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A DIAGRAMMATIC TREATMENT OF SYLLOGISTIC

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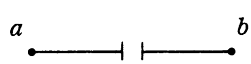
In this paper a method of diagramming subject-predicate propositions, using directed graphs, is presented. By means of the diagrams, all logical consequences of an arbitrary finite set of propositions (of the appropriate form) can be read off. A simple calculation yields the number of valid moods of the n-termed syllogism (sorites).

1. Throughout, we are concerned only with propositions of the forms A, E, I and O, without complex, negative, or empty terms. Each proposition is construed as asserting a relation between two entities of the same type. Thus—if the terms are taken to represent (non-void) *classes*—Aab says that a is a subclass of b, Eab that a and b are disjoint, Iab that a and b have a common subclass, and Oab that a has a subclass which is disjoint from b. The class-interpretation is not essential, however; all that is strictly required is that the entities in question (i.e. whatever is denoted by the terms) should form a quasi-ordered set without zero.

Let R be a finite set of propositions, T the set of terms occurring in propositions of R. R may be represented by a directed graph (with slight additions), as follows. For each term in T, a point is taken as vertex of the graph (with distinct vertices assigned to distinct terms); the vertex assigned to a term a will be described simply as "the vertex a." To each such vertex is attached a loop, i.e. an arc leading from the vertex to itself. Consider now a proposition belonging to R. If the proposition is Aab, we insert in the graph an arc leading from the vertex b to the vertex a. If the proposition is Eab, we insert an 'interrupted arc' between the vertices a, b:



For Iab, we introduce a new vertex x, together with arcs leading from a to x and from b to x. Lastly, if the proposition is Oab, we introduce a new vertex x, a (directed) arc leading from a to x, and an interrupted arc between x and b. This procedure is repeated for each proposition of R in turn, subject to the restriction that all the new vertices introduced in

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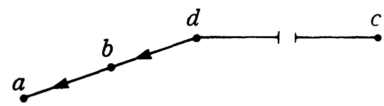
connection with I and O propositions be distinct from each other as well as from the original vertices.

As an example, we diagram the following set of premisses, given by Lewis Carroll:

> Babies are illogical.
> Nobody is despised who can manage a crocodile.
> Illogical persons are despised.

These are of the forms *Aab*, *Ecd*, and *Abd* respectively, and the diagram—omitting loops—is:



In order to describe the 'conclusions' that may be read off from the diagram of a set R of propositions, we need a few definitions. A vertex a is said to be a *descendant* of a vertex b if there is a progression of (directed) arcs leading from b to a; in particular, every vertex is a descendant of itself. Two vertices a, b are *(mutually) excluded* if they are, respectively, the descendants of vertices c, d which are joined by an interrupted arc (in this case we also say that a is *excluded from* b). A *consistent* graph is one in which no term is excluded from itself. (Equivalently, we could say that, in a consistent graph, no pair of vertices joined by an interrupted arc has a common descendant). Finally, a *chain* is a graph consisting of n arcs $l_1, \ldots, l_n$ $(n \geq 1)$ and $n+1$ vertices $a_1, \ldots, a_{n+1}$, such that, for each i $(1 \leq i \leq n)$, l_i joins a_i with a_{i+1}; a_1, a_{n+1} are the *end-points* of the chain.

Let G be a consistent graph. We say that a proposition P may be *read off* from G, or that P is *represented* on G provided (i) P is *Aab*, and a is a descendant of b in G; (ii) P is *Eab*, and a, b are mutually excluded in G; (iii) P is *Iab*, and a, b have a common descendant in G; or (iv) P is *Oab*, and a has a descendant in G that is excluded from b. In the case of an inconsistent graph G, all propositions that can be formed with terms corresponding to the vertices of G are said to be *represented* on G. Example: the following propositions, among others, can be read off from the graph shown above: *Aad*, all babies are despised; and *Eac*, no babies can manage a crocodile.

2. It must now be shown that the conclusions that can be read off from the graph of a set R of propositions are precisely the logical consequences of R. As already indicated, we are assuming that each term denotes an element of a fixed quasi-ordered set $\{Q, \leq\}$ without zero. Let a, b,... be the elements of Q denoted by the terms $a, b, \ldots$, then we are also assuming that:

> *Aab* is true iff a $\leq$ b
> *Eab* is true iff there is no element x of Q such that $x \leq$ a and $x \leq$ b
> *Iab* is true iff there is an element x of Q such that $x \leq$ a and $x \leq$ b
> *Oab* is true iff it is not the case that a $\leq$ b.

Suppose that a set R of true propositions is represented by the graph G. Then it follows from our definitions and assumptions that: if a is a descendant of b, then a $\leqslant$ b; if a, b have a common descendant, then there is an element x of Q such that $x \leqslant$ a and $x \leqslant$ b; if a, b are mutually excluded then there is no element x of Q such that $x \leqslant$ a and $x \leqslant$ b; and if a has a descendant that is excluded from b, it is not the case that a $\leqslant$ b. Thus, every proposition that is represented on G must be true. In other words, the method always yields, from true premisses, true conclusions.

