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DECISION PROCEDURES FOR LEWIS SYSTEM S1 AND RELATED MODAL SYSTEMS

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This paper* contains decision procedures for various modal logics among which is Lewis' system S1. The decidability of these systems is established by showing that they have the finite model property. Proofs, general exposition, and references are given in complete detail so that future results by the author (and others) can be easily incorporated within the framework.

INTRODUCTION

There are several well-known decision procedures[1] for the Classical Propositional Calculus (**CPC**)[2]. The principal advantage in obtaining such procedures for **CPC** lies in the fact that the connectives under consideration have an intended interpretation, i.e., the usual two-valued truth-tables. Kalmar's method[3], for example, consists in proving that a well-formed formula (wff) is a theorem if and only if it is a tautology. After that to decide whether a wff is a theorem one only has to make a simple computation of truth-tables. The same technique also extends to finitely many-valued logics where the connectives again have intended interpretations. There is an increased complexity but the method remains essentially the same.

However, not all propositional calculi are constructed with an interpretation in mind. There are two ways in which we can set up the axioms for such calculi. We can start with matrices—which are our intended interpretation—for the various connectives and then try to formalize these matrices, i.e., give axioms and rules such that the theorems of the system shall coincide with the wffs that are verified by the matrix. **CPC** and other

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finitely many-valued logics fall under this category[4]. The other way is to start with undefined concepts and give axioms and rules using these concepts. The axioms and rules would then constitute a contextual definition of these concepts. Of course, the undefined terms are not entirely arbitrary. They are often supposed to have a meaning in the real world. And the purpose in building up the system is to elucidate this meaning.

Such are the systems of modal logic of Lewis[5]. The undefined concept—besides negation and conjunction which can be given a truth-functional interpretation—is possibility, and the purpose of these logics is to give an exact formal description of this notion. As to be expected many perplexities arise in setting up the axioms and rules. To take just one example: If it is possible that "*p*" is possible, is it possible that "*p*"? Lewis thus constructs five systems S1-S5 in increasing degrees of strength leaving the choice of the system which "really" captures the notion of possibility to the reader. We shall, however, be unconcerned with the philosophical aspects of these systems and approach them in a purely formal manner.

Although the modal systems of Lewis were not constructed with an intended finite matrix for possibility in mind it may be conjectured that a diligent search might reveal one. That such a search will be in vain was shown by Dugundji[6] who, using methods due to Gödel[7], proved that none of the Lewis systems admit of any finite interpretation. It follows, therefore, that we cannot hope to solve the decision problem of these systems by a straightforward truth-table computation.

At this point the paper of McKinsey[8] appeared in which he solved the decision problems for the systems S2 and S4 in a novel manner. The method, in a nutshell, is as follows: first he proves, by using unpublished methods of Lindenbaum and Tarski, that there exists an infinite characteristic matrix for these systems. This matrix happens to be the algebra of formulas of the system with a very natural equivalence relation defined on it. Next he shows that given any matrix that verifies all theorems of a system we can find a finite matrix which mirrors in a significant way the original matrix, i.e., the new finite matrix also verifies all theorems of the system; in addition, it falsifies a class of formulas—whose number of sub-formulas bear a certain relationship with the cardinality of the matrix—that are falsified by the original matrix. Now if a wff A is a non-theorem of the system under consideration it is, of course, falsified by the Lindenbaum-Tarski characteristic matrix. Hence, by the above-mentioned considerations, it is falsified by a certain finite matrix whose cardinality is determined by the number of sub-formulas of A. Therefore all we have to do to know whether a wff A is a theorem is to construct all possible matrices whose cardinality is less than a certain pre-assigned number and which verifies our system and then if A is verified by all these matrices it is a theorem; if not, it is not a theorem. The method is indeed elegant in in conception.

Nevertheless, there are two major shortcomings of the method. Firstly, the number of matrices we have to construct to test for theoremhood even very simple formulas is prohibitively large. But we can ignore this and relegate the problem to computer-technology. The second shortcoming is more serious: we do find out whether a wff A is a theorem; but if it is where is the proof, i.e., the deduction from the axioms and rules? We have given no indication whatsoever how to find one. In other words, we have solved the decision problem but not the deducibility problem[9]. This limitation, however, is partly offset by two factors. Firstly, this method gives us a structural insight into the algebra of formulas of a system as well as into the matrices related to the system. Secondly—and this is more important—the method shows that the system under consideration has, to use a term due to Harrop, the finite model property[10]. Since Gödel[11] and Harrop[12] have constructed propositional calculi which are decidable but do not have the finite model property something more is shown, by McKinsey's method, besides decidability.

After McKinsey's original paper, his method has been applied to a very large group of modal systems[13]. Nevertheless, the weakest of the Lewis systems, S1, has always eluded the McKinsey-attack. Nor has the system yielded to decision-methods subsequently invented to deal with the modal systems, e.g., the semantical method of Kripke[14], the Gentzen method of Ohnishi[15] and Matsumoto[16], the normal-form method of Anderson[17], etc.

Our purpose here is to give a decision procedure for S1 (and some other related systems). The pervasive spirit of this paper is that of McKinsey. We follow the pattern set down by him. Our contribution consists of a technical innovation within his framework that transforms his method into one of greater power. By our method we can do everything that McKinsey can; and more.

Section I introduces the systems. Besides S1° and S1 we shall introduce two new systems: T1° and T1. We shall refer to these as the S-systems and T-systems respectively. The precise motivation of the introduction of the T-systems will be explained in the section. In Section II we shall, by the Lindenbaum-Tarski method, construct characteristic matrices for our systems. It is best to consider Section III as a lemma to Section IV since the main result of Section III, the First Completeness Theorem, is proved in a much stronger form in Section IV, where, finally, we give decision procedures for our systems. Section V gives decision procedures for Lewis' system S2 and Sobociński's system S4°.[18] We have made the assertion that "we can do everything that McKinsey can; and more". The "and more" part is vindicated by our decision procedure for S1. It yet remains to show that "we can do everything that McKinsey can". On our part this is a conviction based on intimate acquaintance with our method. But we have no idea how to prove it since the question under consideration is the applicability of a method. Yet, we can try to convince the reader of the truth of our assertion by inviting him to watch us in action in a special case. Hence S2, since it is known that it yields to McKinsey. We show that it yields to us as well. The reason for including a decision procedure for S4° is simpler: it has not been done before.

I. THE SYSTEMS

In this section we shall introduce two new modal systems: T1 and T1°. It is well-known that there are certain basic differences between the Lewis systems S1(S1°) and S2(S2°)[1]. As to be expected these differences come out in metalogical investigations concerning these systems. But even in the axiomatic level there are indications of things to come e.g., it seems impossible to avoid the actual postulation of the rule of substitutability of strict equivalents if we want to axiomatise S1(S1°) with the Classical Propositional Calculus base[2]. Our motivation in introducing the T-systems is to capture the essential uniqueness of S1(S1°). The T-systems are very simple. Their very simplicity enables us to see clearly the structure of S1(S1°) without the attendant inessential complications that arise if we approach S1(S1°) in a straightforward manner.

The central problem of the kind of systems we shall consider is their decision problem. Leaving aside undecidable calculi, we may classify logical calculi according to the methods that yield decision procedures for these calculi. If we adopt this criterion of classification surprisingly enough most of the modal logics except S1(S1°) and a few other systems fall in the same class. Lemmon, for example, starts with a rather weak system C2 and once he solves its decision problem he is able to rapidly extend it to most standard systems with no further introduction of essential technique.[3] Our T-systems serve the same kind of purpose as C2. They exemplify the method used and once their decision problem is solved the extension to S1(S1°) is purely routine.

There is yet another reason for introducing the systems T1 and T1°. The systems have a close resemblance to Lemmon's E2 and C2[4]; and Lemmon has shown how one can reduce the decision problem of S2 to that of E2[5]. We had initially hoped that we would be able to perform such a reduction. But our efforts in this direction have failed. However, it is not unlikely that such a theorem may be forthcoming at a future date. If it does our results for the T-systems will be readily available for the required reduction.

We now construct our systems. Our primitive connectives are negation, conjunction and possibility to be denoted by '$\sim$', '$\wedge$' and '$\Diamond$' respectively. We write 'p', 'q', 'r', 's', for propositional variables; 'P', 'Q', 'R', 'S', as a syntactical denotation for a formula. We prefix '$\vdash$' to a formula valid in the system considered or to a scheme of formula supposed to be valid in a rule of deduction. The rules of formation of well-formed formula are usual.

We now make some definitions. These definitions are to be understood as mere abbreviations.

DEFINITION I.1: $P \vee Q = \sim(\sim P \wedge \sim Q)$;

DEFINITION I.2: $P \supset Q = \sim(P \wedge \sim Q)$;

DEFINITION I.3: $P \equiv Q = (P \supset Q) \wedge (Q \supset P)$;

DEFINITION I.4: $\Box P = \sim\Diamond\sim P$;

DEFINITION I.5: $P \prec Q = \sim\Diamond(P \wedge \sim Q)$;

DEFINITION I.6: $P \equiv Q = (P \prec Q) \wedge (Q \prec P)$.

§1. THE T-SYSTEMS. As our stock of axioms and rules we list the following:

A1: $p \supset (p \wedge p)$;
A2: $(p \wedge q) \supset p$;
A3: $(p \supset q) \supset (\sim(q \wedge r) \supset \sim(r \wedge p))$;
A4: $(\Box(p \supset q) \wedge \Box(q \supset r)) \supset (\Box(p \supset r))$;
A5: $\Box p \supset p$.

R1: Substitution on propositional variables;
R2: If $\vdash P$ and if $\vdash P \supset Q$, then $\vdash Q$;
R3: Substitutability of material equivalents.

Note that we use the defined symbols: '$\supset$' and '$\Box$' in our axioms and rules. The only reason is typographical convenience and perspicuity. Theoretically they can be entirely dispensed with.

We are now in a position to define our systems:

T1° = {A1 - A4; **R1** - **R3**};
T1 = {A1 - A5; **R1** - **R3**}.

It is to be noted that the system {A1 - A3; **R1**, **R2**} constitutes a formulation of **CPC**[6].

THEOREM I.1: *The systems* T1° *and* T1 *are absolutely consistent, i.e., not all their wff's are theorems.*

Proof: Consider the matrix:

$\wedge$	1	2	$\sim$	$\Diamond$
1	1	2	2	1
2	2	2	1	2

The designated value is 1. The matrix verifies T1°(T1). But it falsifies 'p'.

§2. THE S-SYSTEMS. The systems S1° and S1 are well-known. We adopt the exposition of [15]. We list the following axioms and rules:

30.11: $(p \wedge q) \prec p$;
30.12: $(p \wedge q) \prec (q \wedge p)$;
30.13: $((p \wedge q) \wedge r) \prec (p \wedge (q \wedge r))$;
30.14: $p \prec (p \wedge p)$;
30.15: $((p \prec q) \wedge (q \prec r)) \prec (p \prec r)$;
36.0: $p \prec \Diamond p$.

30.21: Substitution on propositional variables;
30.22: If $\vdash P$ and if $\vdash Q$, then $\vdash P \wedge Q$;

30.23: If $\vdash P$ and if $\vdash P \prec Q$, then $\vdash Q$;
30.24: Substitutability of strict equivalents.

Our systems are as follows:

$S1^\circ = \{30.11 - 30.15; 30.21 - 30.24\}$
$S1 = \{30.11 - 30.15, 36.0; 30.21 - 30.24\}$

The reader is advised to have [15] in hand. In subsequent chapters we shall need theorems of S1° and S1. We shall prove them only if they are not proved in [15]. Otherwise we shall present them with a number from [15].

A crucial difference between the systems of §2 and those of §1 is that in §1 we postulated the substitutability of material equivalents whereas here we postulate the substitutability of strict equivalents. This fact will be of much importance later.

II. ALGEBRAS AND MATRICES

Very well-known is the notion of closure algebras[1] which may be defined as follows (What follows is an informal introduction. We shall presently make all our notions precise.):

DEFINITION II.1: A structure $\mathfrak{M} = \langle M, \cap, -, \mathrm{P} \rangle$ is a *closure algebra* if and only if M is a set of elements closed under the operations $\cap$, -, and P such that

(i) $\mathfrak{M}$ is a Boolean algebra with respect to $\cap$, -;
(ii) for $x, y \in M$, $\mathrm{P}(x \cup y) = \mathrm{P}x \cup \mathrm{P}y$;
(iii) for $x \in M$, $x \leqq \mathrm{P}x$;
(iv) $\mathrm{P}0 = 0$;
(v) for $x \in M$, $\mathrm{PP}x = \mathrm{P}x$.

Closure algebras have proved to be a very powerful device in the investigation of Lewis system S4[2] and intuitionist logic[3] as well as other branches of mathematics, notably topology[4]. In fact, closure algebra bears the same kind of relation to topological spaces as does Boolean algebra to fields of sets[5]. Closure algebras have undergone a process of generalization. These generalized algebras have an elegance all their own and have the perfect right to be studied for their own sake[6]. However, the chief motivation in their construction has been to get decision procedures for modal logics analogous to that obtained for S4 with the help of closure algebras[7]. The weakest of these generalized algebras are called modal algebras[8]:

DEFINITION II.2: A structure $\mathfrak{M} = \langle M, \cap, -, \mathrm{P} \rangle$ is a *modal algebra* if and only if M is a set of elements closed under operations $\cap$, -, and P such that

(i) $\mathfrak{M}$ is a Boolean algebra with respect to $\cap$, -;
(ii) for $x, y \in M$, $\mathrm{P}(x \cup y) = \mathrm{P}x \cup \mathrm{P}y$.

The connection between the modal system C2 of Lemmon and modal algebras is similar to that between S4 and closure algebras[9]. The various other algebras: extension, deontic, epistemic, normal, normal deontic, normal epistemic etc., have an intermediate position between modal and closure algebras; and they correspond to various intermediate logics between C2 and S4[10]. We shall consider here a further generalization of modal algebras, called weak modal algebra, to be used in our study of T1°. Before we define weak modal algebras we shall make a digression to make the notion of an algebra precise and also introduce the notion of matrices[11].

DEFINITION II.3. An *algebra* is a structure $\mathfrak{M} = \langle M, \cap, -, \mathsf{P}\rangle$ such that $\cap$ is dyadic; $-$, P monadic operations on M class-closing on M.

DEFINITION II.4: $x \cup y = -(-x \cap -y)$.

DEFINITION II.5: $x \rightarrow y = -(x \cap -y)$.

DEFINITION II.6: $x \leftrightarrow y = (x \rightarrow y) \cap (y \rightarrow x)$.

