primarily of mathematical interest. Hence the presentation of results will not be totally formalized. The Zermelo-Fraenkel axioms will be assumed as the basis for the set theory considered here. The axiom of regularity will not be employed. The axiom of choice, however, will be relied upon heavily throughout this work. This axiom, in one form or another, is essential to practically all proofs given here. With this axiom we may conclude that $\mathfrak{n} + \mathfrak{m} = \mathfrak{n}\mathfrak{m} = \max(\mathfrak{n}, \mathfrak{m})$ whenever $\mathfrak{m}$ and $\mathfrak{n}$ are cardinal numbers such that at least one is non-finite.

Most of the notation employed will be standard. If x is a set, $|x|$ will represent the cardinality of x and $P(x)$ will denote the power set of x. Also if n is any cardinal number such that $n \leqq |x|$, then $[x]^n = \{y \subset x : |y| = n\}$. For any cardinal number n, $o(n)$ will signify the first ordinal number whose cardinality is n. If δ is an ordinal number, $\overline{\delta}$ represents the cardinality of δ. Let x and y be any two sets. Then $x \cup y$, $x \cap y$, $x - y$ and $x \otimes y$ represent the union, intersection, difference and cartesian product of the set x with the set y, respectively. Let I be any index set such that x_i is a set for each $i \in I$. Then $\bigcup\{x_i : i \in I\}$ will represent the union ranging over all sets x_i such that $i \in I$. Similarly for $\bigcap\{x_i : i \in I\}$. If F is any family of sets one briefly writes $\bigcup F$ to mean $\bigcup\{x : x \in F\}$. The expression $x \subset y$ is taken to mean the set x is included within the set y, improper inclusion not being excluded. $x \not\subset y$ represents the logical negation of this notion.

II. GENERAL DESIGNS ON INFINITE SETS

§1. *Definitions.* It is very natural to view a combinatorial design on a set S in terms of 'covering' one family of subsets of S by another such family.

Definition II.1. Let M be any set and p some non-zero cardinal number and suppose F is some family of subsets of M. A family G is said to be a p-Steiner cover of the family F if every member of F is contained, as a subset, in exactly p members of G.

In terms of Steiner covers it is clear that a set M possesses a (k,n,p)-tactical configuration if and only if the family $[M]^k$ possesses a p-Steiner cover G such that $G \subset [M]^n$. In this way a combinatorial design on a set M represents the covering of a certain family of subsets by another such family. Consequently, an abstract formulation of a combinatorial design on a set could be given as follows.

Definition II.2. Let M be any set. A combinatorial design $\mathfrak{D}$ on the set M is an ordered triple $\langle \mathbf{A}, \mathbf{B}, p\rangle$ where $\mathbf{A}$ and $\mathbf{B}$ are both collections of families of subsets of M (i.e. $\mathbf{A}, \mathbf{B} \subset P^2(M)$) and p a non-zero cardinal number such that for each family $F \in \mathbf{A}$ there exists a family $G \in \mathbf{B}$ which is a p-Steiner cover of F.

Example II.3. Let M be any set. If $\mathbf{A} = \{[M]^k\}$ and $\mathbf{B} = \{F \subset P(M) : |x| = n$ for every $x \in F\}$, where k and n are non-zero cardinal numbers such that $k < n < |M|$, then $\mathfrak{D} = \langle \mathbf{A}, \mathbf{B}, p\rangle$ is nothing other than a (k,n,p)-tactical configuration of M.

§2. *Specialization of the design concept.* For purposes of the present paper the full generality of *II.2* will not be useful. Rather we shall greatly restrict ourselves to combinatorial designs which bear a striking resemblance to the tactical configurations. To do this we introduce the following.

Definition II.4. Let M be any set and p a non-zero cardinal number such that $p \leqq |M|$. A family F of distinct subsets of M is called a p-tuple family of M if 1) $|F| = |M|$, 2) $F \subset [M]^p$ and 3) if $x, y \in F$ such that $x \neq y$, then $x \not\subset y$ and $y \not\subset x$.

Remark II.5. If p is finite the family $[M]^p$ is a p-tuple family of M. This, of course, cannot be said when p is a non-finite cardinal number. In fact, it appears that this is the source of trouble which prohibits us from asserting $\mathbf{C}[v, k, n, p]$ when k is non-finite and $p < v$ (*cf.*, ($\mathbf{B}$)).

With the aid of the p-tuple families we can formulate the sought-for generalization of tactical configurations.

Definition II.6. Let M be any set and k, n and p be non-zero cardinal numbers. A combinatorial design $\langle \mathbf{A}, \mathbf{B}, p\rangle$ of M is said to be a (k, n, p)-generalized configuration of M if the following conditions are satisfied:

i) $\mathbf{A}$ *is the collection of all* k*-tuple families of* M
ii) $\mathbf{B} = \{F : F \subset [M]^n\}$.

It is clear from the above definition that if a set M possesses a (k, n, p)-generalized configuration then any other set M', having the same cardinality as M, will also possess such a configuration. This argues in favor of the following notation.

Definition II.7. Let v, k, n *and* p *be non-zero cardinal numbers. Then the symbol* $\mathbf{C}^*[v, k, n, p]$ *will assert the fact that some (and therefore any) set of cardinality* v *possesses a* (k, n, p)*-generalized configuration.*

Remark II.8. It follows from *II.3* and *II.5* that if k, n and p are finite cardinal numbers, then $\mathbf{C}^*[v, k, n, p]$ implies $\mathbf{C}[v, k, n, p]$ for every non-finite cardinal number v.

The time has come to state the chief result of the present paper. The major concern of section III will be to establish its proof.

Main Theorem. If v, k, n *and* p *are non-zero cardinal numbers such that 1)* v *is non-finite, 2)* $k < n < v$ *and 3)* $p \leqq v$, *then* $\mathbf{C}^*[v, k, n, p]$.

