Notre Dame Journal of Formal Logic
Volume XII, Number 1, January 1971

FINITE MODEL PROPERTY FOR FIVE MODAL CALCULI IN THE NEIGHBOURHOOD OF S3

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That Lewis' system S3 is decidable was shown by Matsumoto in [9]. That it has the finite model property (f.m.p.) has been established only recently by Lemmon in [7]. First it is proved that a weaker system E3 has the f.m.p. and from this it is inferred that S3 also has the same property. There is one disadvantage to this method. It is not clear how to modify it to show that a system which is somewhat stronger (or weaker) than S3 also has the f.m.p. Given a *direct* proof this can be fairly easily done. Halldén, for example, has, in an obvious manner, extended the result from S2 to S6 (compare Theorem 5 of [10] with Theorem 13 of [5]). A similar extension from S3 to S7 is not readily available from Lemmon's treatment; and the same remark applies to weakening the result to, say, S3°.

In this paper I shall give a direct proof of the f.m.p. of S3° and extend it to the systems R3°, S3.1, S7 and S8. The system S3° is due to Sobociński [13]; R3° due to Canty [2]; S3.1, S7 and S8 due to Halldén [5]. The name "S3.1" occurs in [6]; p. 345. In §1 new axiomatizations of these systems will be given. The two important deductions of §1, those of 1.2 and 2.1, are extracted from certain considerations of Lemmon [7], both algebraic and logistical (see pp. 195-196). In §2 the f.m.p. will be established. The results of §2 are simple consequences of the axiomatizations and the author's results of [12] and thorough acquaintance with [12] is presupposed. All the terminology and notation of §2 is that of [12].

§1. AXIOMATICS. We suppose our systems to be N-K-M calculi with the usual definitions. The five systems mentioned are defined as follows:

(1) S3° = {S1°; $\mathfrak{C}\mathfrak{C}pq\mathfrak{C}MpMq$};
(2) R3° = {S3°; $CLpp$};
(3) S3.1 = {S3; $M\mathfrak{C}LpLLp$};
(4) S7 = {S3; MMp};
(5) S8 = {S3; $LMMp$}.

Now consider the following five theses:

V1 $\mathfrak{C}MKMpNMKpNpMp$;

Received December 22, 1969

V2 $CpMp$;
V3 $MNMMKpNp$;
V4 $MMKpNp$;
V5 $NMNMMKpNp$.

It is pointed out by Hughes and Cresswell in [6], p. 269 that S7 can be alternately axiomatized as {S3; *V4*}. (Their remark is for S2 and S6. But, of course, it carries over to S3 and S7.) Similarly, it is easy to see that {S8} $\leftrightarrows$ {S3; *V5*}. Also, clearly {R3°} $\leftrightarrows$ {S3°; *V2*}. We now prove that {S3°} $\leftrightarrows$ {S2°; *V1*} and {S3.1} $\leftrightarrows$ {S3; *V3*}.

Theorem 1. {S3°} $\leftrightarrows$ {S2°; *V1*}.

1.1. First we show that {S3°} $\rightarrow$ {*V1*}.

Z1	$\mathfrak{C}\mathfrak{C}pq\mathfrak{C}MpMq$	[S3°]
Z2	$\mathfrak{C}MKMpNMqMKpNq$	[*Z1*; S1°]
V1	$\mathfrak{C}MKMpNMKpNpMp$	[*Z2*, $q/KpNp$; S1°]

1.2. Next we show that {S2°; *V1*} $\rightarrow$ {S3°}.

Z1	$\mathfrak{C}MKqNqMq$	[S2°]
Z2	$\mathfrak{C}MKpNpMq$	[*Z1*; S1°]
Z3	$\mathfrak{C}NMqNMKpNp$	[*Z2*; S1°]
Z4	$\mathfrak{C}KMpNMqKMpNMKpNp$	[*Z3*; S1°]
Z5	$\mathfrak{C}MKMpNMqMKMpNMKpNp$	[*Z4*; S2°]
Z6	$\mathfrak{C}MKMpNMqMp$	[*Z5*; *V1*; S1°]
Z7	$\mathfrak{C}\mathfrak{C}pqCMpMq$	[S1°; *cf.*33.321 in [4]]
Z8	$\mathfrak{C}NMKpNqANMpMq$	[*Z7*; S1°]
Z9	$\mathfrak{C}KMpNMqMKpNq$	[*Z8*; S1°]
Z10	$\mathfrak{C}KMpNMqKMKpNqNMq$	[*Z9*; S1°]
Z11	$\mathfrak{C}MKMpNMqMKMKpNqNMq$	[*Z10*; S2°]
Z12	$\mathfrak{C}MKMpNMqMKpNq$	[*Z6*, $p/KpNq$; *Z11*; S1°]
Z13	$\mathfrak{C}\mathfrak{C}pq\mathfrak{C}MpMq$	[*Z12*; S1°]