DEFINITION II.7: $\mathsf{N}x = -\mathsf{P}-x$.

DEFINITION II.8: $x \Longrightarrow y = -\mathsf{P}(x \cap -y)$.

DEFINITION II.9: $x \Longleftrightarrow y = (x \Longrightarrow y) \cap (y \Longrightarrow x)$.

DEFINITION II.10: We say $x \leqq y$ if and only if $x \cap y = x$.

DEFINITION II.11: A structure $\mathfrak{M} = \langle M, D, \cap, -, \mathsf{P}\rangle$ is said to be a *matrix* if and only if $\langle M, \cap, -, \mathsf{P}\rangle$ is an algebra and $D \subseteq M$.

DEFINITION II.12: If $\mathfrak{M} = \langle M, D, \cap, -, \mathsf{P}\rangle$ is a matrix then D is said to be the set of *designated* elements of $\mathfrak{M}$.

DEFINITION II.13: A matrix $\mathfrak{M} = \langle M, D, \cap, -, \mathsf{P}\rangle$ is said to be *regular* if and only if

(i) D is a non-empty proper subset of M;
(ii) if $x \in D$ and $(x \rightarrow y) \in D$ and $y \in M$, then $y \in D$;
(iii) if $x \in M$ and $y \in M$ and $(x \leftrightarrow y) \in D$, then $x = y$.

DEFINITION II.14: A matrix $\mathfrak{M} = \langle M, D, \cap, -, \mathsf{P}\rangle$ is said to be σ-*regular* if and only if

(i) D is a non-empty proper subset of M;
(ii) if $x \in D$ and $(x \Longrightarrow y) \in D$ and $y \in M$, then $y \in D$;
(iii) if $x \in D$ and $y \in D$, then $x \cap y \in D$;
(iv) if $x \in M$ and $y \in M$ and $(x \Longleftrightarrow y) \in D$, then $x = y$.

DEFINITION II.15: Let $\mathfrak{M} = \langle M, D, -, \cap, \mathsf{P}\rangle$ be a matrix. We say D is an *additive ideal* of M if and only if

(i) D is a non-empty proper subset of M;
(ii) if $x \in D$ and $y \in D$, then $x \cap y \in D$;
(iii) if $x \in D$ and $y \in M$, then $x \cup y \in D$.

In the three definitions that follow $\|S\|$ denotes the system under consideration.

DEFINITION II.16: A matrix $\mathfrak{M} = \langle M, D, \cap, -, \mathrm{P}\rangle$ is said to *verify* a wff A of $\|S\|$ if and only if every way of evaluating A on the basis of $\mathfrak{M}$, using $\cap, -, \mathrm{P}$ in place of $\wedge, \sim, \Diamond$ leads to an element of D.

DEFINITION II.17: M is called an $\|S\|$-*matrix* if and only if it verifies every provable formula of $\|S\|$.

DEFINITION II.17: By an $\|S\|$-*characteristic matrix* is meant a matrix which verifies every provable formula of $\|S\|$ and which is such, conversely, that every formula which is verified by it is provable in $\|S\|$.

We now introduce our new notion, that of weak modal algebra:

DEFINITION II.19: An algebra $\mathfrak{M} = \langle M, \cap, -, \mathrm{P}\rangle$ is said to be a *weak modal algebra* if and only if

(i) $\mathfrak{M}$ is a Boolean algebra with respect to $\cap, -$;
(ii) for $x,y,z \in M$, $\mathrm{P}(x \cap y) \leqq \mathrm{P}(x \cap z) \cup \mathrm{P}(y \cap -z)$.

THEOREM II.1: *All modal algebras are weak modal.*

Proof. At this point we mention that in this paper we shall constantly use properties of Boolean algebra—without proof. When we do so we shall simply say: "By **BA**". We also adopt all the usual symbolism of Boolean algebras including '0' and '1'.

Let $\mathfrak{M} = \langle M, \cap, -, \mathrm{P}\rangle$ be a modal algebra. Then $\mathfrak{M}$ is a Boolean algebra. Let $x,y,z \in M$. Then, by **BA**,

(1) $(x \cap y) \leqq (x \cap z) \cup (y \cap -z)$

From (1), by **BA** again,

(2) $(x \cap y) \cup ((x \cap z) \cup (y \cap -z)) = (x \cap z) \cup (y \cap -z)$

Hence,

(3) $\mathrm{P}[(x \cap y) \cup ((x \cap z) \cup (y \cap -z))] = \mathrm{P}[(x \cap z) \cup (y \cap -z)]$

By Df. II.2(ii), we have

(4) $\mathrm{P}[(x \cap y) \cup ((x \cap z) \cup (y \cap -z))] = \mathrm{P}(x \cap y) \cup \mathrm{P}((x \cap z) \cup (y \cap -z))$

Then, from (3) and (4),

(5) $\mathrm{P}(x \cap y) \cup \mathrm{P}((x \cap z) \cup (y \cap -z)) = \mathrm{P}((x \cap z) \cup (y \cap -z))$

It follows immediately from (5) by **BA**,

(6) $\mathrm{P}(x \cap y) \leqq \mathrm{P}((x \cap z) \cup (y \cap -z))$

Again by Df. II.2(ii),

(7) $\mathrm{P}((x \cap z) \cup (y \cap -z)) = \mathrm{P}(x \cap z) \cup \mathrm{P}(y \cap -z)$

And, therefore from (6) and (7),

(8) $\mathrm{P}(x \cap y) \leqq \mathrm{P}(x \cap z) \cup \mathrm{P}(y \cap -z)$

This completes the proof. The converse of Theorem II.1. is, however, not true. This is shown by a counterexample. Consider the algebraic system with four distinct elements: x_1, x_2, x_3, x_4. The operations $\cap$, -, P are defined by the following tables[12]:

$\cap$	x_1	x_2	x_3	x_4	-	P
x_1	x_1	x_1	x_1	x_1	x_4	x_3
x_2	x_1	x_2	x_1	x_2	x_3	x_2
x_3	x_1	x_1	x_3	x_3	x_2	x_4
x_4	x_1	x_2	x_3	x_4	x_1	x_4

It can be verified that the above system is a weak modal algebra. But it is not a modal algebra: $P(x_1 \cup x_2) = P(-(-x_1 \cap -x_2)) = P(-(x_4 \cap x_3)) = P(-x_3) = Px_2 = x_2$; but $Px_1 \cup Px_2 = x_3 \cup x_2 = -(-x_3 \cap -x_2) = -(x_2 \cap x_3) = -x_1 = x_4$. It shows that $P(x_1 \cup x_2) \neq Px_1 \cup Px_2$.

Theorem II.1 and the remarks that follow show that a weak modal algebra is strictly weaker than a modal algebra. We have mentioned earlier that there are various algebras between modal and closure algebras i.e., all these algebras are obtained by adding certain axiom (or axioms) to modal algebras. We may now by adding the corresponding axiom (or axioms) to weak modal algebras get weak versions of all these algebras. But we shall not indulge in such a proliferation. We only define:

DEFINITION II.20: An algebra is *weak epistemic* if and only if, in addition to being a weak modal algebra it satisfies the postulate:

(iii) $x \leqq Px$[13].

We shall use weak epistemic algebras in connection with our study of T1. That all weak epistemic algebras are weak modal is immediate. We proceed to show, by counterexample, that the converse is not true. Consider the algebraic system with four distinct elements: x_1, x_2, x_3, x_4. The operations $\cap$, -, P are defined by the following tables:[14]

$\cap$	x_1	x_2	x_3	x_4	-	P
x_1	x_1	x_2	x_3	x_4	x_4	x_1
x_2	x_2	x_2	x_4	x_4	x_3	x_3
x_3	x_3	x_4	x_3	x_4	x_2	x_1
x_4	x_4	x_4	x_4	x_4	x_1	x_3

It can be verified that the above system is a weak modal algebra. However, $Px_2 = x_3$ and $x_2 \cap x_3 = x_4 \neq x_3$ which means that $x_2 \nleqq x_3$, i.e., $x_2 \nleqq Px_2$. It is, therefore, not weak epistemic. Thus weak epistemic algebras are strictly stronger than weak modal algebras.

Before we proceed to prove theorems about matrices, we shall prove a theorem about weak modal algebras which will be useful later.

THEOREM II.2: *Let* $\mathfrak{M} = \langle M, \cap, -, P \rangle$ *be a weak modal algebra. Let* $x \in M$. *Then* $P(x) \leqq P(1) \cup P(0)$.

Proof. Let $x,y,z \in M$. Substituting x for y in Definition II. 19(ii),

(1) $P(x \cap x) \leqq P(x \cap z) \cup P(x \cap -z)$

From (1), by **BA**,

(2) $P(x) \leqq P(x \cap z) \cup P(x \cap -z)$

Substituting z for y and $-z$ for z in Definition II.19(ii),

(3) $P(x \cap z) \leqq P(x \cap -z) \cup P(z \cap --z)$

From (3), by **BA**,

(4) $P(x \cap z) \leqq P(x \cap -z) \cup P(z)$

Due to **BA**, (4) implies

(5) $P(x \cap z) \cup P(x \cap -z) \leqq P(x \cap -z) \cup P(z)$

From (2) and (5) we get, by **BA**,

(6) $P(x) \leqq P(x \cap -z) \cup P(z)$

Substituting 1 for z in (6),

(7) $P(x) \leqq P(x \cap -1) \cup P(1)$

Thus, due to (7) and **BA**, we obtain

(8) $P(x) \leqq P(0) \cup P(1)$

§1. THE T-SYSTEMS.

THEOREM II.3: *There exist characteristic matrices for* T1° *and* T1.

Proof. The following fundamental theorem due to Lindenbaum is well-known: "Let L be a propositional logic, let W_L be the set of its wffs (in terms of connectives $c_1, \ldots, c_n$), and let T_L be the subset of its theorems. Suppose further that T_L is closed under substitution on propositional variables. Then there exists a characteristic matrix $\mathfrak{M}_L$ for L"[15]. Theorem 3 is an immediate consequence thereof. It is to be noted that the elements of $\mathfrak{M}_L$ are the elements of W_L and the designated elements are the members of T_L.

THEOREM II.4: *There exist regular characteristic matrices for* T1° *and* T1.

Proof. Let $\mathfrak{M} = \langle M, D, \cap, -, P\rangle$ be the Lindenbaum matrix of Theorem II.3 for T1°(T1). Define, following McKinsey[16], a relation '$\cong$' on the elements of M as follows: $x \cong y$ if and only if $x \leftrightarrow y \in D$. We show first

(A) '$\cong$' is an equivalence relation.

(1) Since T1°(T1) contains **CPC**, $\vdash_{\overline{T1°(T1)}} p \equiv p$. Since $\mathfrak{M}$ is a characteristic matrix for T1°(T1) we get, for $x \in M$, $x \leftrightarrow x \in D$ i.e., $x \cong x$. So '$\cong$' is reflexive.

(2) Next suppose that for $x,y \in M$, $x \cong y$, i.e., $x \leftrightarrow y \in D$. Hence, since $\mathfrak{M}$ is the Lindenbaum matrix for T1° (T1), we have $\vdash_{\overline{T1°(T1)}} x \equiv y$. Also, by **CPC**, $\vdash_{\overline{T1°(T1)}} x \equiv x$. By the substitutivity of material equivalents, rule **R3**,

$\vdash_{T1°(T1)} y \equiv x$. Hence $y \leftrightarrow x \in D$, i.e., $y \cong x$. So '$\cong$' is symmetric.
(3) Finally, for $x,y,z \in M$, suppose $x \cong y$ and $y \cong z$. Then $x \leftrightarrow y \in D$ and $y \leftrightarrow z \in D$. Hence $\vdash_{T1°(T1)} x \equiv y$ and $\vdash_{T1°(T1)} y \equiv z$. By the substitutivity of material equivalents, $\vdash_{T1°(T1)} x \equiv z$. Hence $x \leftrightarrow z \in D$, i.e., $x \cong z$. Thus '$\cong$' is transitive.

(1), (2) and (3) show that '$\cong$' is an equivalence relation and hence M is partitioned into disjoint classes by this relation. If $x \in M$ we denote the class that contains x by $\mathbf{E}(x)$. We now show that

(B) If $x \in D$ and $y \in \mathbf{E}(x)$, then $y \in D$.

Since $x \in D$, we have $\vdash_{T1°(T1)} x$. Since $y \in \mathbf{E}(x)$, we have $x \cong y$, i.e., $x \leftrightarrow y \in D$. Hence $\vdash_{T1°(T1)} x \equiv y$. Hence, by **CPC**, $\vdash_{T1°(T1)} x \supset y$. By the rule of detachment, $\vdash_{T1°(T1)} y$. Thus $y \in D$.

Now let M_1 be the set of equivalence classes of M. Let D_1 be those elements of M_1 which contain elements of D. We define operations $\cap_1$, $-_1$, and P_1 on M_1 as follows:

$$\mathbf{E}(x) \cap_1 \mathbf{E}(y) = \mathbf{E}(x \cap y)$$
$$-_1\mathbf{E}(x) = \mathbf{E}(-x)$$
$$\mathsf{P}_1(\mathbf{E}(x)) = \mathbf{E}(\mathsf{P}(x))$$

(C) We shall show that the definitions made above are independent of the choice of representatives.

(1) We have to show that if $x_1 \in \mathbf{E}(x)$ and $y_1 \in \mathbf{E}(y)$, then $x_1 \cap y_1 \in \mathbf{E}(x \cap y)$. Since $x_1 \in \mathbf{E}(x)$ we have $x_1 \cong x$, i.e., $x_1 \leftrightarrow x \in D$. It follows $\vdash_{T1°(T1)} x_1 \equiv x$. Similarly, we show that $\vdash_{T1°(T1)} y_1 \equiv y$. Whence, by **CPC**, $\vdash_{T1°(T1)} (x_1 \wedge y_1) \equiv (x \wedge y)$. Hence $(x_1 \cap y_1) \leftrightarrow (x \cap y) \in D$. Hence $(x_1 \cap y_1) \cong (x \cap y)$. Hence $x_1 \cap y_1 \in \mathbf{E}(x \cap y)$.
(2) Next we show that if $x_1 \in \mathbf{E}(x)$ then $-x_1 \in \mathbf{E}(-x)$. Since $x_1 \in \mathbf{E}(x)$ we get $\vdash_{T1°(T1)} x_1 \equiv x$. Hence $\vdash_{T1°(T1)} \sim x_1 \equiv \sim x$. Hence $-x_1 \in \mathbf{E}(-x)$.
(3) Finally, let $x_1 \in \mathbf{E}(x)$. Then $\vdash_{T1°(T1)} x_1 \equiv x$. Also, clearly, $\vdash_{T1°(T1)} \Diamond(x) \equiv \Diamond(x)$. By the substitutivity of material equivalents, rule **R3**, we get $\vdash_{T1°(T1)} \Diamond(x_1) \equiv \Diamond(x)$. Hence $\mathsf{P}(x_1) \in \mathbf{E}(\mathsf{P}(x))$.