III. THEOREMS AND PROOFS

§1. *The proof of* ($\mathbf{B}$). The next theorem will show the non-existence of certain tactical configurations on infinite sets and, in so doing, establish ($\mathbf{B}$).

Theorem III.1. Let k, n *and* v *be any non-zero cardinal numbers such that a)* $n < v$ *and b)* k *is non-finite. Then* $\mathbf{C}[v, k, n, p]$ *is not true for all non-zero cardinal numbers* p *such that* $p < v$.

Proof. Let k, n and v be given as above. Also let p be a non-zero cardinal number such that

(1) $\mathbf{C}[v, k, n, p]$ *is true.*

It will be sufficient for our proof to show that $p \geqq v$ must follow. By (1) there must exist an infinite set M and a family G of subsets of M such that

(2) $|M| = v$
(3) $G \subseteq [M]^n$

and

(4) *for every* $x \in [M]^k$ *there exists at least one* $y \in G$ *such that* $x \subset y$.

The fact that p is a non-zero cardinal number establishes (4). It is immediate from (3) and (4) that

(5) $\bigcup G = M$.

Let x be given such that

(6) $x \in [M]^k$.

We will show that x is contained in at least v members of the family G. By (4) there must exist $y_0 \epsilon G$ such that

(7) $x \subset y_0$.

We now construct a transfinite sequence[4] of type $o(|M|)$, each of whose terms are distinct members of G which contain the set x. Define φ_0 to be y_0. Suppose δ to be an ordinal number such that

(8) $0 < \delta < o(|M|)$.

Moreover, for each $\xi < \delta$, let φ_ξ be given and endowed with the following properties:

(9) $\varphi_\xi \epsilon G$
(10) $x \subset \varphi_\xi$
(11) $\varphi_{\xi_1} \neq \varphi_{\xi_2}$ *whenever* $\xi_1 \neq \xi_2$.

Let $y^* = \bigcup\{\varphi_\xi : \xi < \delta\}$. It is clear from (3) and (9) that

(12) $|y^*| \leqq n\bar{\delta}$.

But a) (8), (12) and the fact that $o(|M|)$ is an initial number give

(13) $|y^*| < |M|$.

Moreover, (13) together with the axiom of choice yields

(14) $|M - y^*| = |M|$.

In particular, there exists an element e such that

(15) $e \epsilon M - y^*$.

Let $x_e = x \cup \{e\}$. Since k is a non-finite cardinal number, (6) and the axiom of choice imply

(16) $|x_e| = |x| = k$.

By (4) there must exist a $z \epsilon G$ such that

(17) $x_e \subset z$.

Finally, set $\varphi_\delta = z$. It is clear from (17) and the construction of the set x_e that

(18) $x \subset z$.

Thus (9) and (10) are satisfied for $\xi = \delta$. But (15), (17) and the construction of x_e indicate

(19) $e \notin \varphi_\xi$ *for each* $\xi < \delta$

4. Once and for all we state that use of transfinite sequences in this paper is justified by appeal to the axiom of choice in the form of the well-ordering principle.

and

(20) $\quad e \in \varphi_\delta$

which, in turn, shows that $\varphi_\xi \neq \varphi_\delta$ for each $\xi < \delta$.

In this manner a sub-family

(21) $\quad G_x = \{\varphi_\xi : \xi < o(|M|)\}$

of G is constructed such that each member φ_ξ of G_x satisfies (9), (10) and (11). Yet from the fact that $\varphi_{\xi_1} \neq \varphi_{\xi_2}$, whenever $\xi_1 \neq \xi_2$, one obtains

(22) $\quad |G_x| = |M|$.

But from this we may conclude $p \geqq |M| = v$. Q.E.D.

§2. *Outline of the proof of the Main Theorem.* To establish the *Main Theorem* we must show $\mathbf{C}^*[v, k, n, p]$ whenever v, k, n and p are non-zero cardinal numbers such that v is non-finite, $k < n < v$ and $p \leqq v$. By *Definitions II.6* and *II.7* this means we must show that there exists a non-finite set M, of cardinality v, such that every k-tuple family of M possesses a p-Steiner cover G contained in $[M]^n$. However, the next theorem shows that a search for a p-Steiner cover may be reduced to finding p many *1*-Steiner covers which are mutually disjoint.

Theorem III.2. Let M be a non-finite set, p a non-zero cardinal number and F a family of subsets of M. For each ordinal number $\xi < o(p)$ let G_ξ be a family of subsets of M which is a 1-Steiner cover of F. If $G_{\xi_1} \cap G_{\xi_2} = \phi$ whenever $\xi_1 \neq \xi_2$, then the family $G = \bigcup\{G_\xi : \xi < o(p)\}$ is a p-Steiner cover of F.

Proof. Let $x \in F$. Now for each $\xi < o(p)$ there exists exactly one member of G_ξ which contains x. Denote this member by $G_\xi(x)$. Therefore

(23) $\quad \{y \in G : x \subset y\} = \{G_\xi(x) : \xi < o(p)\}$.

But since $G_{\xi_1} \cap G_{\xi_2} = \phi$ whenever $\xi_1 \neq \xi_2$ and $G_\xi(x) \in G_\xi$ for each $\xi < o(p)$ it must be that

(24) $\quad G_{\xi_1}(x) \neq G_{\xi_2}(x)$ *whenever* $\xi_1 \neq \xi_2$

which together with (23) yields

(25) $\quad |\{y \in G : x \subset y\}| = p$.

But since (25) holds for each $x \in F$, G is a p-Steiner cover of F. Q.E.D.