For the converse ('completeness' of the method) we show that, roughly speaking, it is possible to construct a counterexample, using finite sets, to any proposition that is not represented on a given graph. Now, in the case of an inconsistent graph, *every* relevant proposition is represented; thus, nothing is lost if we restrict attention to consistent graphs. We use the following notation: if x is any vertex of a consistent graph, $[x]$ is the set of descendants of x. Evidently, $[a] \subseteq [b]$ iff a is a descendant of b; and $[a]$, $[b]$ have non-null intersection iff a, b have a common descendant.

Suppose now that G is the graph of a set R of propositions, T the set of terms occurring in R; and let C be a proposition whose terms belong to T, but which is not represented on G. We want to show that the terms of T may be interpreted, or re-interpreted, in such a way that the propositions of R turn out true, while C is false. In the case that C is an affirmative proposition (A or I), this is achieved simply by assigning to each term a of T the corresponding set $[a]$. If, on the other hand, C is a universal negative proposition, Eab, we extend G to a new graph G' by introducing a new vertex x, together with arcs leading from a to x and from b to x. By hypothesis, Eab is not represented on G, so G' is consistent. Then the desired interpretation is obtained by assigning to each term the set of its descendants in G'. Finally, if C is Oab, we may consistently extend G by inserting an arc leading from a to b; the desired interpretation is obtained from the extended graph, as before.

Remark: The constructions that have just been described consist essentially in adding the negation of C to the graph of R.

From these considerations we conclude that a syllogistic inference $P_1, P_2, \ldots, P_n \Rightarrow C$ is valid if and only if C is represented on the graph of $P_1, P_2, \ldots, P_n$.

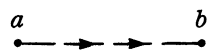
3. We now turn to a more detailed study of the structure of valid syllogisms. This will take the form of an analysis of the conditions under which, for a given pair of distinct terms a, b appearing on a graph G, a proposition Xab can be read off from G (where X is one of the relations A, E, I, O). It is evident that no conclusion Xab can be drawn unless G contains at least one chain having a and b as its end-points. But further: in assessing whether Xab can be read off, we may consider the chains (if any) which connect a with b one at a time. More precisely: if Xab is represented on G, then it is represented on some chain connecting a with b. For example, if Eab is represented on G, G contains a pair of mutually excluded vertices x, y, together with progressions of arcs leading from x to a and from y to b. By taking the arcs of these two progressions and the interrupted arc joining x with y, together with their end-points, we evidently have a chain on which Eab is represented; similar considerations apply if X is A, I, or O.

Thus, without essential loss of generality, we can restrict attention to the problem of drawing a 'conclusion' Xab, given a *chain* between a and b. This is equivalent to the traditional[1] problem of the sorites, which we may formulate as : to test the validity of inferences of the form:

$$X_1 a_1 a_2, \ldots, X_{n-1} a_{n-1} a_n \Rightarrow X_n a_1 a_n$$

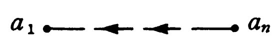
with $n \geqslant 2$ and where, now, each X_i is one of the relations A, E, I, O, the converse[2] of A, or the converse of O, and $a_1, \ldots, a_n$ are n distinct terms.

We will use the method of graphs to compute the number of valid moods of the n-termed syllogism above. (The solution has already been found by C. A. Meredith [4], using a method based on the traditional 'rules' of the syllogism.) In schematizing the relevant graphs, a broken arrow:



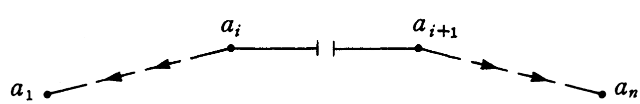
will indicate a progression of zero or more directed arcs (the vertices a, b may coincide). We consider separately the four possible forms of the conclusion, $X_n = A, E, I$, or O:

(1) The conclusion is Aa_1a_n. This is represented on the graph of the premisses only if the graph has the form:



Thus we have one valid mood with the A conclusion.

(2) The conclusion is Ea_1a_n. The graph of the premisses must have the form:



where $1 \leqslant i < n$, giving $n - 1$ possible chains. Each of these chains yields two moods, since we have to distinguish between the propositions Ea_ia_{i+1}, $Ea_{i+1}a_i$; thus we obtain $2(n - 1)$ valid moods.

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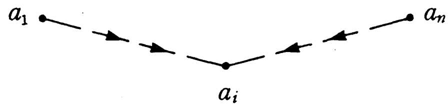
1. Although, if we ignore the trivial 'Aristotelean' and 'Goclenean' sorites, the problem seems not to have been seriously considered prior to Lewis Carroll [1] and Keynes [2].
2. Lorenzen [3] has suggested that these converse relations be treated on an equal footing with A,E,I,O as the basis of syllogistic. If this is done, the calculations below can be simplified even further.