This completes the demonstration of (C). We are therefore justified in considering the structure $\mathfrak{M}_1 = \langle M_1, D_1, \cap_1, -_1, \mathsf{P}_1 \rangle$ as a matrix. We now show:

(D) $\mathfrak{M}_1$ is a regular characteristic matrix for T1°(T1).

To show $\mathfrak{M}_1$ is a characteristic matrix for T1°(T1) let $\vdash_{T1°(T1)} A$. Let $p_1, \ldots, p_n$ be the distinct variables in A. Let $\mathbf{E}(x_1), \ldots, \mathbf{E}(x_n)$ be an assignment from $\mathfrak{M}_1$ to $p_1, \ldots, p_n$. Since $\vdash_{T1°(T1)} A$ and $\mathfrak{M}$ is a characteristic matrix for T1°(T1), the assignment $x_1, \ldots, x_n$ to $p_1, \ldots, p_n$ leads to an element of D. Hence it is easy to show by (C) that the assignment $\mathbf{E}(x_1), \ldots, \mathbf{E}(x_n)$ leads to an element $\mathbf{E}(y)$ where $y \in D$. Hence $\mathbf{E}(y) \in D_1$. Hence A is verified by $\mathfrak{M}_1$. Conversely, suppose a wff A is a non-theorem of T1°(T1). Then there exists an assignment $x_1, \ldots, x_n$ of elements of $\mathfrak{M}$

such that if we evaluate A on this basis it does not lead to an element of D. Then if we evaluate A on the basis of $\mathsf{E}(x_1), \ldots, \mathsf{E}(x_n)$ it will lead to an element $\mathsf{E}(y)$ where $y \notin D$. From (B) it follows that every element of $\mathsf{E}(y)$ is an element of $M - D$. Hence $\mathsf{E}(y) \notin D_1$. So A is not verified by $\mathfrak{M}_1$. It follows that $\mathfrak{M}_1$ is a characteristic matrix for T1°(T1).

We next show that the matrix is regular. We verify the three conditions of Definition II.13.

(1) Obviously, at least one theorem can be proved in T1°(T1). So D and hence D_1 is non-empty. Again by Theorem I.1. there is at least one wff which is not a theorem of T1°(T1). So D is a proper subset of M. It follows that D_1 is a proper subset of M_1. Thus D_1 is a non-empty proper subset of M_1.

(2) Now suppose $\mathsf{E}(x) \in D_1$ and $-_1(\mathsf{E}(x) \cap_1 -_1\mathsf{E}(y)) \in D_1$. Hence $\mathsf{E}(-(x \cap -y)) \in D_1$, i.e., $\mathsf{E}(x \rightarrow y) \in D_1$. Then $x \in D$ and $x \rightarrow y \in D$. Hence $\vdash_{\overline{\mathrm{T1°(T1)}}} x$ and $\vdash_{\overline{\mathrm{T1°(T1)}}} x \supset y$. Hence $\vdash_{\overline{\mathrm{T1°(T1)}}} y$. Hence $y \in D$. Thus $\mathsf{E}(y) \in D_1$.

(3) Suppose that $-_1(\mathsf{E}(x) \cap_1 -_1\mathsf{E}(y)) \cap_1 -_1(\mathsf{E}(y) \cap_1 -_1\mathsf{E}(x)) \in D_1$. It follows that $\mathsf{E}(-(x \cap -y) \cap -(y \cap -x)) \in D_1$, i.e., $\mathsf{E}(x \leftrightarrow y) \in D_1$, so that $x \leftrightarrow y \in D$, i.e., $x \cong y$. Now if z is an element of $\mathsf{E}(x)$ we have $z \cong x$ and hence $z \cong y$ and, therefore, $z \in \mathsf{E}(y)$. So $\mathsf{E}(x)$ is a subset of $\mathsf{E}(y)$. In a similar way we see that $\mathsf{E}(y)$ is a subset of $\mathsf{E}(x)$. Hence $\mathsf{E}(x) = \mathsf{E}(y)$.

Hence our characteristic matrix $\mathfrak{M}_1 = \langle M_1, D_1, \cap_1, -_1, \mathsf{P}_1 \rangle$ is regular. This completes the proof.

THEOREM II.5: *There exist regular characteristic matrices for* T1° *and* T1 *such that only one element is designated.*

Proof. Let $\mathfrak{M}_1 = \langle M_1, D_1, \cap_1, -_1, \mathsf{P}_1 \rangle$ be the regular characteristic matrix of Theorem II.4. We know that D_1 consists of those elements of M_1 which contain elements of D. By Theorem II.4(B) it follows that D_1 consists of elements of M_1 which only contain elements of D. Let $x,y \in D$. Then $\vdash_{\overline{\mathrm{T1°(T1)}}} x$ and $\vdash_{\overline{\mathrm{T1°(T1)}}} y$. From the theorem '$p \supset (q \supset p)$' of **CPC** it follows easily that $\vdash_{\overline{\mathrm{T1°(T1)}}} x \equiv y$. So $x \cong y$. From this it follows immediately that D_1 has only one member. This completes the proof.

§2. THE S-SYSTEMS. We now desire theorems for the S-systems analogous to those obtained in §1 for the T-systems (*cf.* Theorems II.3. - II.5). We are able to prove the analogues of Theorem II.3 and Theorem II.4. but not that of Theorem II.5. Since substitutability of material equivalents was a rule of the T-systems we were able to form the matrix of Theorem II.4 by identifying materially equivalent elements in the Lindenbaum matrix of Theorem II.3. But our rule for the S-systems is the substitutability of strict equivalents and so to construct the desirable kind of matrix we shall have to identify strictly equivalent elements in the Lindenbaum matrix. Also in this article we shall construct σ-regular matrices instead of regular matrices. These factors do not permit us to make a straightforward imitation of the proofs of §1. However, the structure of the proofs remain essentially the same. For this reason our proofs of this article will be somewhat abbreviated and we shall make frequent reference to §1.

THEOREM II.6: *There exist characteristic matrices for* S1° *and* S1.

Proof. Completely analogous to the "proof" of Theorem II.3.

THEOREM II.7: *There exist σ-regular characteristic matrices for* S1° *and* S1.

Proof. Let $\mathfrak{M} = \langle M, D, \cap, -, \mathsf{P} \rangle$ be the Lindenbaum matrix of Theorem II.6. (*cf.* Th. II.3.) for S1°(S1). Define a relation '≅' on the elements of M as follows: $x \cong y$ if and only if $x \Leftrightarrow y \in D$. We show that

(A) '≅' is an equivalence relation

To prove (A) we first note that $\vdash_{\overline{S1°(S1)}} p \equiv p$ (*cf.* 31.13 [15]). Next observe that substitutability of strict equivalents is a rule of S1°(S1). Now *cf.* Th. II.4. The proof is clear.

Thus M is partitioned into disjoint classes. If $x \in M$ we denote the class that contains x by $\mathsf{E}(x)$. We now show that

(B) if $x \in D$ and $y \in \mathsf{E}(x)$, then $y \in D$.

Since $x \in D$, we have $\vdash_{\overline{S1°(S1)}} x$. Since $y \in \mathsf{E}(x)$, we have $x \cong y$, i.e., $x \Leftrightarrow y \in D$. Hence $\vdash_{\overline{S1°(S1)}} x \equiv y$. Hence by 31.16 [15], $\vdash_{\overline{S1°(S1)}} x \prec y$. By 30.23[15], $\vdash_{\overline{S1°(S1)}} y$. Hence $y \in D$.

We now define M_1, D_1, $\cap_1$, $-_1$ and P_1 as in Th. II.4.

(C) We shall show that the definitions made above are independent of the choice of representatives.

It is easy to see (*cf.* proof of Th. II.4) that in order to demonstrate (C) we must prove the following rule for our systems S1° and S1.

If $\vdash P \equiv Q$ and if $\vdash R \equiv S$, then $\vdash (P \wedge R) \equiv (Q \wedge S)$.

We prove it as follows:

(1) $P \equiv Q$ by Hp.
(2) $(P \wedge R) \equiv (P \wedge R)$ by 31.12 [15]
(3) $(P \wedge R) \equiv (Q \wedge R)$ by (1), (2), 30.24 [15]
(4) $R \equiv S$ by Hp.
(5) $(P \wedge R) \equiv (Q \wedge S)$ by (3), (4), 30.24 [15]

We now show:

(D) $\mathfrak{M}_1$ is a σ-regular characteristic matrix for S1°(S1)

That $\mathfrak{M}_1$ is a characteristic matrix for S1°(S1) is easily seen (*cf.* Th. II.4). We show that the matrix is σ-regular. We verify the four conditions of Definition II.14.

(i) *cf.* Th. II.4. The proof is exactly similar.
(ii) *cf.* Th. II.4. The proof is analogous: use strict detachment in place of material detachment.
(iii) Suppose $\mathsf{E}(x) \in D_1$ and $\mathsf{E}(y) \in D_1$. Then $x \in D$ and $y \in D$. Hence $\vdash_{\overline{S1°(S1)}} x$ and $\vdash_{\overline{S1°(S1)}} y$. By 30.22 [15], $\vdash_{\overline{S1°(S1)}} x \wedge y$. Hence $x \cap y \in D$. Hence $\mathsf{E}(x \cap y) \in D_1$. Hence, $\mathsf{E}(x) \cap_1 \mathsf{E}(y) \in D_1$.

(iv) *cf*. Th. II.4. The proof is analogous.

Hence our characteristic matrix is σ-regular. This completes the proof.

We conclude this section by remarking that the reason we are unable to prove the analogue of Theorem II.5. for our S-systems is that any two theorems of the T-systems are materially equivalent whereas the corresponding statement about the S-systems, i.e., any two theorems are strictly equivalent is not true. We demonstrate this by an example:

Both '$p \supset p$' and '$\Box(p \supset p)$' are theorems of S1°(S1): *cf.* 34.1 and 34.3 [15]. But $(p \supset p) \equiv \Box(p \supset p)$ is not a theorem:

Consider the matrix[17]:

$\wedge$	1	2	3	4	$\sim$	$\Diamond$
1	1	2	3	4	4	1
2	2	2	4	4	3	2
3	3	4	3	4	2	1
4	4	4	4	4	1	3

The designated values are 1 and 2. The matrix verifies S1°(S1). But $(1 \supset 1) \equiv \Box(1 \supset 1) = \sim (1 \wedge \sim 1) \equiv \sim \Diamond \sim\sim (1 \wedge \sim 1) = 1 \equiv 2 = 4$.

III. FIRST COMPLETENESS THEOREM

We shall now establish certain correlations between our systems and the appropriate kind of algebras (or matrices). The theorem which does this we shall call: First Completeness Theorem. Although the theorems themselves are not entirely devoid of interest, it is more appropriate to consider them as a prelude to the stronger completeness theorems of the next chapter which, finally, will lead to decision procedures for the systems concerned.

§1. THE T-SYSTEMS.

THEOREM III.1. *Let* $\mathfrak{M} = \langle M, D, \cap, -, P \rangle$ *be a regular* T1°(T1)*-matrix. Let* $x, y \in M$. *Then* $x \leqq y$ *if and only if* $x \rightarrow y \in D$.

Proof. In this and subsequent sections we shall have frequent occasion to prove theorems about matrices like Theorem III.1. We shall give detailed analysis of the proof of Theorem III.1 to make the spirit of such proofs clear. After that we shall present the proofs as is usually done in logical calculi, i.e., we shall state the reasons for each step to its right enclosing them with parentheses. We now proceed with our proof.

I. First suppose that:

(1) $x \leqq y$

From (1), by Definition II. 10,

(2) $x \cap y = x$

Now **CPC** gives,

(3) $x \rightarrow x \in D$

From (2) and (3),

(4) $x \rightarrow (x \cap y) \in D$

Again from **CPC**,

(5) $(x \rightarrow (x \cap y)) \rightarrow (x \rightarrow y) \in D$

Thus by Definition II.13(ii), from (4) and (5),

(6) $x \rightarrow y \in D$.

II. Next suppose that:

(7) $x \rightarrow y \in D$

By **CPC**,

(8) $(x \rightarrow y) \rightarrow (x \leftrightarrow (x \cap y)) \in D$

From (7) and (8), by Definition II.13(ii),

(9) $x \leftrightarrow (x \cap y) \in D$

So Definition II.13(iii) gives, from (9),

(10) $x = x \cap y$

By Definition II.10,

(11) $x \leqq y$

This completes the proof.

THEOREM III.2. *Let* $\mathfrak{M} = \langle M, D, \cap, -, P\rangle$ *be a regular* T1°(T1)*-matrix. Then* $\langle M, \cap, -\rangle$ *is a Boolean algebra.*

Proof. For the purpose of demonstrating that $\langle M, \cap, -\rangle$ is a Boolean algebra we shall show that each of the axioms of the following well-known[1] axiom system for Boolean algebras is satisfied:

A structure $\langle M, \cap, -\rangle$ is a Boolean algebra if and only if

P1: M contains at least two elements;
P2: if $x,y \in M$, then $x \cap y = y \cap x$;
P3: if $x,y,z \in M$, then $(x \cap y) \cap z = x \cap (y \cap z)$;
P4: if $x,y,z \in M$, then $(x \cap -y) = (z \cap -z)$ if and only if $x \cap y = x$.

P1: By Definition II. 13(i), D is a non-empty proper subset of M. So M contains at least two elements.

P2: Let $x,y,z \in M$.

Then,

(1) $(x \cap y) \leftrightarrow (y \cap x) \in D$ [**CPC**]
(2) $x \cap y = y \cap x$ [(1);Df. II.13(iii)]

P3: Let $x, y, z \in M$.

Then,

(3) $(x \cap y) \cap z \leftrightarrow x \cap (y \cap z) \in D$ [CPC]
(4) $(x \cap y) \cap z = x \cap (y \cap z)$ [(3);Df.II.13(iii)]

P4: Let $x,y,z \in M$.