Our attack upon the *Main Theorem* now becomes clear. First, for any non-zero cardinal numbers k, n and v such that $k < n < v$ we prove $\mathbf{C}^*[v, k, n, 1]$. In other words we show that for a non-finite set M, of cardinality v, every k-tuple family of M possesses a *1*-Steiner cover contained in $[M]^n$. Then to complete the job we assert, for any $p \leqq v$, $\mathbf{C}^*[v, k, n, p]$, by exhibiting p many mutually disjoint *1*-Steiner covers contained in $[M]^n$ of any k-tuple family of M. We begin our program by establishing certain properties of k-tuple families which will be crucial to what follows.

§3. *A preparation lemma.* The following result about k-tuple families will be used throughout this section.

Lemma III.3. Let M be any non-finite set and k and n any non-zero cardinal numbers such that $k, n < |M|$. Let $F = \{x_\xi : \xi < o(|M|)\}$ be a family of distinct subsets of M such that $|x_\xi| = k$ for each $\xi < o(|M|)$. Then there exists a well defined set function Δ, defined on F, such that a) $\Delta(x) \in [M]^n$ for each $x \in F$, b) $\Delta(x) \cap \Delta(y) = \phi$ whenever $x \neq y$ and c) $\Delta(x_\delta) \cap x_\xi = \phi$ for each $\xi \leqq \delta$, where δ is any ordinal number less than $o(|M|)$.

Remark. The reader will observe that any k-tuple family of M satisfies the hypotheses of *Lemma III.3.*

Proof. It is clear that $|x_0| = k$. Since M is non-finite and $k, n < |M|$, there exists a set $N \subset M$ such that $|N| = n$ and $N \cap x_0 = \phi$. Define $\Delta(x_0) = N$. Now suppose δ to be an ordinal number such that

(26) $\quad 0 < \delta < o(|M|)$

and, in addition, suppose Δ is defined on the segment $F_\delta = \{x_\xi : \xi < \delta\}$ of F such that Δ possesses properties a) - c) of *III.3.* Let

(27) $\quad X = \bigcup \{\Delta(x_\xi) : \xi < \delta\}$

and

(28) $\quad Y = \bigcup \{x_\xi : \xi \leqq \delta\}$

From (27), (28) and the fact that $\Delta(x_\xi) \in [M]^n$ for each $\xi < \delta$, one obtains

(29) $\quad |X| \leqq n\overline{\delta}$

and

(30) $\quad |Y| \leqq k\overline{\delta}.$

But since $o(|M|)$ is an initial number, (26), (29), (30) and the axiom of choice yield $|X \cup Y| < |M|$. Consequently

(31) $\quad |M - (X \cup Y)| = |M|.$

This shows there must exist a set $N^* \subset M - (X \cup Y)$ such that

(32) $\quad |N^*| = n.$

Put $\Delta(x_\delta) = N^*$. It is clear that Δ is now defined on the segment $F_{\delta+1} = \{x_\xi : \xi < \delta + 1\}$. However, (32) and the induction assumption show Δ, defined on $F_{\delta+1}$, possesses property a). Properties b) and c) follow from the induction assumption and the fact that the set $\Delta(x_\delta)$ was constructed to satisfy

(33) $\quad \Delta(x_\delta) \cap (X \cup Y) = \phi.$

In this way the principle of transfinite induction insures Δ to be defined on all of F with the required properties a) – c). Q.E.D.

§4. *The case when* $p = 1$. The present paragraph will show $\mathbf{C}^*[v, k, n, 1]$ whenever v, k and n are non-zero cardinal numbers such that v is non-finite and $k \leqq n \leqq v$. We always assume M to be a set of cardinality v. It is easy to verify the following observations:

(34) *if* $n = v$, *the family* $G = \{M\}$ *is a 1-Steiner cover for every* k*-tuple family of* M

and

(35) *if* $k = n$, *every* k*-tuple family of* M *is, in fact, a 1-Steiner cover of itself.*

Thus, in light of (34) and (35), to prove $\mathbf{C}^*[v, k, n, 1]$ when v is non-finite and $k \leqq n \leqq v$ it is only necessary to consider the case when $k < n < v$. In pursuit of this end we prove a slightly weaker result.

Theorem III.4. Let M *be a non-finite set of cardinality* v *and suppose* k *and* n *are non-zero cardinal numbers such that* $k < n < v$. *Then every* k*-tuple family* F *of* M, *such that* $|\bigcup F| = |M|$, *possesses a 1-Steiner cover contained in* $[M]^n$.

Remark. This theorem shows that certain k-tuple families F of M (*viz.*, those with $|\bigcup F| = |M|$) possess the appropriate 1-Steiner cover. This restriction will be dispensed with in *Theorem III.5.*

Proof. Since F is a k-tuple family of M it is possible to express

(36) $F = \{x_\xi : \xi < o(v)\}$

and define, in virtue of *Lemma III.3*, a set $\Delta(x_\xi) \in [M]^{n-k}$, for each $x_\xi \in F$, such that

(37) $\Delta(x_{\xi_1}) \cap \Delta(x_{\xi_2}) = \phi$ *whenever* $\xi_1 \neq \xi_2$

and

(38) $\Delta(x_\delta) \cap x_\xi = \phi$ *for each* $\xi \leqq \delta$

One observes that *III.3* applies in this case since for all non-zero cardinal numbers n and k such that $k < n < v$, it must be that $n-k$ is a non-zero cardinal number $< v$. Continuing, one constructs the family $G' = \{y_\xi : \xi < o(v)\}$, where $y_\xi = x_\xi \cup \Delta(x_\xi)$. It is clear that for each $\xi < o(v)$ one has

(39) $|y_\xi| = |x_\xi \cup \Delta(x_\xi)| = k + (n-k) = n$.