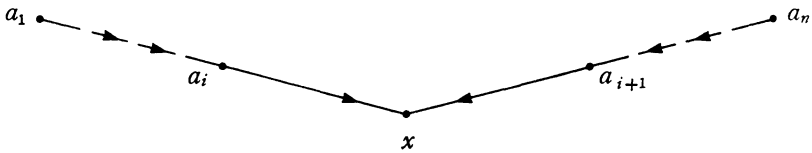
(3) For the conclusion Ia_1a_n we have:

(i) the chains in which only the vertices $a_1, \dots, a_n$ are used:



where $1 \leqslant i \leqslant n$; and

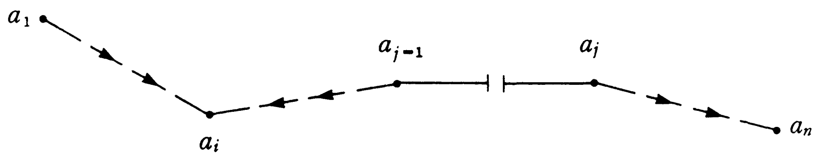
(ii) chains in which an auxiliary vertex is introduced (corresponding to an I proposition in the premisses):



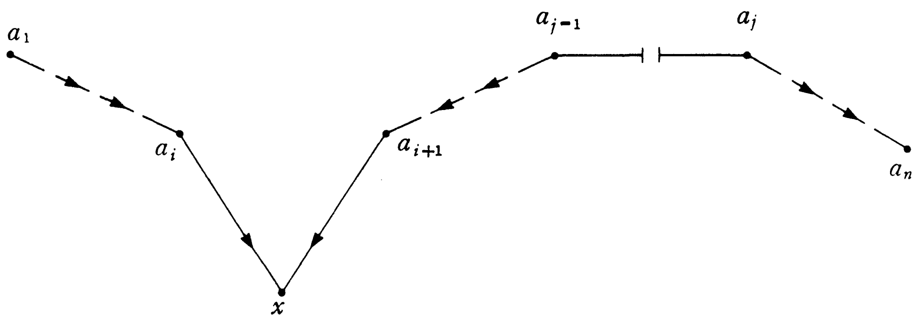
where $1 \leqslant i < n$. Each of the latter chains yields two moods, since we distinguish between Ia_ia_{i+1} and $Ia_{i+1}a_i$; hence we obtain $n + 2(n - 1) = 3n - 2$ valid moods with an I conclusion.

(4) The most complex case is the conclusion Oa_1a_n. The possible types of chains are:

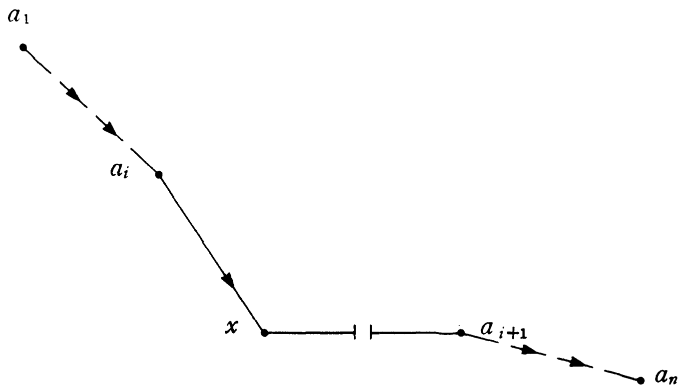
(i) those in which only the vertices $a_1, \dots, a_n$ appear $(1 \leqslant i < j \leqslant n)$:



(ii) those in which an auxiliary vertex x results from an I proposition in the premisses $(1 < i + 1 < j \leqslant n)$



(iii) those in which an auxiliary vertex x results from an O proposition in the premisses $(1 \leqslant i < n)$:



It is readily seen that (i), (ii), (iii) are the only chains fulfilling the required condition, viz. that a_1 have a descendant which is excluded from a_n. Each chain of type (i) gives 2 valid moods, each chain of type (ii) 4 valid moods; hence the number of valid moods yielding the O conclusion is:

$$2.\ {}^{n}C_2 + 4.\ {}^{n-1}C_2 + n - 1 = 3(n-1)^2 .$$

The total number of valid moods is therefore:

$$1 + 2(n-1) + 3n - 2 + 3(n-1)^2 = 3n^2 - n .$$

REFERENCES

[1] Carroll, Lewis, *Symbolic Logic,* 3rd Ed., London (1896).

[2] Keynes, J. N., *Studies and Exercises in Formal Logic,* 2nd Ed., London (1887).

[3] Lorenzen, P., *Formale Logik,* Berlin (1958).

[4] Meredith, C. A., "The figures and moods of the n-term Aristotelean syllogism," *Dominican Studies,* vol. 6 (1953), pp. 42-47.

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