I. First suppose that $(x \cap -y) = (z \cap -z)$. Then,

(5) $(x \cap -y) = (z \cap -z)$ [Hp.]
(6) $(x \cap -y) \rightarrow (x \cap -y) \in D$ [CPC]
(7) $(x \cap -y) \rightarrow (z \cap -z) \in D$ [(5),(6)]
(8) $\{(x \cap -y) \rightarrow (z \cap -z)\} \rightarrow \{x \leftrightarrow (x \cap y)\} \in D$ [CPC]
(9) $x \leftrightarrow x \cap y \in D$ [(7),(8);Df.II.13(ii)]
(10) $x = x \cap y$ [(9);Df.II.13(iii)]

II. Next suppose that $x = x \cap y$. Then,

(11) $x = x \cap y$ [Hp.]
(12) $x \rightarrow x \in D$ [CPC]
(13) $x \rightarrow (x \cap y) \in D$ [(11),(12)]
(14) $(x \rightarrow (x \cap y)) \rightarrow ((x \cap -y) \leftrightarrow (z \cap -z)) \in D$ [CPC]
(15) $(x \cap -y) \leftrightarrow (z \cap -z) \in D$ [(13),(14);Df.II.13(ii)]
(16) $x \cap -y = z \cap -z$ [(15);Df.II.13(iii)]

This completes the proof of Theorem III.2.

THEOREM III.3: $\mathfrak{M} = \langle M, \{d\}, \cap, -, \mathrm{P}\rangle$ *is a regular* T1°-*matrix if and only if* $\langle M, \cap, -, \mathrm{P}\rangle$ *is a weak modal algebra and* $d = 1$.

Proof. Let $\mathfrak{M} = \langle M, \{d\}, \cap, -, \mathrm{P}\rangle$ be a regular T1°-matrix. By Theorem III.2, $\langle M, \cap, -\rangle$ is a Boolean algebra. It remains to show that condition (ii) of Definition II.19 is satisfied. Substituting $\sim q$ for r and $\sim r$ for q in A4 (section I) (*cf.* rule **R1**), we get,

(1) $\vdash_{\mathrm{T1}^\circ} (\Box(p \supset \sim r) \wedge \Box(\sim r \supset \sim q)) \supset (\Box(p \supset \sim q))$

From (1), by Definition I.2 and Definition I.4,

(2) $\vdash_{\mathrm{T1}^\circ} (\sim \Diamond \sim (\sim (p \wedge \sim\sim r)) \wedge \sim \Diamond \sim (\sim(\sim r \wedge \sim\sim q))) \supset (\sim \Diamond \sim (\sim (p \wedge \sim\sim q)))$

By **CPC**, (2) and **R3**,

(3) $\vdash_{\mathrm{T1}^\circ} (\sim\Diamond(p \wedge r) \wedge \sim\Diamond(q \wedge \sim r)) \supset (\sim\Diamond(p \wedge q))$

It follows from (3) and **CPC**,

(4) $\vdash_{\mathrm{T1}^\circ} \Diamond(p \wedge q) \supset (\Diamond(p \wedge r) \vee \Diamond(q \wedge \sim r))$

Hence, since $\mathfrak{M}$ is a T1°-matrix, (4) shows that for $x,y,z \in M$,

(5) $\mathrm{P}(x \cap y) \rightarrow \{\mathrm{P}(x \cap z) \cup \mathrm{P}(y \cap -z)\} \in \{d\}$

So, by Theorem III.1, we obtain from (5):

(6) $\mathrm{P}(x \cap y) \leqq \mathrm{P}(x \cap z) \cup \mathrm{P}(y \cap -z)$

Thus $<M, \cap, -, \mathrm{P}>$ is a weak modal algebra. Next observe that $\vdash_{T1°} p \vee \sim p$. Hence if $x \in M$, $x \cup -x \in \{d\}$, i.e., $x \cup -x = d$. But in a Boolean algebra $x \cup -x = 1$. Hence $d = 1$.

Conversely, let $\mathfrak{M} = <M, \cap, -, \mathrm{P}>$ be a weak modal algebra. Consider the structure $<M, \{1\}, \cap, -, \mathrm{P}>$. That $<M, \{1\}, \cap, -, \mathrm{P}>$ is a matrix is immediate. We observe that $\mathfrak{M}$ being a weak modal algebra, $<M, \cap, ->$ is a Boolean algebra. We first show that the matrix is regular by verifying the three conditions of Definition II.13.

(i) $<M, \cap, ->$ being a Boolean algebra, M contains at least two elements. Hence clearly $\{1\}$ is a non-empty proper subset of M.
(ii) Let $x,y \in M$. Let $x = 1$ and $x \rightarrow y = 1$. By Definition II.5, $-(x \cap -y) = 1$. Hence $x \cap -y = 0$. It follows $x \leqq y$. But $x = 1$. Therefore $y = 1$.
(iii) Finally suppose that $x,y \in M$ and $x \leftrightarrow y = 1$. By Definition II.6, $(x \rightarrow y) \cap (y \rightarrow x) = 1$. Hence $x \rightarrow y = 1$ and $y \rightarrow x = 1$. By Definition II.5, $-(x \cap -y) = 1$ and $-(y \cap -x) = 1$. Hence $x \cap -y = 0$ and $y \cap -x = 0$. Therefore $x \leqq y$ and $y \leqq x$. Thus $x = y$.

Thus our matrix is regular.

We next show that our matrix verifies the axioms of T1°. The axioms A1, A2, A3 written in primitive notation are:

A1: $\sim (p \wedge \sim (p \wedge p))$
A2: $\sim ((p \wedge q) \wedge \sim p)$
A3: $\sim (\sim (p \wedge \sim q) \wedge \sim\sim (\sim (q \wedge r) \wedge \sim\sim (r \wedge p)))$

Let $x,y,z \in M$. Then,

A1: $-(x \cap -(x \cap x)) = -(x \cap -x) = -0 = 1$
A2: $-((x \cap y) \cap -x) = -((x \cap -x) \cap y) = -(0 \cap y) = -0 = 1$
A3: $-\{-(x \cap -y) \cap --(-(y \cap z) \cap --(z \cap x))\}$
$= -\{ -(x \cap -y) \cap (-(y \cap z) \cap (z \cap x))\}$
$= -\{(-x \cup y) \cap (-y \cup -z) \cap (z \cap x)\}$
$= -\{(-x \cup y) \cap ((-y \cap z \cap x) \cup (-z \cap z \cap x))\}$
$= -\{(-x \cup y) \cap (-y \cap z \cap x)\}$
$= -\{(-x \cap -y \cap z \cap x) \cup (y \cap -y \cap z \cap x)\}$
$= -0$
$= 1$

Thus A1, A2, A3 are verified. Before proceeding to show A4 is verified we make the following observation about weak modal algebras. Let $x,y,z \in M$. Then we have,

(7) $\mathrm{P}(x \cap y) \leqq \mathrm{P}(x \cap z) \cup \mathrm{P}(y \cap -z)$

Substituting $-z$ for y and $-y$ for z in (7),

(8) $\mathrm{P}(x \cap -z) \leqq \mathrm{P}(x \cap -y) \cup \mathrm{P}(-z \cap --y)$

It follows immediately from (8),

(9) $-\{\mathrm{P}(x \cap -y) \cup \mathrm{P}(y \cap -z)\} \leqq -\mathrm{P}(x \cap -z)$

whence,

(10) $-\mathrm{P}(x \cap -y) \cap -\mathrm{P}(y \cap -z) \leqq -\mathrm{P}(x \cap -z)$

Hence,

(11) $-\mathrm{P}(x \cap -y) \cap -\mathrm{P}(y \cap -z) \cap --\mathrm{P}(x \cap -z) = 0$

And, therefore,

(12) $-\mathrm{P}(x \cap -y) \cap -\mathrm{P}(y \cap -z) \cap \mathrm{P}(x \cap -z) = 0$

Now we write A4 in primitive notation:

A4: $-[\{-\mathrm{P}--(x \cap -y) \cap -\mathrm{P}--(y \cap -z)\} \cap --\mathrm{P}--(x \cap -z)]$
$= -[-\mathrm{P}(x \cap -y) \cap -\mathrm{P}(y \cap -z) \cap \mathrm{P}(x \cap -z)]$
$= -0$ (*cf.* (12))
$= 1$

Thus $<M, \{1\}, \cap, -, \mathrm{P}>$ is a matrix that verifies the axioms of T1° and, in addition, is regular. Note that the conditions of regularity: Definition II.13(i), (ii), (iii) correspond respectively to the consistency of T1°, the rule **R2** and the rule **R3**. Note also that the rule **R1** corresponds to the rule of substitution in algebras of which we have made no explicit mention. Consequently our matrix verifies all the provable formulas of T1°, i.e., it is a T1°-matrix. This completes the proof.

We have introduced the concept of verifiability in Section II in connection with matrices (cf. Df.II.16). We shall now extend it to algebras that are at least Boolean, i.e., the algebra has one binary and one unary operation: $\cap$ and -, it satisfies the axioms for Boolean algebras; it may also have other operations and other axioms.

DEFINITION III.1. An algebra (that is at least Boolean) is said to verify a wff A of a system $||S||$ if and only if the matrix which can be constructed from the algebra by taking 1 as the sole designated element verifies A.

THEOREM III.4. $\mathfrak{M} = <M, \{d\}, \cap, -, \mathrm{P}>$ *is a regular* T1-*matrix if and only if* $<M, \cap, -, \mathrm{P}>$ *is a weak epistemic algebra and* $d = 1$.

Proof. Let $\mathfrak{M} = <M, \{d\}, \cap, -, \mathrm{P}>$ be a regular T1-matrix. *A fortiori*, it is a regular T1°-matrix. By Theorem III. 3, $<M, \cap, -, \mathrm{P}>$ is a weak modal algebra and $d = 1$. It remains to show that $x \leqq \mathrm{P}x$ (*cf.* Df. II.20). We have, by A5, $\vdash_{\mathrm{T1}} \Box p \supset p$, i.e., $\vdash_{\mathrm{T1}} \sim\Diamond\sim p \supset p$. It is easy to deduce $\vdash_{\mathrm{T1}} p \supset \Diamond p$. Hence for $x \in M$, $x \rightarrow \mathrm{P}x \in \{d\}$, i.e., $x \rightarrow \mathrm{P}x = 1$. By Theorem III.1, $x \leqq \mathrm{P}x$. Conversely suppose $\mathfrak{M} = <M, \cap, -, \mathrm{P}>$ is a weak epistemic algebra. As in Theorem III.3, $<M, \{1\}, \cap, -, \mathrm{P}>$ is a regular T1°-matrix. It remains to show that it verifies A5. We observe: $\mathrm{N}x \rightarrow x = -\mathrm{P}-x \rightarrow x = -(-\mathrm{P}-x \cap -x) = \mathrm{P}-x \cup x \geqq -x \cup x$ (Since $-x \leqq \mathrm{P}-x$). So $\mathrm{N}x \rightarrow x \geqq 1$. Hence $\mathrm{N}x \rightarrow x = 1$. This completes the proof.

We are now in a position to state and prove our First Completeness Theorem.

THEOREM III.5. (*First Completeness Theorem*). $\vdash_{\mathrm{T1}^\circ(\mathrm{T1})} A$ if and only if A *is verified by all weak modal (weak epistemic) algebras.*

Proof. First suppose that $\vdash_{\mathrm{T1}^\circ} A$. Let $\mathfrak{M} = \langle M, \cap, -, \mathrm{P}\rangle$ be a weak modal algebra. By Theorem III.3 $\langle M, \{1\}, \cap, -, \mathrm{P}\rangle$ is a regular T1°-matrix, i.e., A is verified by $\mathfrak{M}$. Conversely, let A be any non-theorem of T1°. It is therefore not verified by the regular characteristic matrix of Theorem II.5 which we recall has only one designated element. By Theorem III.3, the regular characteristic matrix is a weak modal algebra and the designated element is 1. So A is not verified by a certain weak modal algebra. Consequently, if A is verified by all weak modal algebras, then $\vdash_{\mathrm{T1}^\circ} A$.

The case of T1 similarly uses Theorem II.5 and Theorem III.4.

§2. THE S-SYSTEMS. In this article we shall prove completeness theorems for the S-systems similar to those obtained for the T-systems in the previous article. However, we shall not be able to dispense with the notion of matrices—more appropriately, matrices with more than one designated element—and reduce our discussion to algebras as we have done for the T-systems. As will be revealed in the proofs of theorems that follow, this stems from the fact that we could not prove the existence of a characteristic matrix for the S-systems with only one designated element. In proofs that follow we shall make frequent use of theorems of [15]. To avoid repetition, we shall not refer to [15] when we use a theorem from [15]: we shall simply state the number of the theorem as in [15].

THEOREM III.6. *Let* $\mathfrak{M} = \langle M, D, \cap, -, \mathrm{P}\rangle$ *be a* σ*-regular* S1°(S1)*-matrix. Then* $x \leqq y$ *if and only if* $x \Longrightarrow y \in D$.

Proof. First suppose that $x \leqq y$. Then,

(1) $x \leqq y$ [Hp.]
(2) $x \cap y = x$ [(1);Df.II.10]
(3) $x \Longrightarrow x \in D$ [31.11]
(4) $x \Longrightarrow (x \cap y) \in D$ [(2),(3)]
(5) $(x \cap y) \Longrightarrow y \in D$ [31.23]
(6) $[(x \Longrightarrow (x \cap y)) \cap ((x \cap y) \Longrightarrow y)] \in D$ [(4),(5);Df.II.14(iii)]
(7) $[(x \Longrightarrow (x \cap y)) \cap ((x \cap y) \Longrightarrow y)] \Longrightarrow (x \Longrightarrow y) \in D$ [30.15]
(8) $x \Longrightarrow y \in D$ [(6),(7);Df.II.14(ii)]

Conversely suppose that $x \Longrightarrow y \in D$. Then

(9) $x \Longrightarrow y \in D$ [Hp.]
(10) $x \Longrightarrow x \in D$ [31.11]
(11) $(x \Longrightarrow x) \cap (x \Longrightarrow y) \in D$ [(9),(10);Df.II.14(iii)]
(12) $(x \cap y) \Longrightarrow (x \cap y) \in D$ [31.11]
(13) $[((x \cap y) \Longrightarrow (x \cap y)) \cap ((x \Longrightarrow x) \cap (x \Longrightarrow y))] \in D$
[(11),(12);Df.II.14(iii)]
(14) $[((x \cap y) \Longrightarrow (x \cap y)) \cap (x \Longrightarrow x) \cap (x \Longrightarrow y)] \Longrightarrow (x \Longrightarrow (x \cap y)) \in D$
[35.22]
(15) $x \Longrightarrow (x \cap y) \in D$ [(13),(14);Df.II.14(ii)]

(16) $(x \cap y) \Rightarrow x \in D$ [30.11]
(17) $(x \Rightarrow (x \cap y)) \cap ((x \cap y) \Rightarrow x) \in D$ [(15),(16);Df.II.14(iii)]
(18) $x \Leftrightarrow (x \cap y) \in D$ [17 ;Df.II.9]
(19) $x = x \cap y$ [(18);Df.II.14(iv)]
(20) $x \leqq y$ [(19);Df.II.10]

THEOREM III.7. *Let* $\mathfrak{M} = \langle M, D, \cap, -, \mathrm{P} \rangle$ *be a* σ*-regular* S1°(S1)*-matrix. Then* $\langle M, \cap, - \rangle$ *is a Boolean algebra.*

Proof. Let $x,y,z \in M$. We have stated the axioms for Boolean algebra in §1. We shall verify them below.