This follows from the fact that x_ξ and $\Delta(x_\xi)$ are disjoint sets. Hence the family G' is contained in $[M]^n$ and covers F in the sense that every member of F is contained in, *at least*, one member of G'. The aim now is to select a subfamily G of G' which will be a 1-Steiner cover of F. To accomplish this one uses a transfinite sequence. Let $\varphi_0 = 0$. Now suppose δ to be an ordinal number such that

(40) $\quad 0 < \delta < o(v)$.

Assume φ_ξ has been defined for all $\xi < \delta$. Let φ_δ be the smallest ordinal number $\mu < o(v)$ satisfying

(41) $\quad x_\mu \not\subset y_{\varphi_\xi}$ *for each* $\xi < \delta$.

To show the sequence $\{\varphi_\xi\}_{\xi<o(v)}$ is well-defined it is expedient to proceed by contradiction. Suppose there exists an ordinal number $\delta < o(v)$ such that the statement

(42) $\quad x_\mu \subset y_{\varphi_\xi}$ *for some* $\xi < \delta$

holds for each $\mu < o(v)$. However, (42) gives

(43) $\quad \bigcup\{x_\xi : \xi < o(v)\} \subset \bigcup\{y_{\varphi_\xi} : \xi < \delta\}$.

But it is clear from (39) that

(44) $\quad \left|\bigcup\{y_{\varphi_\xi} : \xi < \delta\}\right| \leqq n\bar{\delta} < v$.

Yet by hypothesis

(45) $\quad \left|\bigcup\{x_\xi : \xi < o(v)\}\right| = |M| = v$

which clearly contradicts (43) and (44). Thus the sequence $\{\varphi_\xi\}_{\xi<o(v)}$ is well-defined.

In addition, this sequence is strictly increasing. To establish this assume, to the contrary, either of the following:

Case 1° $\quad \eta < \lambda < o(v)$ *and* $\varphi_\eta = \varphi_\lambda$
Case 2° $\quad \eta < \lambda < o(v)$ *and* $\varphi_\eta > \varphi_\lambda$

Suppose *Case 1°* occurs. Now φ_λ is defined to be the smallest ordinal number $\mu < o(v)$ such that

(46) $\quad x_\mu \not\subset y_{\varphi_\xi}$ *for each* $\xi < \lambda$.

But since $\eta < \lambda$, (46) yields

(47) $\quad x_{\varphi_\lambda} \not\subset y_{\varphi_\eta}$.

Here $y_{\varphi_\eta} = y_{\varphi_\lambda}$. Hence we have $x_{\varphi_\lambda} \not\subset y_{\varphi_\lambda}$, contradicting the construction of y_{φ_λ}. Therefore *Case 1°* cannot obtain.

Suppose *Case 2°*. By construction, φ_η is the smallest ordinal number $\mu < o(v)$ such that

(48) $\quad x_\mu \not\subset y_{\varphi_\xi}$ *for each* $\xi < \eta$.

But since $\eta < \lambda$, it must be that $x_{\varphi_\lambda} \not\subset y_{\varphi_\xi}$ for each $\xi < \eta$. In this case, however, $\varphi_\eta < \varphi_\lambda$. Thus (48) is satisfied for $\mu = \varphi_\lambda$ which is less than φ_η, a contradiction. Therefore *Case 2°* cannot obtain. Neither of the cases holding, the sequence $\{\varphi_\xi\}_{\xi<o(v)}$ must be strictly increasing.

Proceed now by defining the sub-family

(49) $\quad G = \{x\varphi_\xi \cup \Delta(x\varphi_\xi) : \xi < o(v)\}$

of G'. One can show G to be a *1*-Steiner cover of F. Let $x \in F$. In light of (36) there exists an ordinal number $\mu < o(v)$ such that $x = x_\mu$. Since the sequence $\{\varphi_\xi\}_{\xi < o(v)}$ is strictly increasing and of type $o(v)$ there must exist an ordinal number $\delta < o(v)$ such that

(50) $\quad \varphi_\delta > \mu.$

By the construction of φ_δ, an ordinal number $\lambda < \delta$ must exist such that $x_\mu \subset y\varphi_\lambda$; otherwise, by (50), μ would be less than φ_δ such that $x_\mu \not\subset y\varphi_\xi$ for each $\xi < \delta$. But this would contradict the construction of φ_δ. Hence

(51) $\quad x_\mu \subset y\varphi_\lambda$

and, consequently, every member of the family F is contained in, *at least*, one member of the family G.

To show G is a *1*-Steiner cover of F assume, to the contrary, the existence of an $x = x_\mu \in F$ such that

(52) $\quad x_\mu \subset (x\varphi_\eta \cup \Delta(x\varphi_\eta)) \cap (x\varphi_\lambda \cup \Delta(x\varphi_\lambda))$

for $\eta < \lambda < o(v)$. But since $\{\varphi_\xi\}_{\xi < o(v)}$ is strictly increasing, it must be that

(53) $\quad \varphi_\eta < \varphi_\lambda < o(v)$.

Now (52) implies

(54) $\quad x_\mu \subset (x\varphi_\eta \cap x\varphi_\lambda) \cup (x\varphi_\eta \cap \Delta(x\varphi_\lambda)) \cup (\Delta(x\varphi_\eta) \cap x\varphi_\lambda) \cup (\Delta(x\varphi_\eta) \cap \Delta(x\varphi_\lambda))$.

However, (37), (38), (53) and (54) yield

(55) $\quad x_\mu \subset (x\varphi_\eta \cap x\varphi_\lambda) \cup (\Delta(x\varphi_\eta) \cap x\varphi_\lambda)$

from which follows $x_\mu \subset x\varphi_\lambda$. But F is a k-tuple family which forces

(56) $\quad x_\mu = x\varphi_\lambda$.

