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PREMISSES ARE NOT AXIOMS

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On page 35 of his excellent *Mathematical Logic* [1], Stephen C. Kleene writes: "For the propositional calculus applied to infer formulas from assumptions $A_1, \ldots, A_m$, the formulas $A_1, \ldots, A_m$ are in effect allowed to function as axioms also." Kleene does not go on to call assumption formulas axioms. Still his suggestion that assumption formulas function as axioms is misleading. And I think that Kleene's way of putting what we do when we use assumption formulas is fairly common amongst logic instructors. Hence, this note.

For instance, Kleene's suggestion could lead someone to the following misconception about what they do when they establish that $(P \supset Q)$, $(Q \supset R) \therefore (P \supset R)$ is a derived rule in some axiomatization of classical propositional calculus. Let us say that they—those with the misconception—have axiom schemata which together with *Modus Ponens* suffices for a deductively complete consistent axiomatization of classical propositional logic. Call the set of axiom schemata AS. They may think that when they establish that a formula schema $(P \supset R)$ can be derived from formula schemata $(P \supset Q)$ and $(Q \supset R)$, they add $(P \supset Q)$ and $(Q \supset R)$ as axiom schemata to AS to get a larger set of axiom schemata AS' and have $(P \supset R)$ as a theorem schema form AS'. Now, of course, this has to be a totally erroneous conception of what we do when we establish that $(P \supset Q)$, $(Q \supset R) \therefore (P \supset R)$ is a derived rule in an axiomatization of classical propositional calculus. It has to be erroneous because a derivation of $(P \supset R)$ from $(P \supset Q)$ and $(Q \supset R)$ does not require an inconsistent system. But it is well known that if a non-tautologous formula schema is added as an axiom schema to a complete, consistent classical propositional calculus the resulting system in inconsistent. (For a proof of this see Kleene [2], page 134.)

When we use assumption formulas they do not function as axioms in the sense that we first add assumption formulas to our axiom schemata and then proceed to construct demonstrations. We do not add assumption formulas to the axiom schemata. We first place the assumption formulas directly into demonstrations and then add axioms and use rules of proof to

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construct demonstrations. For instance, if our assumption formula is $(P \supset Q)$ and we add it to our axioms as an axiom schema, we are entitled to put $(\sim P \supset P)$ into demonstrations as an axiom schema, and that will show that we have an inconsistent system. However, if we put $(P \supset Q)$ directly into demonstrations without first regarding it as an axiom schema, we will not be able to say that any schema with the same form as $(P \supset Q)$ is an axiom schema. Assumption formulas function in demonstrations like axioms only in that their inclusion in a demonstration need not be justified by saying that they are derived by legitimate rules from preceding lines in the demonstration.

Of course, my remarks do not apply only to axiomatizations by means of axiom schemata. If we had an axiomatization of the propositional calculus by means of proper axioms the addition of non-tautologous formulas as axioms would lead to inconsistency as noted by Hilbert and Ackermann on page 43 of [3] and by Church on page 110 of [4]. We would get the inconsistency by applying the variable substitution rule to non-tautologous formulas. In closing I should note that when Church defines demonstration from assumption formulas on page 87 of [4], he is careful about making it clear that assumption formulas are not axioms because he does not allow variable substitution to be applied to assumption formulas.

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[3] Hilbert, D., and W. Ackermann, *Principles of Mathematical Logic,* Chelsea Publishing Co. (1950).

[4] Church, Alonzo, *Introduction to Mathematical Logic,* Vol. I, Princeton (1956).

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