P1: By Definition II.14(i), D is a non-empty proper subset of M. So M contains at least two elements.

P2:

(1) $(x \cap y) \Leftrightarrow (y \cap x) \in D$ [31.21]
(2) $x \cap y = y \cap x$ [(1);Df.II.14(iv)]

P3:

(3) $(x \cap (y \cap z)) \Leftrightarrow ((x \cap y) \cap z) \in D$ [31.24]
(4) $x \cap (y \cap z) = (x \cap y) \cap z$ [(3);Df.II.14(iv)]

P4: Suppose first that $x \cap -y = z \cap -z$. Then,

(5) $x \cap -y = z \cap -z$ [Hp.]
(6) $(x \cap -y) \Rightarrow (x \cap -y) \in D$ [31.11]
(7) $(x \cap -y) \Rightarrow (z \cap -z) \in D$ [(5),(6)]
(8) $[(x \cap -y) \Rightarrow (z \cap -z)] \Leftrightarrow [-(z \cap -z) \Rightarrow -(x \cap -y)] \in D$ [31.34]
(9) $(x \cap -y) \Rightarrow (z \cap -z) = -(z \cap -z) \Rightarrow -(x \cap -y)$ [(8);Df.II.14(iv)]
(10) $-(z \cap -z) \Rightarrow -(x \cap -y) \in D$ [(7),(9)]
(11) $\mathrm{N}[-(z \cap -z)] \in D$ [34.1]
(12) $[-(z \cap -z) \Rightarrow -(x \cap -y)] \cap \mathrm{N}[-(z \cap -z)] \in D$ [(10),(11);Df.II.14(iii)]
(13) $\{[-(z \cap -z) \Rightarrow -(x \cap -y)] \cap \mathrm{N}[-(z \cap -z)]\} \Rightarrow \mathrm{N}[-(x \cap -y)] \in D$ [33.31]
(14) $\mathrm{N}[-(x \cap -y)] \in D$ [(12),(13);Df.II.14(ii)]
(15) $-\mathrm{P}-[-(x \cap -y)] \in D$ [(14);Df.II.7]
(16) $--(x \cap -y) \Leftrightarrow (x \cap -y) \in D$ [31.32]
(17) $--(x \cap -y) = (x \cap -y)$ [(16);Df.II.14(iv)]
(18) $-\mathrm{P}(x \cap -y) \in D$ [(15),(17)]
(19) $x \Rightarrow y \in D$ [(18);Df.II.8]
(20) $(y \cap x) \Rightarrow (y \cap x) \in D$ [31.11]
(21) $[\{(y \cap x) \Rightarrow (y \cap x)\} \cap (x \Rightarrow y)] \in D$ [(19),(20);Df.II.14(iii)]
(22) $[\{(y \cap x) \Rightarrow (y \cap x)\} \cap (x \Rightarrow y)] \Rightarrow [(x \cap x) \Rightarrow (x \cap y)] \in D$ [35.21]
(23) $(x \cap x) \Rightarrow (x \cap y) \in D$ [(21),(22);Df.II.14(ii)]
(24) $x \Leftrightarrow (x \cap x) \in D$ [31.22]
(25) $x = x \cap x$ [(24);Df.II.14(iv)]
(26) $x \Rightarrow (x \cap y) \in D$ [(23),(25)]
(27) $(x \cap y) \Rightarrow x \in D$ [30.11]
(28) $(x \Rightarrow (x \cap y)) \cap ((x \cap y) \Rightarrow x) \in D$ [(26),(27);Df.II.14(iii)]

(29) $x \Leftrightarrow (x \cap y) \in D$ [(28);Df.II.9]
(30) $x = x \cap y$ [(29);Df.II.14(iv)]

So we have shown that if $(x \cap -y) = (z \cap -z)$, then $x \cap y = x$. Conversely suppose that $x \cap y = x$. Then

(31) $x \cap y = x$ [Hp.]
(32) $x \Rightarrow x \in D$ [31.11]
(33) $x \Rightarrow (x \cap y) \in D$ [(31),(32)]
(34) $(x \cap y) \Rightarrow y \in D$ [31.23]
(35) $(x \Rightarrow (x \cap y)) \cap ((x \cap y) \Rightarrow y) \in D$ [(33),(34);Df.II.14(iii)]
(36) $((x \Rightarrow (x \cap y)) \cap ((x \cap y) \Rightarrow y)) \Rightarrow (x \Rightarrow y) \in D$ [30.15]
(37) $x \Rightarrow y \in D$ [(35),(36);Df.II.14(ii)]
(38) $(x \cap -(z \cap -z)) \Rightarrow x \in D$ [30.11]
(39) $((x \cap -(z \cap -z)) \Rightarrow x) \cap (x \Rightarrow y) \in D$ [(37),(38);Df.II.14(iii)]
(40) $[((x \cap -(z \cap -z)) \Rightarrow x) \cap (x \Rightarrow y)] \Rightarrow [(x \cap -(z \cap -z)) \Rightarrow y] \in D$
[30.15]
(41) $(x \cap -(z \cap -z)) \Rightarrow y \in D$ [(39),(40);Df.II.14(ii)]
(42) $--y \Leftrightarrow y \in D$ [31.32]
(43) $--y = y$ [(42);Df.II.14(iv)]
(44) $(x \cap -(z \cap -z)) \Rightarrow --y \in D$ [(41),(43)]
(45) $[(x \cap -y) \Rightarrow (z \cap -z)] \Leftrightarrow [(x \cap -(z \cap -z)) \Rightarrow --y] \in D$ [32.11]
(46) $(x \cap -y) \Rightarrow (z \cap -z) = (x \cap -(z \cap -z)) \Rightarrow --y$ [(45);Df.II.14(iv)]
(47) $(x \cap -y) \Rightarrow (z \cap -z) \in D$ [(44),(46)]
(48) $(z \cap -(x \cap -y)) \Rightarrow z \in D$ [30.11]
(49) $((z \cap -(x \cap -y)) \Rightarrow z) \Leftrightarrow ((z \cap -z) \Rightarrow --(x \cap -y)) \in D$ [32.11]
(50) $(z \cap -(x \cap -y)) \Rightarrow z = (z \cap -z) \Rightarrow --(x \cap -y)$ [(49);Df.II.14(iv)]
(51) $(z \cap -z) \Rightarrow --(x \cap -y) \in D$ [(48),(50)]
(52) $--(x \cap -y) \Leftrightarrow (x \cap -y) \in D$ [31.32]
(53) $--(x \cap -y) = (x \cap -y)$ [(52);Df.II.14(iv)]
(54) $(z \cap -z) \Rightarrow (x \cap -y) \in D$ [(51),(53)]
(55) $[(x \cap -y) \Rightarrow (z \cap -z)] \cap [(z \cap -z) \Rightarrow (x \cap -y)] \in D$
[(47),(54);Df.II.14(iii)]
(56) $(x \cap -y) \Leftrightarrow (z \cap -z) \in D$ [(55);Df.II.9]
(57) $x \cap -y = z \cap -z$ [(56);Df.II.14(iv)]

Thus we have also shown that if $x \cap y = x$, then $x \cap -y = z \cap -z$. This completes the proof.

In subsequent proofs we shall make constant use of the fact that a σ-regular S1°(S1)-matrix is a Boolean algebra.

THEOREM III.8. *Let* $\mathfrak{M} = \langle M, D, \cap, -, \mathrm{P} \rangle$ *be a* σ*-regular* S1°(S1)*-matrix. Let* $x, y, z \in M$. *Then,*

(A) $-\mathrm{P}(0) \in D$;
(B) *If* $-\mathrm{P}(x) \in D$, *then* $x = 0$;
(C) *If* $x \in D$ *and* $x \leqq y$, *then* $y \in D$;
(D) $0 \notin D$;
(E) $1 \in D$;
(F) $\mathrm{P}(x \cap y) \leqq \mathrm{P}(x \cap z) \cup \mathrm{P}(y \cap -z)$.

Proof. A:

(1) $0 \Longrightarrow 0 \in D$ [31.11]
(2) $-\mathrm{P}(0 \cap -0) \in D$ [(1);Df.II.8]
(3) $-\mathrm{P}(0) \in D$ [(2);**BA**]

B: Suppose that $-\mathrm{P}(x) \in D$. Then,

(4) $-\mathrm{P}(x) \in D$ [Hp.]
(5) $-\mathrm{P}(x \cap -0) \in D$ [(4);**BA**]
(6) $x \Longrightarrow 0 \in D$ [(5);Df.II.8]
(7) $x \leqq 0$ [(6);Th.III.6]
(8) $x = 0$ [(7);**BA**]

C: Suppose that $x \in D$ and $x \leqq y$. Then,

(9) $x \in D$ [Hp.]
(10) $x \leqq y$ [Hp.]
(11) $x \Longrightarrow y \in D$ [(10);Th.III.6]
(12) $y \in D$ [(9),(11);Df.II.14(ii)]

D: Let us assume that

(13) $0 \in D$.

Then, if $x \in M$,

(14) $0 \leqq x$ [**BA**]
(15) $x \in D$ [(13),(14);Th.III.8(C)]

Thus $M \subseteq D$. But by Definition II. 11, $D \subseteq M$. Hence $D = M$. This violates Definition II. 14(i) which says that D is a proper subset of M. So our assumption (13) is false. Therefore,

(16) $0 \notin D$.

E:

(17) $0 \cup -0 \in D$ [34.3]
(18) $1 \in D$ [(17);**BA**]

F:

(19) $[(x \Longrightarrow -z) \cap (-z \Longrightarrow -y)] \Longrightarrow (x \Longrightarrow -y) \in D$ [30.15]
(20) $[(x \Longrightarrow -z) \cap (-z \Longrightarrow -y)] \leqq (x \Longrightarrow -y)$ [(19);Th.III.6]
(21) $-(x \Longrightarrow -y) \leqq -[(x \Longrightarrow -z) \cap (-z \Longrightarrow -y)]$ [(20);**BA**]
(22) $--\mathrm{P}(x \cap --y) \leqq -[-\mathrm{P}(x \cap --z) \cap -\mathrm{P}(-z \cap --y)]$ [(21);Df.II.8]
(23) $\mathrm{P}(x \cap y) \leqq \mathrm{P}(x \cap z) \cup \mathrm{P}(y \cap -z)$ [(22);**BA**]

This completes the proof of Theorem III.8.

THEOREM III.9. $\mathfrak{M} = \langle M, D, \cap, -, \mathrm{P} \rangle$ *is a* σ*-regular* S1°*-matrix if and only if*

(A) $\langle M, \cap, -, \mathrm{P} \rangle$ *is a weak modal algebra*;
(B) D *is an additive ideal of* M;
(C) $x = 0$ *if and only if* $-\mathrm{P}(x) \in D$.

Proof.

I. First suppose that $\mathfrak{M} = \langle M, D, \cap, -, \mathrm{P}\rangle$ is a σ-regular S1°-matrix. Then,

(A): By Theorem III.6 it is a Boolean algebra. Hence condition (i) of Definition II.19 is satisfied. By Theorem III.8(F) condition (ii) of Definition II.19 is satisfied. Thus $\langle M, \cap, -, \mathrm{P}\rangle$ is a weak modal algebra.

(B): We verify the three conditions of Definition II.15. Let $x, y \in M$.

(i) By Definition II.14(i), D is a non-empty proper subset of M.
(ii) By Definition II.14(iii), if $x \in D$ and $y \in D$, then $x \cap y \in D$.
(iii) Let

(1) $x \in D$.

Then,

(2) $x \leqq x \cup y$ [**BA**]
(3) $x \cup y \in D$ [(1),(2);Th.III.8(C)]

(C): This follows immediately from Theorem III. 8(A) and Theorem III. 8(B).

II. Conversely suppose $\mathfrak{M} = \langle M, D, \cap, -, \mathrm{P}\rangle$ is a matrix satisfying conditions (A), (B) and (C).

We first show that $\mathfrak{M}$ is σ-regular (cf. Df. II.14).
(i) By Definition II. 15(i), D is a non-empty proper subset of M.

(ii) Let $x, y \in M$. Suppose

(4) $x \in D$,

and

(5) $x \Rightarrow y \in D$.

Then,

(6) $-\mathrm{P}(x \cap -y) \in D$ [(5);Df.II.8]
(7) $x \cap -y = 0$ [(6);(C)]
(8) $x \leqq y$ [(7);**BA**]
(9) $y = x \cup y$ [(8);**BA**]
(10) $x \cup y \in D$ [(4);Df.II.15(iii);$y \in M$]
(11) $y \in D$ [(9),(10)]

(iii) This is immediate from Definition II.15.
(iv) Let $x, y \in M$.

Suppose

(12) $x \Leftrightarrow y \in D$.