This completes the proof. There are a number of things to notice about the thesis *V1*: (1) Note its similarity to the condition for transitive algebras in [7], p. 196. (2) The proper axiom of S3°(S3) (*Z13* above) when added to S1°(S1) gives us S3°(S3). In other words its addition to S2°(S2) makes the proper axiom of S2°(S2), $\mathfrak{C}MKpqMp$, non-independent. But *V1* has to be added to S2°(S2) to give S3°(S3). Group V of [8], p. 494 verifies S1°(S1) and *V1* but falsifies $\mathfrak{C}MKpqMp$. (3) In [1] Åqvist constructs a system S3.5. "S3.5 is put forward to stand to S5 as S3 stands to S4 and S2 to T" (See [3], p. 58). A similar system on the S1-side, i.e., a system which stands to S1 as S3 stands S2 and S4 to T can be constructed by adding *V1* to S1. And we can call it S1.5. (4) *V1* can be thought of as a sort of incomplete form of the proper axiom of S4°(S4), $\mathfrak{C}MMpMp$, since erasing $NMKpNp$ from *V1* gives us $\mathfrak{C}MMpMp$. (5) In [8] mention is made of "*T*-principles" of S1, viz., theorems of S1 of the form $\mathfrak{C}K\alpha T\beta$ where T is a theorem of S1 but $\mathfrak{C}\alpha\beta$ is not. Apparently Lewis and Langford thought that only S1 has T-principles (See p. 151). However, it is noted by Hughes and Cresswell in [6], p. 230,

n. 209 that S2 also has T-principles and their argument clearly shows that even S3 has these principles. Now $V1$ is a theorem of S3 which may well be called a T-principle but of a different sort than the ones mentioned above, i.e., S3 contains theorems of the form $\mathfrak{C}MK\alpha T\beta$ where T is a theorem of S3 but $\mathfrak{C}M\alpha\beta$ is not.

Theorem 2. $\{S3.1\} \rightleftarrows \{S3;\ V3\}$.

2.1 First we show that $\{S3.1\} \rightarrow \{V3\}$.

$Z1$	$\mathfrak{C}KMMKpNpNMKpNpKMMKpNpNMKpNp$	[S1°]
$Z2$	$\mathfrak{C}MMKpNpCNMKpNpKMMKpNpNMKpNp$	[$Z1$; S1°]
$Z3$	$\mathfrak{C}MMKpNpAMKpNpKMMKpNpNMKpNp$	[$Z2$; S1°]
$Z4$	$\mathfrak{C}MKpNpMKMMKpNpNMKpNp$	[S2°; *cf.* $Z2$ of **1.2** above]
$Z5$	$\mathfrak{C}KMMKpNpNMKpNpMKMMKpNpNMKpNp$	[S1]
$Z6$	$\mathfrak{C}AMKpNpKMMKpNpNMKpNpMKMMKpNpNMKpNp$	[$Z4$; $Z5$; S1°]
$Z7$	$\mathfrak{C}MMKpNpMKMMKpNpNMKpNp$	[$Z6$; S1°]
$Z8$	$\mathfrak{C}NMKMMKpNpNMKpNpNMMKpNp$	[$Z7$; S1°]
$Z9$	$\mathfrak{C}MNMKMMKpNpNMKpNpMNMMKpNp$	[$Z8$; S2°]
$Z10$	$M\mathfrak{C}LpLLp$	[S3.1]
$Z11$	$M\mathfrak{C}MMpMp$	[$Z10$, p/Np; S1°]
$Z12$	$MNMKMMKpNpNMKpNp$	[$Z11$, $p/KpNp$; S1°]
$V3$	$MNMMKpNp$	[$Z12$; $Z9$; S1°]

2.2 Next we show that $\{S3;\ V3\} \rightarrow \{S3.1\}$.

$Z1$	$\mathfrak{C}LLq\mathfrak{C}LpLLp$	[S3; *cf.* TS3.7 in [6], p. 235]
$Z2$	$\mathfrak{C}NMMKpNp\mathfrak{C}LpLLp$	[$Z1$, $q/NKpNp$; S1°]
$Z3$	$\mathfrak{C}MNMMKpNpM\mathfrak{C}LpLLp$	[$Z2$; S2°]
$Z4$	$M\mathfrak{C}LpLLp$	[$Z3$; $V3$; S1°]