Consequently (52) and (56) give

(57) $\quad x\varphi_\lambda \subset (x\varphi_\eta \cup \Delta(x\varphi_\eta)) = y\varphi_\eta$.

However (57) and the fact that $\eta < \lambda$ contradict the very construction of $x\varphi_\lambda$, showing (52) cannot obtain. Thus G is a *1*-Steiner cover of F. Q.E.D.

We now strengthen the preceding theorem with the following

Theorem III.5. Let M by any non-finite set of cardinality v and let k and n be any non-zero cardinal numbers such that $k < n < v$. Then every k-tuple family F of M possesses a 1-Steiner cover contained in $[M]^n$.

Proof. In light of *III.4* it is only necessary to consider the case when $|\bigcup F| < v$. Then, with the aid of the axiom of choice, it is known that $|M - \bigcup F| = v$. Since $vk = v$, it is possible to decompose the set $M - \bigcup F$

into a family of cardinality v, of disjoint subsets, each of whose cardinality is k. Let

$$(58) \qquad M - \bigcup F = \bigcup \{N_\xi : \xi < o(v)\}$$

represent such a decomposition of the set $M - \bigcup F$. Now consider the family $F^* = \{N_\xi : \xi < o(v)\} \cup F$. Since F is a k-tuple family and since the family $\{N_\xi : \xi < o(v)\}$ consists of disjoint subsets which are elements of $[M]^k$ and whose union does not meet any member of F, it follows that F^* is a k-tuple family of M. But it is clear from (58) that $\bigcup F^* = M$ and hence $|\bigcup F^*| = |M| = v$. Thus, by *Theorem III.4,* there exists a family G^* contained in $[M]^n$ which is a *1*-Steiner cover of F^*. Yet F being a subfamily of F^* forces G^* to be a *1*-Steiner cover for F. Q.E.D.

In view of remarks (34) and (35), *Theorem III.5* allows us to realize the aim of the present paragraph:

Theorem III.6. Let v, k and n be any non-zero cardinal numbers such that v is non-finite and $k \leqq n \leqq v$. Then $\mathbf{C}^*[v, k, n, 1]$.

§**5.** *Proof of the Main Theorem.* Let v, k, n and p be non-zero cardinal numbers such that v is non-finite, $k < n < v$ and $p \leqq v$. To show $\mathbf{C}^*[v, k, n, p]$ it is sufficient to prove, given any set M of cardinality v and any k-tuple family F of M, the existence of a family $G \subset [M]^n$ which is a p-Steiner cover of F. But, in view of *Theorem III.2,* it is sufficient to exhibit p many *1*-Steiner covers of F which are contained in $[M]^n$ and mutually disjoint. The next theorem does exactly this.

Theorem III.7. Let M be any non-finite set of cardinality v and let k, n and p be non-zero cardinal numbers such that a) $k < n < v$ and b) $p \leqq v$. Let F be any k-tuple family of M. Then for each ordinal number $\xi < o(p)$ there exists a family G_ξ of subsets of M such that

i) $G_\xi \subset [M]^n$
ii) G_ξ *is a 1-Steiner cover of* F

and

iii) $G_{\xi_1} \cap G_{\xi_2} = \phi$ *whenever* $\xi_1 \neq \xi_2$.

Remark. To prove the above theorem it is sufficient to consider the case when $|\bigcup F| = |M|$. For if $|\bigcup F| < |M|$ we may embed the family F into a larger family F^* with the property $\bigcup F^* = M$ and then every *1*-Steiner cover G_ξ of F^* would also be a *1*-Steiner cover of F. This procedure was followed in the proof of *Theorem III.5.*

Proof. Constructing the G_ξ's will be accomplished by transfinite induction. In doing this a series of lemmas will be employed. They will be stated and proved within the body of the proof of *Theorem III.7* and will be considered under the hypotheses of this same theorem.

The conditions of *Theorem III.7* satisfy the hypotheses of *Theorem III.5* from which one may conclude the existence of a family G_0 of subsets of M such that

(59) $G_0 \subset [M]^n$

and

(60) G_0 *is a 1-Steiner cover of* F.

Now let γ be an arbitrary ordinal number (which will remain fixed throughout the proof of *III.7*) such that

(61) $0 < \gamma < o(p)$

and suppose for each $\xi < \gamma$ a family G_ξ of subsets of M has been constructed such that

(62) $G_\xi \subset [M]^n$
(63) G_ξ *is a 1-Steiner cover of* F

and

(64) $G_{\xi_1} \cap G_{\xi_2} = \phi$ *whenever* $\xi_1 \neq \xi_2$.

It remains now to construct the family G_γ. The following three lemmas will be employed to help arrive at this construction. (One will observe that many of the subsequent arguments will closely follow those given in §4.) On the basis of (62), (63) and (64) for each ordinal number $\xi < \gamma$ a family G_ξ is assumed to be a given *1*-Steiner cover of F. This justifies

Definition III.8. Let ξ *be an arbitrary ordinal number less than* γ. *Then for each* $x \in F$, *let the symbol* $G_\xi(x)$ *represent the unique member of the 1-Steiner cover* G_ξ *of* F *which contains* x.

Lemma III.9. Let the k-tuple family F *be expressed as* $F = \{x_\xi : \xi < o(v)\}$. *Then there exists a well-defined set-function* Δ *such that*

i) $\Delta(x) \in [M]^{n-k}$ *for each* $x \in F$

ii) $\Delta(x) \cap \Delta(y) = \phi$ *whenever* $x \neq y$

iii) $\Delta(x_\nu) \cap x_\xi = \phi$ *for each* $\xi \leqq \nu$

and

iv) $\Delta(x_\nu) \cap \bigcup\{G_\xi(x_\nu) : \xi < \gamma\} = \phi$ *for each* $\nu < o(v)$.