Then

(13) $(x \Rightarrow y) \cap (y \Rightarrow x) \in D$ [(12);Df.II.9]
(14) $\{(x \Rightarrow y) \cap (y \Rightarrow x)\} \cup (x \Rightarrow y) \in D$ [(13);Df.II.15(iii);$x \Rightarrow y \in M$]

(15) $\{(x \Rightarrow y) \cap (y \Rightarrow x)\} \cup (x \Rightarrow y) = x \Rightarrow y$ [**BA**]
(16) $x \Rightarrow y \in D$ [(14),(15)]
(17) $-\mathrm{P}(x \cap -y) \in D$ [(16);Df.II.8]
(18) $x \cap -y = 0$ [(17);(C)]
(19) $x \leqq y$ [(18);**BA**]
(20) $\{(x \Rightarrow y) \cap (y \Rightarrow x)\} \cup (y \Rightarrow x) = y \Rightarrow x$ [**BA**]
(21) $y \Rightarrow x \in D$ [(14),(20)]
(22) $-\mathrm{P}(y \cap -x) \in D$ [(21);Df.II.8]
(23) $y \cap -x = 0$ [(22);(C)]
(24) $y \leqq x$ [(23);**BA**]
(25) $x = y$ [(19),(24);**BA**]

This completes the demonstration of the σ-regularity of the matrix. To prove that it is an S1°-matrix, it now suffices to show that the axioms of S1° are verified. To see that 30.11 is verified, we note that:

$(x \cap y) \Rightarrow x$
$= -\mathrm{P}((x \cap y) \cap -x)$ [Df.II.8]
$= -\mathrm{P}(0 \cap y)$ [**BA**]
$= -\mathrm{P}(0) \in D$ [by (C)]

It can be similarly seen that 30.12, 30.13, 30.14 are verified. To see 30.15 is verified, we obtain from (A) (*cf.* Df.II.19(ii)),

(26) $\mathrm{P}(x \cap -z) \leqq \mathrm{P}(x \cap -y) \cup \mathrm{P}(-z \cap --y)$
(27) $-\mathrm{P}(x \cap -y) \cap -\mathrm{P}(y \cap -z) \leqq -\mathrm{P}(x \cap -z)$ [(26);**BA**]
(28) $(x \Rightarrow y) \cap (y \Rightarrow z) \leqq (x \Rightarrow z)$ [(27);Df.II.8]
(29) $\{(x \Rightarrow y) \cap (y \Rightarrow z)\} \cap -(x \Rightarrow z) = 0$ [(28);**BA**]
(30) $-\mathrm{P}[\{(x \Rightarrow y) \cap (y \Rightarrow z)\} \cap -(x \Rightarrow z)] \in D$ [(29);(C)]
(31) $\{(x \Rightarrow y) \cap (y \Rightarrow z)\} \Rightarrow (x \Rightarrow z) \in D$ [(30);Df.II.8]

This completes the proof of Theorem III.9.

THEOREM III.10. *$\mathfrak{M} = \langle M, D, \cap, -, \mathrm{P}\rangle$ is a σ-regular* S1*-matrix if and only if*

(A) *$\langle M, \cap, -, \mathrm{P}\rangle$ is a weak epistemic algebra*;
(B) *D is an additive ideal of M*;
(C) *$x = 0$ if and only if $-\mathrm{P}(x) \in D$.*

Proof. First suppose that $\mathfrak{M}$ is a σ-regular S1-matrix. *A fortiori*, by Definition II.17, it is a σ-regular S1°-matrix. By Theorem III.9, conditions (B) and (C) are satisfied. To show condition (A) is satisfied, it remains to show, by Theorem III.9, that $x \leqq \mathrm{P}x$. By 36.0, $x \Rightarrow \mathrm{P}x \in D$. Hence, by Theorem III.6, $x \leqq \mathrm{P}x$.

Conversely suppose $\mathfrak{M} = \langle M, D, \cap, -, \mathrm{P}\rangle$ satisfies conditions (A), (B), (C). Since all weak epistemic algebras are weak modal, *a fortiori*, $\mathfrak{M}$ satisfies conditions (A), (B), (C) of Theorem III.9. Hence, by Theorem III.9, $\mathfrak{M}$ is a σ-regular S1°-matrix. It remains to show that the matrix verifies 36.0. By condition (A) (*cf.* Df.II.20) $x \leqq \mathrm{P}x$. By Theorem III.6, $x \Rightarrow \mathrm{P}x \in D$.

THEOREM III.11. (*First Completeness Theorem*). $\vdash_{\overline{S1^\circ(S1)}} A$ *if and only if A is verified by all matrices* $\mathfrak{M} = \langle M, D, \cap, -, \mathrm{P}\rangle$ *such that*

(i) $\langle M, \cap, -, \mathrm{P}\rangle$ *is a weak modal* (*weak epistemic*) *algebra*;
(ii) *D is an additive ideal of M*;
(iii) $x = 0$ *if and only if* $-\mathrm{P}(x) \in D$.

Proof. First suppose that $\vdash_{\overline{S1^\circ}} A$. Let $\mathfrak{M}$ satisfy the three conditions enumerated above, By Theorem III.9, $\mathfrak{M}$ is a σ-regular S1°-matrix, i.e., A is verified by $\mathfrak{M}$. Conversely, let A be a non-theorem of S1°. It is therefore not verified by the σ-regular characteristic matrix of Theorem II.7. By Theorem III.9, the characteristic matrix satisfies the three conditions of our theorem. So A is not verified by a certain matrix satisfying our three conditions. Hence if A is verified by all matrices which satisfy (i), (ii), (iii), then $\vdash_{\overline{S1^\circ}} A$. The case of S1 similarly employs Theorem II.7 and Theorem III.10.

IV. SECOND COMPLETENESS THEOREM: DECISION PROCEDURES

We have now established all the necessary apparatus to prove the main theorem (Theorem IV.1) of our paper which, finally, will lead to decision procedures for the systems concerned. From our point of view the theorem is the main lemma to establish the decidability of our systems; but it has a purely independent interest in the theory of Boolean algebras with operators. We shall call this theorem the Finite Embedding Theorem. It should also be noted that given any propositional logic to show it is decidable it is sufficient to establish a correlation between it and the appropriate kind of algebras—the theorem which does this we have called the First Completeness Theorem—and then to prove a Finite Embedding Theorem for these algebras. This may be termed the "algebraic method".

§1. THE T-SYSTEMS.

THEOREM IV.1. (*Finite Embedding Theorem*). *Let* $\mathfrak{M} = \langle M, \cap, -, \mathrm{P}\rangle$ *be a weak modal* (*weak epistemic*) *algebra, and let* $a_1, a_2, \ldots, a_r$ *be a finite sequence of elements of M. Then there is a finite weak modal* (*weak epistemic*) *algebra* $\mathfrak{M}_1 = \langle M_1, \cap_1, -_1, \mathrm{P}_1\rangle$ *with at most* $2^{2^{r+2}}$ *elements such that*

(i) *for* $1 \leqq i \leqq r$, $a_i \in M_1$;
(ii) *for* $x, y \in M_1$, $x \cap_1 y = x \cap y$;
(iii) *for* $x \in M_1$, $-_1 x = -x$;
(iv) *for* $x \in M_1$ *such that* $\mathrm{P}x \in M_1$, $\mathrm{P}_1 x = \mathrm{P}x$.

Proof. Let M_1 be the set of elements of M obtained from $\mathrm{P}0$, $\mathrm{P}1$, $a_1, a_2, \ldots, a_r$ by any finite number of applications of the operations - and $\cap$. Then from the theory of Boolean algebras we know that $\langle M_1, \cap, -\rangle$ is a Boolean algebra and that M_1 has at most $2^{2^{r+2}}$ elements. We put $\cap_1$, $-_1$ equal to $\cap$,- restricted to M_1. Then, trivially, $\langle M_1, \cap_1, -_1\rangle$ is a Boolean algebra with at most $2^{2^{r+2}}$ elements. It is also immediate that (i) - (iii) of the theorem are satisfied. We now propose to define an operation P_1 on M_1.

We introduce some terminology and notation. The symbols A_i ($i = 1,2,\dots$) shall denote non-empty subsets (proper or improper) of M_1. Note that since M_1 is finite the number of elements in any A_i is finite. If $A_i = \{x_1, x_2, \dots, x_n\}$ ($n \leqq 2^{2^{r+2}}$) where $x_1, x_2, \dots, x_n$ are distinct members of M_1, by definition we set

$$\mathbf{P}A_i \text{ equal to } \mathsf{P}x_1 \cup \mathsf{P}x_2 \cup \dots \cup \mathsf{P}x_n.$$

Observe that $\mathbf{P}A_i$ is an element of M. We say that an element x of M_1 is *covered* by $A_i = \{x_1, x_2, \dots, x_n\}$ if $\mathsf{P}x_1, \mathsf{P}x_2, \dots, \mathsf{P}x_n \in M_1$ and $\mathsf{P}x \leqq \mathbf{P}A_i$. It is to be noted that if $x \in M_1$ and A_i covers x, then $\mathbf{P}A_i \in M_1$; because it follows from the definition of covering that if $A_i = \{x_1, x_2, \dots, x_n\}$, then $\mathsf{P}x_1, \mathsf{P}x_2, \dots, \mathsf{P}x_n \in M_1$; and hence $\mathbf{P}A_i = \mathsf{P}x_1 \cup \mathsf{P}x_2 \cup \dots \cup \mathsf{P}x_n \in M_1$.

Now consider the set $\{1,0\}$. Clearly $\{1,0\} \subseteq M_1$. We next notice that by our construction $\mathsf{P}0, \mathsf{P}1 \in M_1$. Further, since M is a weak modal algebra and $M_1 \subseteq M$, by Theorem II.2, for $x \in M_1$, we have $\mathsf{P}x \leqq \mathsf{P}1 \cup \mathsf{P}0$, i.e., $\mathsf{P}x \leqq \mathbf{P}\{1,0\}$. Hence, by the definition of the preceeding paragraph, every element of M_1 is covered by some non-empty subset of M_1.

Next, let $x \in M_1$. Observe that M_1 has only finitely many distinct subsets. Let $A_1, A_2, \dots, A_m$ be those non-empty subsets of M_1 that cover x. We then set

(1) $\mathsf{P}_1x = \mathbf{P}A_1 \cap \mathbf{P}A_2 \cap \dots \cap \mathbf{P}A_m$

As remarked earlier, since the A_i's ($i = 1,2,\dots,m$) cover x and $x \in M_1$, $\mathbf{P}A_i$ ($i = 1,2,\dots,m$) $\in M_1$. Hence $\mathbf{P}A_1 \cap \mathbf{P}A_2 \cap \dots \cap \mathbf{P}A_m \in M_1$. Consequently, from (1), $\mathsf{P}_1x \in M_1$.

Further, since x is covered by A_i ($i = 1,\dots,m$), $\mathsf{P}x \leqq \mathbf{P}A_i$ ($i = 1,2,\dots,m$). Hence $\mathsf{P}x \leqq \mathbf{P}A_1 \cap \mathbf{P}A_2 \cap \dots \cap \mathbf{P}A_m$. Therefore, from (1),

(2) $\mathsf{P}x \leqq \mathsf{P}_1x$

It is important to note that (2) holds in general, i.e., we have made no presupposition about $\mathsf{P}x$: it may or may not be an element of M_1.

We now wish to show that condition (iv) of our theorem is satisfied. Let $x \in M_1$ such that $\mathsf{P}x \in M_1$. Consider the set $\{x\}$. Clearly $\{x\} \subseteq M_1$, $\mathsf{P}x \in M_1$ and $\mathsf{P}x \leqq \mathbf{P}\{x\} = \mathsf{P}x$. So $\{x\}$ covers x. Let $B_1, \dots, B_k$ be the other non-empty subsets of M_1 which cover x. Then from (1), we get,

(3) $\mathsf{P}_1x = \mathsf{P}x \cap \mathbf{P}B_1 \cap \dots \cap \mathbf{P}B_k$

By Boolean algebra,

(4) $\mathsf{P}_1x \leqq \mathsf{P}x$

From (2) and (4) we conclude,

(5) $\mathsf{P}x = \mathsf{P}_1x$

Thus (iv) is satisfied. It remains to show that if $\mathfrak{M} = \langle M, \cap, -, \mathsf{P} \rangle$ is a weak modal (weak epistemic) algebra, then $\mathfrak{M}_1 = \langle M_1, \cap_1, -_1, \mathsf{P}_1 \rangle$ is a weak modal (weak epistemic) algebra. To avoid a notation that would be cumbersome we cease to distinguish between $\cap_1$, $-_1$ and $\cap$, -; since $\cap_1$ and $-_1$ are merely the restrictions of $\cap$ and - to M_1.

We begin with weak modal algebras. We have already seen that $< M_1, \cap_1, -_1 >$ is a Boolean algebra. We have therefore only to show that for $x,y,z \in M_1$,

(6) $P_1(x \cap y) \leqq P_1(x \cap z) \cup P_1(y \cap -z)$

Let:

$x \cap y$ be covered by $A_1, \dots, A_r$
$x \cap z$ be covered by $B_1, \dots, B_s$
$y \cap -z$ be covered by $C_1, \dots, C_t$

Then, by definition (*cf.* (1)),

(7) $P_1(x \cap y) = \mathbf{P}A_1 \cap \dots \cap \mathbf{P}A_r$
(8) $P_1(x \cap z) = \mathbf{P}B_1 \cap \dots \cap \mathbf{P}B_s$
(9) $P_1(y \cap -z) = \mathbf{P}C_1 \cap \dots \cap \mathbf{P}C_t$

From (8) and (9) it follows by the dual distributive law of Boolean algebra that

(10) $P_1(x \cap z) \cup P_1(y \cap -z)$
$= (\mathbf{P}B_1 \cap \dots \cap \mathbf{P}B_s) \cup (\mathbf{P}C_1 \cap \dots \cap \mathbf{P}C_t)$
$= (\mathbf{P}B_1 \cup \mathbf{P}C_1) \cap (\mathbf{P}B_1 \cup \mathbf{P}C_2) \cap \dots \cap (\mathbf{P}B_s \cup \mathbf{P}C_t)$

Now, let:

(11) $B_1 = \{x_1, x_2, \dots, x_v\}$
(12) $C_1 = \{y_1, y_2, \dots, y_w\}$

Since B_1 covers $x \cap z$ and C_1 covers $y \cap -z$, we get,

(13) $P(x \cap z) \leqq \mathbf{P}B_1$

and

(14) $P(y \cap -z) \leqq \mathbf{P}C_1$

From (11), (12), (13) and (14) we get,

(15) $P(x \cap z) \leqq Px_1 \cup Px_2 \cup \dots \cup Px_v$

and

(16) $P(y \cap -z) \leqq Py_1 \cup Py_2 \cup \dots \cup Py_w$

Since $\mathfrak{M}$ is a weak modal algebra,

(17) $P(x \cap y) \leqq P(x \cap z) \cup P(y \cap -z)$

We conclude, from (15), (16) and (17),

(18) $P(x \cap y) \leqq Px_1 \cup \dots \cup Px_v \cup Py_1 \cup \dots \cup Py_w$

Now let $z_1, \dots, z_p$ be the distinct elements of $x_1, \dots, x_v, y_1, \dots, y_w$. Then, obviously, $Pz_1, Pz_2, \dots Pz_p \in M_1$. And

(19) $\mathsf{P}(x \cap y) \leqq \mathsf{P}z_1 \cup \mathsf{P}z_2 \cup \ldots \cup \mathsf{P}z_p$

We define:

(20) $D = \{z_1, \ldots, z_p\}$

It follows from the foregoing remarks that D covers $x \cap y$. We therefore conclude that D is one of the A_i's ($i = 1,2,\ldots,r$). It is then immediate that

$\mathsf{P}A_1 \cap \ldots \cap \mathsf{P}A_r \leqq \mathsf{P}D$
i.e., $\mathsf{P}A_1 \cap \ldots \cap \mathsf{P}A_r \leqq \mathsf{P}z_1 \cup \ldots \cup \mathsf{P}z_p$
i.e., $\mathsf{P}A_1 \cap \ldots \cap \mathsf{P}A_r \leqq \mathsf{P}x_1 \cup \ldots \cup \mathsf{P}x_v \cup \mathsf{P}y_1 \cup \ldots \cup \mathsf{P}y_w$
i.e., $\mathsf{P}A_1 \cap \ldots \cap \mathsf{P}A_r \leqq \mathsf{P}B_1 \cup \mathsf{P}C_1$

From (7), it follows,

(21) $\mathsf{P}_1(x \cap y) \leqq (\mathsf{P}B_1 \cup \mathsf{P}C_1)$

It can be seen in an exactly analogous manner that

$$\begin{array}{lll} \mathsf{P}_1(x \cap y) \leqq & (\mathsf{P}B_1 \cup & \mathsf{P}C_2) \\ * & * & * \\ * & * & * \\ * & * & * \\ \mathsf{P}_1(x \cap y) \leqq & (\mathsf{P}B_s \cup & \mathsf{P}C_t) \end{array} \tag{22}$$

(21) and (22) gives us

(23) $\mathsf{P}_1(x \cap y) \leqq (\mathsf{P}B_1 \cup \mathsf{P}C_1) \cap (\mathsf{P}B_1 \cup \mathsf{P}C_2) \cap \ldots \cap (\mathsf{P}B_s \cup \mathsf{P}C_t)$

From (10) and (23) we finally conclude,

(24) $\mathsf{P}_1(x \cap y) \leqq \mathsf{P}_1(x \cap z) \cup \mathsf{P}_1(y \cap -z)$

So $< M_1, \cap_1, -_1, \mathsf{P}_1 >$ is a weak modal algebra.