This completes the proof. Halldén in [5] proved two intersection results: (1) α is a theorem of S3 if and only if α is a theorem of both S4 and S7; (2) α is a theorem of S3 if and only if α is a theorem of both S3.1 and S8. It is well-known that $\{S4\} \rightleftarrows \{S3;\ NMMKpNp\}$ and we saw earlier that $\{S7\} \rightleftarrows \{S3;\ MMKpNp\}$. Also we have just shown that $\{S3.1\} \rightleftarrows \{S3;\ MNMMKpNp\}$ whereas $\{S8\} \rightleftarrows \{S3;\ NMNMMKpNp\}$. It is interesting that in both cases we can find a thesis A such that the two intersecting calculi can be axiomatized by adding A and NA respectively to S3.

We therefore have the following alternative axiomatizations which we now write in a different notation:

(1) $S3^\circ = \{S2^\circ;\ \Diamond(\Diamond p \wedge \sim\Diamond(p \wedge \sim p)) ⥽ \Diamond p\}$;
(2) $R3^\circ = \{S3^\circ;\ p \supset \Diamond p\}$;
(3) $S3.1 = \{S3;\ \Diamond\sim\Diamond\Diamond(p \wedge \sim p)\}$;
(4) $S7 = \{S3;\ \Diamond\Diamond(p \wedge \sim p)\}$;
(5) $S8 = \{S3;\ \sim\Diamond\sim\Diamond\Diamond(p \wedge \sim p)\}$.

§2. FINITE MODEL PROPERTY. As in [12] we shall use matrices $\mathfrak{M} = \langle M, D, \cap, -, \mathrm{P}\rangle$ in our investigation. As our stock of conditions on these matrices we list the following:

(A) $\langle M, \cap, -, \mathsf{P}\rangle$ is a weak modal algebra;
(B) D is an additive ideal of M;
(C) $x = 0$ if and only if $-\mathsf{P}(x) \in D$;
(D) $\mathsf{P}0 \leq \mathsf{P}x$;
(E) $\mathsf{P}(\mathsf{P}x \cap -\mathsf{P}0) \leq \mathsf{P}x$;
(F) $x \rightarrow \mathsf{P}x \in D$;
(G) $x \leq \mathsf{P}x$;
(H) $\mathsf{P}-\mathsf{P}\mathsf{P}0 \in D$;
(I) $\mathsf{P}\mathsf{P}0 \in D$;
(J) $-\mathsf{P}-\mathsf{P}\mathsf{P}0 \in D$.

We omit the proof of the three theorems that follow:

Theorem 3. There exists a σ-regular characteristic matrix for S3°(R3°, S3.1, S7, S8).

Theorem 4. $\mathfrak{M} = \langle M, D, \cap, -, \mathsf{P}\rangle$ *is a σ-regular* S3°(R3°, S3.1, S7, S8)-*matrix if and only if*

(1) (A) —— (E);
(2) (A) —— (F);
(3) (A) —— (H);
(4) (A) —— (G), (I);
(5) (A) —— (G), (J).

Theorem 5. $\vdash_{\overline{\mathrm{S3}^\circ(\mathrm{R3}^\circ,\ \mathrm{S3.1},\ \mathrm{S7},\ \mathrm{S8})}} A$ *if and only if A is verified by all matrices* $\mathfrak{M} = \langle M, D, \cap, -, \mathsf{P}\rangle$ *such that condition (1)((2), (3), (4), (5)) of Theorem 4 is satisfied.*

Theorem 6. Let $\mathfrak{M} = \langle M, D, \cap, -, \mathsf{P}\rangle$ *be a σ-regular* S3°(R3°, S3.1, S7, S8)-*matrix, and let* $a_1, \ldots, a_r$ *be a finite sequence of elements of M. Then there is a finite σ-regular* S3°(R3°, S3.1, S7, S8)-*matrix* $\mathfrak{M}_1 = \langle M_1, D_1, \cap_1, -_1, \mathsf{P}_1\rangle$ *with at most* 2^{2^r+4} *elements such that*

(*i*) *for* $1 \leq i \leq r$, $a_i \in M_1$;
(*ii*) *for* $x, y \in M_1$, $x \cap_1 y = x \cap y$;
(*iii*) *for* $x \in M_1$, $-_1 x = -x$;
(*iv*) *for* $x \in M_1$ *such that* $\mathsf{P}x \in M_1$, $\mathsf{P}_1 x = \mathsf{P}x$;
(*v*) *for* $x \in M_1$, *if* $x \in D_1$, *then* $x \in D$.