Proof. Let ν be any ordinal number less than $o(v)$. Then set

(65) $Z_\nu = \bigcup\{G_\xi(x_\nu) : \xi < \gamma\}$.

From (62) and *Definition III.8* one may obtain

(66) $|Z_\nu| \leqq n\overline{\gamma}$

which with (61) and the fact that $p \leqq v$ yields

(67) $|Z_\nu| < v$ *for each* $\nu < o(v)$.

Since $F \subset [M]^k$ it is clear that $|x_0| = k$. Therefore in view of the axiom of choice, (67) and the facts that M is non-finite and k, $n < v$, there exists a set $N \subset M - (x_0 \cup Z_0)$ such that $|N| = n - k$. Define $\Delta(x_0) = N$. Now suppose δ to be an arbitrary ordinal number such that

(68) $\quad 0 < \delta < o(v)$

and, in addition, suppose Δ is defined on the segment $F_\delta = \{x_\xi : \xi < \delta\}$ of F in such a way that Δ possesses properties *i) - iv)* of *III.9*. Let

(69) $\quad X = \bigcup \{\Delta(x_\xi) : \xi < \delta\}$

and

(70) $\quad Y = \bigcup \{x_\xi : \xi \leqq \delta\}.$

From (69), (70) and the fact that $\Delta(x_\xi) \in [M]^{n-k}$ for each $\xi < \delta$ (by induction assumption) one obtains

(71) $\quad |X| \leqq (n-k)\overline{\delta}$

and

(72) $\quad |Y| \leqq k\overline{\delta}$

But since $o(v)$ is an initial number (67), (68), (71), (72) and the axiom of choice yield

(73) $\quad |X \cup Y \cup Z_\delta| < v$

and consequently

(74) $\quad |M - (X \cup Y \cup Z_\delta)| = v.$

By (74) the set $M - (X \cup Y \cup Z_\delta)$ is not empty and since $n - k < v$ there must exist a set $N_\delta \subset M - (X \cup Y \cup Z_\delta)$ such that

(75) $\quad |N_\delta| = n - k\,.$

Let $\Delta(x_\delta) = N_\delta$. It is clear that Δ is now defined on the segment $F_{\delta+1} = \{x_\xi : \xi < \delta + 1\}$ of F. Hence (75) and the induction assumption show Δ, defined of $F_{\delta+1}$, possesses property *i)*. Properties *ii)*, *iii)* and *iv)* follow from the induction assumption and the fact that the set $\Delta(x_\delta)$ was constructed to satisfy

(76) $\quad \Delta(x_\delta) \cap (X \cup Y \cup Z_\delta) = \phi.$

Thus the principle of transfinite induction insures Δ to be defined on the whole of F with the properties *i) - iv)* and consequently *Lemma III.9* is proved.

Continuing with the proof of *Theorem III.7* and, in particular, with the construction of the family G_γ, one defines a family of subsets of M by letting

(77) $\quad G^*_\gamma = \{y_\xi = x_\xi \cup \Delta(x_\xi) : \xi < o(v)\}.$

It is immediate from property *iii)* of *Lemma III.7* that

(78) $x_\xi \cap \Delta(x_\xi) = \phi$ *for each* $\xi < o(v)$

and consequently

(79) $|x_\xi \cup \Delta(x_\xi)| = k + (n-k) = n.$

Therefore (77) and (79) yield

(80) $G^*_\gamma \subset [M]^n.$

Another important property of this family is exhibited in the following lemma.

Lemma III.10. Let ν *be any ordinal number less than* γ. *Then* $G_\nu \cap G^*_\gamma = \phi$.

Proof. The conclusion of *III.10* will be established if one can show no member of G^*_γ is identical with any member of G_ν. To this end let

(81) $x \in G^*_\gamma.$

By (77) there must exist a $\lambda < o(v)$ such that

(82) $x = x_\lambda \cup \Delta(x_\lambda)$ *where* $x_\lambda \in F$.

However G_ν is assumed in (63) to be a *1*-Steiner cover of F. Thus there exists a unique $G_\nu(x_\lambda) \in G_\nu$ such that

(83) $x_\lambda \subset G_\nu(x_\lambda).$

Therefore one has

(84) $x_\lambda \not\subset y$ *for each* $y \in G_\nu - \{G_\nu(x_\lambda)\}$.

But by property *iv)* of *III.9* one obtains

(85) $\Delta(x_\lambda) \cap \bigcup\{G_\xi(x_\lambda) : \xi < \gamma)\} = \phi$

which *a fortiori* yields

(86) $\Delta(x_\lambda) \cap G_\nu(x_\lambda) = \phi$

since ν is given to be an ordinal number less than γ. By (84) one observes that the element $x \in G^*_\gamma$ cannot be identical with any member of the family $G_\nu - \{G_\nu(x_\lambda)\}$. Yet (82) and (86) insure that x cannot be identical with the elements $G_\nu(x_\lambda)$ of G_ν and consequently cannot be identical with any member of G_ν. This proves *Lemma III.10.*

Although G^*_γ itself is not a *1*-Steiner cover of F it possesses a sub-family which is. This is seen in the following

Lemma III.11. There exists a sub-family G *of* G^*_γ *such that* G *is a 1-Steiner cover of* F.

Proof. The argument used here will follow very closely the one used in the proof of *III.4.* The Sierpiński sequence[5] will again be the main tool.