For weak epistemic algebras, we need to show that, given $x \leqq \mathsf{P}x$, then $x \leqq \mathsf{P}_1x$. But by (2), $\mathsf{P}x \leqq \mathsf{P}_1x$, whence $x \leqq \mathsf{P}_1x$. This completes the proof.

DEFINITION IV.1. By a subformula of a wff A we mean a wff which occurs as a part (proper or improper) of A.

THEOREM IV.2. (*Second Completeness Theorem*). *Let A be a wff with r subformulas. Then $\vdash_{\overline{T1^\circ(T1)}} A$ if and only if A is verified by all weak modal (weak epistemic) algebras with at most $2^{2^{r+2}}$ elements.*

Proof. If $\vdash_{\overline{T1^\circ(T1)}} A$, then by Theorem III.5 (First Completeness Theorem), A is verified by all weak modal (weak epistemic) algebras. *A fortiori*, A is verified by all weak modal (weak epistemic) algebras with at most $2^{2^{r+2}}$ elements.

Conversely suppose that A is a non-theorem of T1°(T1) with r subformulas. Then A is not verified by the appropriate regular characteristic matrix of Theorem II.5, say $\mathfrak{M} = < M, \{1\}, \cap, -, \mathsf{P} >$. Let the propositional variables in A be $v_1, v_2, \ldots, v_n$ and let $a_1, a_2, \ldots, a_n$ be elements of M which form an assignment to $v_1, \ldots v_n$ which does not verify A. Suppose that for this assignment the values of the subformulas of A other than $v_1, v_2, \ldots, v_n$ are $a_{n+1}, \ldots, a_r$. We may assume that the last such subformula is A itself, so that $a_r \neq 1$. By Theorem III.3 and Theorem III.4, $\langle M, \cap, -, \mathrm{P}\rangle$ is a weak modal (weak epistemic) algebra so that by Theorem IV.1 there is a finite weak modal (weak epistemic) algebra $\mathfrak{M}_1$ with at most $2^{2^{r+2}}$ elements satisfying the four conditions of that theorem. By condition (i), we can consider the same assignment $a_1, \ldots, a_n$ to $v_1, \ldots, v_n$ in the matrix $\mathfrak{M}_1$. By conditions (ii) - (iv), it is clear that this assignment assigns the same value to A in $\mathfrak{M}_1$ as it assigned in $\mathfrak{M}$, namely $a_r \neq 1$. Thus A is not verified by $\mathfrak{M}_1$. Consequently if A is verified by all weak modal (weak epistemic) algebras with not more than $2^{2^{r+2}}$ elements, then $\vdash_{\overline{\mathrm{T1}^\circ(\mathrm{T1})}} A$. This completes the proof.

Theorem IV.2 gives us a decision procedure for T1°(T1): Let A be a given wff. Then the number, r, of its subformulas can be found in a constructive way. Now all weak modal (weak epistemic) algebras with not more than $2^{2^{r+2}}$ elements can be constructed. Then we can determine, again by a constructive method, whether A is verified by all these algebras. If it is verified by all of them, then, by Theorem IV.2, it is a theorem. If not, it is not a theorem. We thus have,

THEOREM IV.3. *The systems* T1° *and* T1 *are decidable.*

§2. THE S-SYSTEMS. We shall now obtain decision procedures for the S-systems. The method is essentially the same as that of the previous section. So we shall use the terminology introduced in Theorem IV.1. However, additional complications set in because of the designated elements in a σ-regular S1°(S1)-matrix.

THEOREM IV.4. (*Finite Embedding Theorem*). *Let* $\mathfrak{M} = \langle M, D, \cap, -, \mathrm{P}\rangle$ *be a* σ*-regular* S1°(S1)*-matrix, and let* $a_1, a_2, \ldots, a_r$ *be a finite sequence of elements of* M. *Then there is a finite* σ*-regular* S1°(S1)*-matrix* $\mathfrak{M}_1 = \langle M_1, D_1, \cap_1, -_1, \mathrm{P}_1\rangle$ *with at most* $2^{2^{r+2}}$ *elements such that*

(i) for $1 \leqq i \leqq r$, $a_i \in M_1$;
(ii) *for* $x, y \in M_1$, $x \cap_1 y = x \cap y$;
(iii) *for* $x \in M_1$, $-_1 x = -x$;
(iv) *for* $x \in M_1$ *such that* $\mathrm{P}x \in M_1$, $\mathrm{P}_1 x = \mathrm{P}x$;
(v) *for* $x \in M_1$, *if* $x \in D_1$, *then* $x \in D$.

Proof. By Theorem III.8 and Theorem III.9, $\langle M, \cap, -, \mathrm{P}\rangle$ is a weak modal (weak epistemic) algebra. By Theorem IV.1, there is a finite weak modal (weak epistemic) algebra with at most $2^{2^{r+2}}$ elements such that conditions (i), (ii), (iii), (iv) are satisfied.

To avoid needless repetition we shall assume that we have made a construction analogous to that of Theorem IV.1. We set

(1) D_1 to be equal to the intersection of D and M_1.

Then, clearly (v) is satisfied. It remains to show that $\mathfrak{M}_1$ is a σ-regular S1°(S1)-matrix. In order to do this, we verify the three conditions of

Theorem III.9 (Theorem III.10). We have already seen that (A) is satisfied.

(B): (i) By Theorem III.8(E), $1 \in D$. Also, clearly, $1 \in M_1$. Hence by (1) $1 \in D_1$. By Theorem III.8(D), $0 \notin D$. Hence, by (1), $0 \notin D_1$. Also, clearly, $0 \in M_1$. So D_1 is a non-empty proper subset of M_1.
(ii) Next, if $x \in D_1$ and $y \in D_1$, by (1), $x \in D$, $x \in M_1$, $y \in D$ and $y \in M_1$. Hence, since $\mathfrak{M}$ is σ-regular (*cf.* Df. II.14), $x \cap y \in D$; also, clearly, $x \cap y \in M_1$. By (1), $x \cap y \in D_1$.
(iii) Finally, if $x \in D_1$ and $y \in M_1$, then $x \in D$ and $y \in M$. By Theorem III.9 (Theorem III.10), D is an additive ideal of M. By Definition II.15(iii), $x \cup y \in D$. Also, obviously $x \cup y \in M_1$. Hence, by (1), $x \cup y \in D_1$.

This shows that (B) is satisfied.

(C): We first show that $-_1\mathrm{P}_1(0) \in D_1$. We have already seen that conditions (i) - (v) of our theorem are satisfied. Since $0 \in M_1$ (obviously) and $\mathrm{P}(0) \in M_1$ (by construction), by (iv), we conclude, $\mathrm{P}(0) = \mathrm{P}_1(0)$. By (iii), $-\mathrm{P}(0) = -_1\mathrm{P}_1(0)$. By Theorem III.8(A), $-\mathrm{P}(0) \in D$. Hence $-_1\mathrm{P}_1(0) \in D$. Also $\mathrm{P}(0) \in M_1$ and hence $-\mathrm{P}(0) \in M_1$. Since $-\mathrm{P}(0) = -_1\mathrm{P}_1(0)$, $-_1\mathrm{P}_1(0) \in M_1$. We conclude from (1), $-_1\mathrm{P}_1(0) \in D_1$. Next suppose that $-_1\mathrm{P}_1(x) \in D_1$. Then $-_1\mathrm{P}_1(x) \in D$. We know that $\mathrm{P}x \leqq \mathrm{P}_1 x$ (*cf.* Th.IV.1(2) and the remark that follows). Hence $-\mathrm{P}_1(x) \leqq -\mathrm{P}(x)$. Since $\mathrm{P}_1(x) \in M_1$, $-\mathrm{P}_1(x) \in M_1$ and therefore $-_1\mathrm{P}_1(x) = -\mathrm{P}_1(x)$. Hence $-_1\mathrm{P}_1(x) \leqq -\mathrm{P}(x)$. By Theorem III.8 (C), $-\mathrm{P}(x) \in D$. By Theorem III.8 (B), $x = 0$. Thus (C) is satisfied.

This completes the proof of Theorem IV.4.

THEOREM IV.5. (*Second Completeness Theorem*). *Let A be a wff with r subformulas. Then* $\vdash_{\overline{\mathrm{S1^\circ(S1)}}} A$ *if and only if A is verified by all matrices* $\mathfrak{M} = \langle M, D, \cap, -, \mathrm{P}\rangle$ *with at most $2^{2^{r+2}}$ elements such that*

(i) $\langle M, \cap, -, \mathrm{P}\rangle$ *is a weak modal* (*weak epistemic*) *algebra*;
(ii) *D is an additive ideal of M*;
(iii) $x = 0$ *if and only if* $-\mathrm{P}(x) \in D$.

Proof. First suppose that $\vdash_{\overline{\mathrm{S1^\circ(S1)}}} A$. Then by Theorem III.11, A is verified by all matrices which satisfies the three conditions of our theorem. *A fortiori*, it is verified by those matrices which have at most $2^{2^{r+2}}$ elements.

Conversely, suppose A is a non-theorem of S1°(S1). Then A is not verified by the appropriate σ-regular characteristic matrix of Theorem III.7, say, $\mathfrak{M} = \langle M, D, \cap, -, \mathrm{P}\rangle$. Let the propositional variables in A be $v_1, v_2, \ldots, v_n$, and let $a_1, \ldots, a_n$ be elements of M which form an assignment to $v_1, \ldots, v_n$ which does not verify A. Suppose that for this assignment the values of the sub-formulas of A other than $v_1, \ldots, v_n$ are $a_{n+1}, \ldots, a_r$. Without loss of generality we can suppose that $a_r \notin D$.

Then we see, by Theorem IV.4, that there exists a finite σ-regular S1°(S1)-matrix $\mathfrak{M}_1 = \langle M_1, D_1, \cap_1, -_1, \mathrm{P}_1\rangle$ with at most $2^{2^{r+2}}$ elements such that

(i) for $1 \leqq i \leqq r$, $a_i \in M_1$;
(ii) for $x, y \in M_1$, $x \cap_1 y = x \cap y$;

(iii) for $x \in M_1$, $-_1 x = -x$;
(iv) for $x \in M_1$ such that $\mathrm{P}x \in M_1$, $\mathrm{P}_1 x = \mathrm{P}x$;
(v) for $x \in M_1$, if $x \notin D$, then $x \notin D_1$.

By condition (i), we can consider the same assignment $a_1, a_2, \ldots, a_n$ to $v_1, \ldots, v_n$ in the matrix $\mathfrak{M}_1$. By conditions (ii) - (iv), it is clear that this assignment assigns the same value to A in $\mathfrak{M}_1$ as it assigned in $\mathfrak{M}$, i.e., a_r. By condition (v), $a_r \notin D_1$. Thus A is not verified by $\mathfrak{M}_1$. But $\mathfrak{M}_1$ is a σ-regular S1°(S1)-matrix. By Theorem III.9 (Th. III.10), it satisfies the three conditions of our theorem. Thus A is not verified by a certain matrix with at most $2^{2^{r+2}}$ elements which satisfies the three conditions of our theorem. Hence if A is verified by all matrices with at most $2^{2^{r+2}}$ elements satisfying our three conditions, then $\vdash_{\overline{\text{S1°(S1)}}} A$. This completes the proof.

It follows from Theorem IV.5 (*cf.* remarks following Th. IV.2) that

THEOREM IV.6. *The systems* S1° *and* S1 *are decidable.*

V. S2 AND S4°

§1. THE SYSTEM S2. The motivation for including a decision procedure for S2 has been explained in the introduction. The results of this article viewed as a whole are not new. However, in the course of our proofs, we shall give a characterization of σ-regular S2-matrices. This characterization, to the best of our knowledge, is new[1].

THEOREM V.1. *There exists a* σ*-regular characteristic matrix for* S2.

We omit the proof of Theorem V.1. Its proof can be obtained by repeating, almost verbatim, the proof of Theorem II.7.

THEOREM V.2. $\mathfrak{M} = \langle M, D, \cap, -, \mathrm{P}\rangle$ *is a* σ*-regular* S2*-matrix if and only if*
(A) $\langle M, \cap, -, \mathrm{P}\rangle$ *is a weak epistemic algebra,*
(B) D *is an additive ideal of* M;
(C) $x = 0$ *if and only if* $-\mathrm{P}(x) \in D$;
(D) $\mathrm{P}0 \leqq \mathrm{P}x$.