Proof. See Theorem IV.1 [12] and Theorem IV.4 [12]. Include now in the construction of M_1, $\mathsf{P}\mathsf{P}0$ and $\mathsf{P}-\mathsf{P}\mathsf{P}0$ as well. This does not affect the proofs of the theorems but changes the "2^{2^r+2}" in their statements to "2^{2^r+4}." It is clear that the only thing which remains to be shown is that $\mathfrak{M}_1$ satisfies conditions (D) —— (J) given that $\mathfrak{M}$ satisfies the corresponding conditions.

D: Let $\mathsf{P}0 \leq \mathsf{P}x$. But $\mathsf{P}0 = \mathsf{P}_1 0$ and $\mathsf{P}x \leq \mathsf{P}_1 x$. So $\mathsf{P}_1 0 \leq \mathsf{P}_1 x$.

We pause now and note that this shows that $\mathfrak{M}_1$ is a S2°-matrix given that $\mathfrak{M}$ is one (see the axiomatization of S2° given in [11]). Also note that each of our systems contain S2°. We shall use this fact in what follows.

E: Let $\mathsf{P}(\mathsf{P}x \cap -\mathsf{P}0) \leq \mathsf{P}x$. Let x be covered by $A_1, \ldots, A_n$. Let $\mathsf{P}_1 x \cap -\mathsf{P}_1 0$ be covered by $B_1, \ldots, B_p$. Let $A_1 = \{x_1, \ldots, x_s\}$. Then $\mathsf{P}_1 x \leq \mathsf{P}A_1$. Hence

$P_1x \cap -P_10 \leqslant PA_1 \cap -P0 = (Px_1 \cup \ldots \cup Px_s) \cap -P0 = (Px_1 \cap -P0) \cup \ldots \cup (Px_s \cap -P0)$. Now proceeding exactly as in Theorem V. 10 [12] (observe that properties of S2°-matrices are used in the proof) we get, $P_1(P_1x \cap -P_10) \leqslant P_1x$.

F: Let $x \varepsilon M_1$ and $x \rightarrow Px \varepsilon D$. Now $Px \leqslant P_1x$. Hence $x \cap -P_1x \leqslant x \cap -Px$. Hence $-(x \cap -Px) \leqslant -(x \cap -P_1x)$. By Definition II.5 [12] and Theorem III.6 [12], $(x \rightarrow Px) \rightarrow (x \rightarrow P_1x) \varepsilon D$. By Definition II.14(ii) [12], $x \rightarrow P_1x \varepsilon D$. Also, clearly, $x \rightarrow P_1x \varepsilon M_1$. Therefore, $x \rightarrow P_1x \varepsilon D_1$.

G: Let $x \leqslant Px$. But $Px \leqslant P_1x$. So $x \leqslant P_1x$.

H: Let $P-PP0 \varepsilon D$. Now $P0 = P_10$. Hence $PP0 = PP_10$. Also $P_10 \varepsilon M_1$ and $PP_10 = PP0 \varepsilon M_1$ (by construction). Hence by condition (iv) of the theorem, $P_1P_10 = PP_10 = PP0$. Hence $-PP0 = -P_1P_10$. So $P-PP0 = P-P_1P_10$. Again, $-P_1P_10 \varepsilon M_1$ and $P-P_1P_10 = P-PP0 \varepsilon M_1$ (by construction). By condition (iv), $P_1-P_1P_10 = P-P_1P_10 = P-PP0$. Hence $P_1-P_1P_10 \varepsilon D$. And clearly $P_1-P_1P_10 \varepsilon M_1$. Therefore $P_1-P_1P_10 \varepsilon D_1$.

I: Let $PP0 \varepsilon D$. We have $P0 \leqslant P_10$. Hence $PP0 \leqslant PP_10$ (by the algebraic variant of Becker's Rule, which, of course, holds in S2°-matrices). Also $PP_10 \leqslant P_1P_10$. So $PP0 \leqslant P_1P_10$. By arguing as in (F), $P_1P_10 \varepsilon D_1$.

J: Let $-P-PP0 \varepsilon D$. The arguing as in (H), $P_1-P_1P_10 = P-PP0$. Hence $-P-PP0 = -P_1-P_1P_10$. So $-P_1-P_1P_10 \varepsilon D$. And $-P_1-P_1P_10 \varepsilon M_1$. Therefore, $-P_1-P_1P_10 \varepsilon D_1$.

This completes the proof of Theorem 6. It follows that our systems have the f.m.p. and so are decidable. For the systems S3°, R3° and S3.1, the decidability results are new. It is known, however, that S7 and S8 are decidable (see [6], pp. 282-284), but the proof that they have the f.m.p. is new. And, of course, implicit in Theorem 6 is another proof that S3 has the f.m.p.

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