5. For a complete discussion of the Sierpiński sequence see the Appendix.

Repeating (77) one has

(87) $\quad G^*_\gamma = \{ y_\xi = x_\xi \cup \Delta(x_\xi) : \xi < o(v)\}.$

Constructing the Sierpiński sequence one sets $\varphi_0 = 0$. Then suppose δ to be an ordinal number such that

(88) $\quad 0 < \delta < o(v)\,.$

Assume φ_ξ has been defined for all $\xi < \delta$. Let φ_δ be the smallest ordinal number $\mu < o(v)$ satisfying

(89) $\quad x_\mu \not\subset y_{\varphi_\xi}$ *for each* $\xi < \delta\,.$

To show the sequence $\{\varphi_\xi\}_{\xi < o(v)}$ is well-defined proceed by contradiction. Suppose there exists an ordinal number $\delta < o(v)$ such that the statement

(90) $\quad x_\mu \subset y_{\varphi_\xi}$ *for some* $\xi < \delta$

holds for each $\mu < o(v)$. However, (90) gives

(91) $\quad \bigcup \{x_\xi : \xi < o(v)\} \subset \bigcup \{ y_{\varphi_\xi} : \xi < \delta \}.$

But it is clear from (79) and (88) that

(92) $\quad \left| \bigcup \{ y_{\varphi_\xi} : \xi < \delta \}\right| \leq n\bar{\delta} < v.$

Yet by the properties of F one obtains

(93) $\quad \left| \bigcup \{x_\xi : \xi < o(v)\} \right| = \left| \bigcup F \right| = v$

which clearly contradicts (91) and (92). Hence the transfinite sequence $\{\varphi_\xi\}_{\xi < o(v)}$ is well-defined. Moreover, the sequence can be shown to be strictly increasing in exactly the same manner as was done in *III.4*. Continuing one defines a sub-family

(94) $\quad G = \{ y_{\varphi_\xi} = x_{\varphi_\xi} \cup \Delta(x_{\varphi_\xi}) : \xi < o(v)\}$

of G^*_γ. One can show G to be a *1*-Steiner cover of F. Let $x \in F$. Then there must exist an ordinal number $\mu < o(|F|) = o(v)$ such that $x = x_\mu$. Since the sequence $\{\varphi_\xi\}_{\xi < o(v)}$ is strictly increasing and of type $o(v)$ there must also exist an ordinal number $\delta < o(v)$ such that

(95) $\quad \varphi_\delta > \mu\,.$

By the definition of φ_δ, an ordinal number $\lambda < \delta$ must exist such that $x_\mu \subset y_{\varphi_\lambda}$; otherwise, by (95), μ would be less than φ_δ such that $x_\mu \not\subset y_{\varphi_\xi}$ for each $\xi < \delta$. But this would contradict the construction of φ_δ. Hence

(96) $\quad x_\mu \subset y_{\varphi_\lambda}$

and, consequently, every member of the family F is contained in, at least, one member of the family G.

To show G is a *1*-Steiner cover of F, assume, to the contrary, the existence of an $x = x_\mu \in F$ such that

(97) $\quad x_\mu \subset (x_{\varphi_\eta} \cup \Delta(x_{\varphi_\eta})) \cap (x_{\varphi_\lambda} \cup \Delta(x_{\varphi_\lambda}))$

for $\eta < \lambda < o(v)$. But since $\{\varphi_\xi\}_{\xi < o(v)}$ is strictly increasing, it must be that

(98) $\varphi_\eta < \varphi_\lambda < o(v)$.

Now (97) implies

(99) $x\mu \subset (x\varphi_\eta \cap x\varphi_\lambda) \cup (x\varphi_\eta \cap \Delta(x\varphi_\lambda)) \cup (\Delta(x\varphi_\eta) \cap x\varphi_\lambda) \cup (\Delta(x\varphi_\eta) \cap \Delta(x\varphi_\lambda))$.

However, (98), (99) and the properties of the set-function Δ yield

(100) $x\mu \subset (x\varphi_\eta \cap x\varphi_\lambda) \cup (\Delta(x\varphi_\eta) \cap x\varphi_\lambda)$

from which follows $x\mu \subset x\varphi_\lambda$. But F is given to be a k-tuple family of M thus forcing

(101) $x\mu = x\varphi_\lambda$.

Consequently (97) and (101) give

(102) $x\varphi_\lambda \subset (x\varphi_\eta \cup \Delta(x\varphi_\lambda)) = y\varphi_\eta$.

However (102) and the fact that $\eta < \lambda$ contradict the very construction of $x\varphi_\lambda$, showing (97) cannot obtain. Thus G is a *1*-Steiner cover of F. This completes the proof of *Lemma III.11.*

With the establishment of *Lemmas III.9*, *III.10* and *III.11* one is in a position to complete the construction of the G_ξ's which was initiated at the outset of the proof of *Theorem III.7*. To do this it only remains to define a family $G\gamma$ having the following properties:

(103) $G\gamma \subset [M]^n$

(104) $G\gamma$ *is a 1-Steiner cover of* F

and

(105) $G_\xi \cap G\gamma = \phi$ *for each* $\xi < \gamma$.

Set $G\gamma = G$. *Lemma III.11* and (80) immediately show G to satisfy (103) and (104). That G satisfies (105) is seen from *Lemma III.10* and the fact that G is a sub-family of G^*_γ. Thus one has produced a $G\gamma$ with the required properties. In this manner, for each $\xi < o(p)$ a family G_ξ of subsets of M can be constructed such that

(106) $G_\xi \subset [M]^n$

(107) G_ξ *is a 1-Steiner cover of* F

and

(108) $G_{\xi_1} \cap G_{\xi_2} = \phi$ *whenever* $\xi_1 \neq \xi_2$.

Q.E.D.

From *III.7* we can now assert our chief existence result of combinatorial designs on infinite sets.

Main Theorem III.12. Let v, k, n *and* p *be non-zero cardinal numbers such that 1)* v *is non-finite, 2)* $k < n < v$ *and 3)* $p \leqq v$. *Then* $\mathbf{C}^*[v,k,n,p]$.

In light of *II.7* is is clear that the above theorem verfies (**A**).

IV. APPENDIX: THE SIERPIŃSKI SEQUENCE

Without exaggeration it can be said that the entire theory of block designs on infinite sets, presented in this work, developed as an outgrowth of Sierpiński's original four-page note [11]. The transfinite sequence which he employs there with great success to construct a Steiner triple system can be considered the cornerstone of the entire theory presented. These facts justify a separate investigation into the nature of Sierpiński's sequence.

To show a non-finite set M possesses a Steiner triple system it is necessary to construct a family F of triples of M such that every two elements $a, b \in M$ are together in one and only one member of F. Essentially, Sierpiński first forms a family F^* of triples of M with the property that every two elements of M are together in, *at least*, one member of this family. Then, using a transfinite sequence, he selects a sub-family of F^* which has the desired property to solve the Steiner problem.

Being more explicit, with the aid of the axiom of choice it may be supposed that

(1) $$M = \{\xi : \xi < o(|M|)\}.$$

That is, M is the set of all ordinal numbers less than the first ordinal number whose cardinality is $|M|$. Then Sierpiński considers the set

(2) $$P = \{<\alpha,\beta> : \alpha < \beta < o(|M|)\}$$

of all ordered pairs of distinct ordinal numbers α and β which are elements of M. Using a well-known ordering Sierpiński remarks that the set P can be well-ordered according to the ordinal number $o(|M|)$. Hence the set P can be expressed as

(3) $$P = \{<\alpha_\xi,\beta_\xi> : \xi < o(|M|)\}.$$

Now to construct the family F^* it is sufficient to augment to each $<\alpha_\xi, \beta_\xi>$ an element of M not equal to either α_ξ or β_ξ. Using standard theorems concerning ordinal addition the ordinal number $\alpha_\xi + \beta_\xi + \xi'$, where ξ' is some ordinal number $> \xi$, can be shown to be an element of M which has this property. In view of this, he constructs the family

(4) $$F^* = \{\{\alpha_\xi, \beta_\xi, \alpha_\xi + \beta_\xi + \xi'\} : \xi < o(|M|)\}.$$

It becomes immediate that such a family has the property that every two distinct elements $a, b \in M$ are together in, *at least*, one member of F^*.

The aim, now, is to discard from F^* members which make this family redundant with respect to the above property. Sierpiński's solution to this problem reduces to this. The two distinct elements, constituting the coordinates of the first ordered pair of the well-ordered set P (i.e. the elements α_1 and β_1) are together in the member $\{\alpha_1, \beta_1, \alpha_1 + \beta_1 + 1'\}$. This member of F^* will be retained. Now look to the elements α_2 and β_2 of M. If these two elements are not together in the member $\{\alpha_1, \beta_1, \alpha_1 + \beta_1 + 1'\}$ their "private cover" given in F^*, namely $\{\alpha_2, \beta_2, \alpha_2 + \beta_2 + 2'\}$, will also be kept to form the new family. If, however, $\{\alpha_2, \beta_2\} \subset \{\alpha_1, \beta_1, \alpha_1 + \beta_1 + 1'\}$ one then removes from F^* the cover $\{\alpha_2, \beta_2, \alpha_2 + \beta_2 + 2'\}$ and goes on to look at the elements α_3 and β_3 of M. If the set $\{\alpha_3, \beta_3\}$ is contained on one of the previously retained members of F^* the member $\{\alpha_3, \beta_3, \alpha_3 + \beta_3 + 3'\} \in F^*$ may be discarded. If not, one must retain this member. In this way one may construct, using the retained members of F^*, a sub-family F of F^* which inherits the property of F^* that every two distinct elements of M are together in, *at least*, one member of the family F. The trick that really solves the problem is that if the assignment of a third element to every two distinct elements of M is made "nice enough" (here one assigns to the elements α_ξ and β_ξ the element $\alpha_\xi + \beta_\xi + \xi'$) the family F also enjoys the crucial property that every two distinct elements of M are together in, *at most*, one member of F. In this way F can be shown to be a Steiner triple system of M.

Extracting the family F from F^* is the work of the Sierpiński sequence. Precisely, it is defined in the following manner. Let $\varphi_1 = 1$. Suppose δ is an arbitrary ordinal number such that

$$(5) \qquad 1 < \delta < o(|M|)$$

and suppose φ_ξ is defined for all $\xi < \delta$. Then let φ_δ be the smallest ordinal number μ which enjoys the property

$$(6) \qquad \{\alpha_\mu, \beta_\mu\} \not\subset \{\alpha_{\varphi_\xi}, \beta_{\varphi_\xi}, \alpha_{\varphi_\xi} + \beta_{\varphi_\xi} + \xi'\} \textit{ for each } \xi < \delta .$$

Thus in the manner of transfinite induction, the sequence $\{\varphi_\xi\}_{\xi < o(|M|)}$ is defined. Using very standard properties of ordinal numbers this definition is shown to be non-vacuous and the sequence defined by it strictly increasing. In virtue of all this it can then be established that the family

$$(7) \qquad F = \{\{\alpha_{\varphi_\xi}, \beta_{\varphi_\xi}, \alpha_{\varphi_\xi} + \beta_{\varphi_\xi} + \xi'\} : \xi < o(|M|)\}$$

is a Steiner triple system of M.

All generalizations obtained in this paper concerning the existence of block designs can be considered as an elaboration, to a greater or less degree, of the very simple, but powerful, idea which is at the core of the Sierpiński sequence. Probably more than any other result presented in this work, the preparation lemma, given in III, §3, exemplifies this point. There, the notion of "niceness of the assignment", mentioned above, is rigorously precised. Once this is accomplished the Sierpiński sequence becomes the chief weapon employed in almost all the proofs of section III.

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