Proof. First suppose that $\mathfrak{M}$ is a σ-regular S2-matrix. *A fortiori*, it is a σ-regular S1-matrix. By Theorem III.10., conditions (A), (B) and (C) are satisfied. Next, by 40.1, $\mathrm{P}(x \cap -x) \Longrightarrow \mathrm{P}x \in D$, i.e., $\mathrm{P}0 \Longrightarrow \mathrm{P}x \in D$. By Theorem III.6., $\mathrm{P}0 \leqq \mathrm{P}x$. Conversely suppose $\mathfrak{M} = \langle M, D, \cap, -, \mathrm{P}\rangle$ satisfies conditions (A), (B), (C) and (D). It thus satisfies conditions (A), (B) and (C) of Theorem III.10. By Theorem III.10., $\mathfrak{M}$ is a σ-regular S1-matrix. It remains to show that the matrix verifies 40.1. Let $x, y \in M$. Then[2],

(1) $[((y \cap x) \Longrightarrow y) \cap \mathrm{P}(y \cap x)] \Longrightarrow \mathrm{P}(x) \in \mathrm{D}$ [35.32]
(2) $[-\mathrm{P}((y \cap x) \cap -y) \cap \mathrm{P}(y \cap x)] \Longrightarrow \mathrm{P}(x) \in \mathrm{D}$ [(1);Df.II.8]
(3) $[-\mathrm{P}(0) \cap \mathrm{P}(x \cap y)] \Longrightarrow \mathrm{P}(x) \in \mathrm{D}$ [(2);**BA**]
(4) $-\mathrm{P}(0) \cap \mathrm{P}(x \cap y) \leqq \mathrm{P}(x)$ [(3);Th.III.6]
(5) $\mathrm{P}(x \cap y) \leqq \mathrm{P}(0) \cup \mathrm{P}(x)$ [(4);**BA**]

(6) $P(0) \leqq P(x)$ [(D)]
(7) $P(0) \cup P(x) = P(x)$ [(6);**BA**]
(8) $P(x \cap y) \leqq P(x)$ [(5),(7)]
(9) $P(x \cap y) \Longrightarrow P(x) \in D$ [(8);Th.III.6]

This completes the proof.

THEOREM V.3. (*First Completeness Theorem*). $\vdash_{S2} A$ *if and only if A is verified by all matrices* $\mathfrak{M} = \langle M, D, \cap, -, P \rangle$ *such that*

(A) $\langle M, \cap, -, P \rangle$ *is a weak epistemic algebra;*
(B) *D is an additive ideal of M;*
(C) $x = 0$ *if and only if* $-P(x) \in D$;
(D) $P0 \leqq Px$.

We omit the proof. The proof is similar to that of Theorem III.11: we employ in our case Theorem V.1 and Theorem V.2.

THEOREM V.4. (*Finite Embedding Theorem*). *Let* $\mathfrak{M} = \langle M, D, \cap, -, P \rangle$ *be a* σ*-regular* S2*-matrix, and let* $a_1, a_2, \ldots, a_r$ *be a finite sequence of elements of M. Then there is a finite* σ*-regular* S2*-matrix* $\mathfrak{M}_1 = \langle M_1, D_1, \cap_1, -_1, P_1 \rangle$ *with at most* $2^{2^{r+2}}$ *elements such that*

(i) *for* $1 \leqq i \leqq r$, $a_i \in M_1$;
(ii) *for* $x, y \in M_1$, $x \cap_1 y = x \cap y$;
(iii) *for* $x \in M_1$, $-_1 x = -x$;
(iv) *for* $x \in M_1$ *such that* $Px \in M_1$, $P_1 x = Px$;
(v) *for* $x \in M_1$, *if* $x \in D_1$, *then* $x \in D$.

Proof. $\mathfrak{M}$ is a σ-regular S2-matrix. It is then, of course, a σ-regular S1-matrix. By Theorem IV.4., there is a finite σ-regular S1-matrix $\mathfrak{M}_1 = \langle M_1, D_1, \cap_1, -_1, P_1 \rangle$ with at most $2^{2^{r+2}}$ elements such that (i) - (v) is satisfied. We now want to show that $\mathfrak{M}_1$ is a σ-regular S2-matrix. In order to do this we shall demonstrate that conditions (A), (B), (C), (D) of Theorem V.2 are satisfied. Since $\mathfrak{M}_1$ is a σ-regular S1-matrix, by Theorem III.10 conditions (A), (B) and (C) of Theorem V.2 are satisfied. It thus remains to show that (D) is satisfied, i.e., for $x \in M_1$, $P_1(0) \leqq P_1(x)$. Since $\mathfrak{M}$ is a σ-regular S2-matrix, by Theorem V.2 we have, $P0 \leqq Px$. In the course of proving Theorem IV.4 we have seen that $P(0) = P_1(0)$. By Theorem IV.1(2), $Px \leqq P_1(x)$. From these facts we conclude: $P_1(0) \leqq P_1(x)$. This completes the proof.

THEOREM V.5. (*Second Completeness Theorem*). *Let A be a wff with r subformulas. Then* $\vdash_{S2} A$ *if and only if A is verified by all matrices* $\mathfrak{M} = \langle M, D, \cap, -, P \rangle$ with at *most* $2^{2^{r+2}}$ *elements such that*

(i) $\langle M, \cap, -, P \rangle$ *is a weak epistemic algebra;*
(ii) *D is an additive ideal of M;*
(iii) $x = 0$ *if and only if* $-P(x) \in D$;
(iv) $P0 \leqq Px$.

We omit the proof. The proof is obtained by imitating the proof of

Theorem IV.5. We employ here Theorem V.1, Theorem V.2, Theorem V.3, Theorem V.4. As an immediate corollary we get,

THEOREM V.6. *The system* S2 *is decidable.*

§2. THE SYSTEM S4°. The system S4° is due to Sobociński[3]. It is obtained by adding to the system S1° the axiom:

60.01. $\Diamond\Diamond p \dashv\!\!\!\!\!\!\; \Diamond p$

It is well-known that S2° is a sub-system of S4°. We mention this because we shall use theorems of S2° proved in [15] in connection with S4°-matrices.

THEOREM V.7. *There exists a* σ-*regular characteristic matrix for* S4°.

We omit the proof.

THEOREM V.8. $\mathfrak{M} = \langle M, D, \cap, -, \mathrm{P}\rangle$ is *a* σ-*regular* S4°-*matrix if and only if*

(A) $\langle M, \cap, -, \mathrm{P}\rangle$ *is a weak modal algebra;*
(B) D *is an additive ideal of* M;
(C) $x = 0$ *if and only if* $-\mathrm{P}(x) \in D$;
(D) $\mathrm{P}\,\mathrm{P}\,x \leqq \mathrm{P}x$.

Observe how we have extended Theorem III.9. to Theorem III.10. Theorem V.8 can be obtained in a similar manner from Theorem III.9. We omit the obvious details.

THEOREM V.9. (*First Completeness Theorem*). $\vdash_{\overline{S4^\circ}} A$ *if and only if* A *is verified by all matrices* $\mathfrak{M} = \langle M, D, \cap, -, \mathrm{P}\rangle$ *such that*

(A) $\langle M, \cap, -, \mathrm{P}\rangle$ *is a weak modal algebra;*
(B) D *is an additive ideal of* M;
(C) $x = 0$ *if and only if* $-\mathrm{P}(x) \in D$;
(D) $\mathrm{P}\,\mathrm{P}x \leqq \mathrm{P}x$.

We omit the proof.

THEOREM V.10. (*Finite Embedding Theorem*). *Let* $\mathfrak{M} = \langle M, D, \cap, -, \mathrm{P}\rangle$ *be a* σ-*regular* S4°-*matrix and let* $a_1, a_2, \ldots, a_r$ *be a finite sequence of elements of* M. *Then there is a finite* σ-*regular* S4°-*matrix* $\mathfrak{M}_1 = \langle M_1, D_1, \cap_1, -_1, \mathrm{P}_1\rangle$ *with at most* $2^{2^{r+2}}$ *elements such that*

(i) *for* $1 \leqq i \leqq r$, $a_i \in M_1$;
(ii) *for* $x, y \in M_1$, $x \cap_1 y = x \cap y$;
(iii) *for* $x \in M_1$, $-_1 x = -x$;
(iv) *for* $x \in M_1$ *such that* $\mathrm{P}x \in M_1$, $\mathrm{P}_1 x = \mathrm{P}x$;
(v) *for* $x \in M_1$, *if* $x \in D_1$, *then* $x \in D$.

Proof: We shall assume that we have made a construction similar to Theorem IV.1. supplemented by constructions similar to Theorem IV.4. It is easy to see, by Theorem V.9, that the only thing that remains to be shown is $\mathrm{P}_1\mathrm{P}_1 x \leqq \mathrm{P}_1 x$. Let x be covered by $A_1, A_2, \ldots, A_n$. Then

(1) $P_1x = \mathbf{P}A_1 \cap \mathbf{P}A_2 \cap \ldots \cap \mathbf{P}A_n$

Let P_1x be covered by $B_1, B_2, \ldots, B_p$ Then

(2) $P_1 P_1 x = \mathbf{P}B_1 \cap \mathbf{P}B_2 \cap \ldots \cap \mathbf{P}B_p$

From (1), it follows, by **BA**,

(3) $P_1x \leqq \mathbf{P}A_1$

Let:

(4) $A_1 = \{x_1, x_2, \ldots, x_s\}$

From (3) and (4) we get,

(5) $P_1x \leqq Px_1 \cup Px_2 \cup \ldots \cup Px_s$

$\mathfrak{M}$ is a σ-regular S4°-matrix and hence a σ-regular S1°-matrix. Hence from Theorem III.6.,

(6) $P_1x \Rightarrow (Px_1 \cup Px_2 \cup \ldots \cup Px_s) \in D$.

(7) $(P_1x \Rightarrow (Px_1 \cup \ldots \cup Px_s)) \Leftrightarrow [P_1x \Leftrightarrow (P_1x \cap (Px_1 \cup \ldots \cup Px_s))] \in D$ [45.21]

(8) $P_1x \Rightarrow (Px_1 \cup \ldots \cup Px_s) = P_1x \Leftrightarrow (P_1x \cap (Px_1 \cup \ldots \cup Px_s))$ [(7);Df.II.14(iv)]

(9) $P_1x \Leftrightarrow (P_1x \cap (Px_1 \cup \ldots \cup Px_s)) \in D$ [(6),(8)]

(10) $P_1x = P_1x \cap (Px_1 \cup \ldots \cup Px_s)$ [(9);Df.II.14(iv)]

(11) $P(P_1x \cap (Px_1 \cup \ldots \cup Px_s)) \Rightarrow P(Px_1 \cup \ldots \cup Px_s) \in D$ [41.3(3)]

(12) $PP_1x \Rightarrow P(Px_1 \cup \ldots \cup Px_s) \in D$ [(10),(11)]

(13) $P(Px_1 \cup (Px_2 \cup \ldots \cup Px_s)) \Leftrightarrow (PPx_1 \cup P(Px_2 \cup \ldots \cup Px_s)) \in D$ [44.4]

(14) $P(Px_1 \cup (Px_2 \cup \ldots \cup Px_s)) = PPx_1 \cup P(Px_2 \cup \ldots \cup Px_s)$ [(13);Df.II.14(iv)]

Proceeding in a similar vein, we finally get,

(15) $P(Px_1 \cup \ldots \cup Px_s) = PPx_1 \cup \ldots \cup PPx_s$

From (12) and (15) we conclude,

(16) $PP_1x \Rightarrow PPx_1 \cup \ldots \cup PPx_s \in D$

(17) $PP_1x \leqq PPx_1 \cup \ldots \cup PPx_s$ [(16);Th.III.6]

Since $\mathfrak{M}$ is a σ-regular S4°-matrix, by Theorem V.8(D),

(18) $PPx_i \leqq Px_i$ $(i = 1, 2, \ldots, s)$

From (17) and (18),

(19) $PP_1x \leqq Px_1 \cup \ldots \cup Px_s$

From (4) and (19) we conclude,

(20) $PP_1x \leqq \mathbf{P}A_1$

It follows from (20) that A_1 covers P_1x. Then A_1 is one of the B_i's ($i = 1, 2, \ldots, p$). Hence,

(21) $\mathsf{P}B_1 \cap \mathsf{P}B_2 \cap \ldots \cap \mathsf{P}B_p \leqq \mathsf{P}A_1$

Hence, from (2),

(22) $\mathsf{P}_1\mathsf{P}_1x \leqq \mathsf{P}A_1$

In an exactly similar manner we deduce,

(23) $\mathsf{P}_1\mathsf{P}_1x \leqq \mathsf{P}A_i$ $(i = 2, 3, \ldots, n)$

From (1), (22) and (23),

(24) $\mathsf{P}_1\mathsf{P}_1x \leqq \mathsf{P}_1x$

This completes the proof of Theorem V.10.

THEOREM V.11. (*Second Completeness Theorem.*) *Let A be a wff with r subformulas. Then* $\vdash_{\overline{S4^\circ}} A$ *if and only if A is verified by all matrices* $\mathfrak{M} = \langle M, D, \cap, -, \mathsf{P}\rangle$ *with at most* $2^{2^{r+2}}$ *elements such that*

(i) $\langle M, \cap, -, \mathsf{P}\rangle$ *is a weak modal algebra;*
(ii) *D is an additive ideal of M;*
(iii) $x = 0$ *if and only if* $-\mathsf{P}(x) \in D$;
(iv) $\mathsf{P}\mathsf{P}x \leqq \mathsf{P}x$.

We omit the proof. As a corollary we have,

THEOREM V.12. *The system* S4° *is decidable*.[4]

NOTES

INTRODUCTION

1. For an explanation of the term "decision procedure" and discussions concerning it, *cf.* [10], pp. 99-100.

2. For just two of them, variants of Kalmar's method, *cf.* [43] and [44].

3. An exposition of the method is to be found in [10], pp. 97-99.

4. To be accurate, we should mention that sometimes we can even start with an infinite matrix and give axioms and rules such that the theorems coincide with the wffs verified by the matrix. *cf.* [12]. But in order to be able to do this the infinite matrix has to be especially "nice".

5. *Cf.* [31].

6. *Cf.* [11].

7. *Cf.* [17].

8. *Cf.* [33].

9. The term "deducibility problem" was proposed by Church in order to distinguish it from the decision problem. *Cf.* [10], p. 100, n. 184.

10. *Cf.* [20], p. 8. "We shall say that a calculus P has the *finite model property* if there is a finite model counter-example to each non-provable formula of P, that is, given an arbitrary unprovable formula X of P, there is a finite model of P in which X is not valid". For a fascinating general survey of problems connected with the finite model property *cf.* [22].

11. *Cf.* [16].

12. *Cf.* [20] and [21].

13. *Cf.* [3], [4], [5], [6], [7], [9], [13], [18], [28], [29], and [30]. For applications to algebra and topology *cf.* [34] and [35].

14. *Cf.* [24], [25], and [26].

15. *Cf.* [37], [38], and [39]. *cf.* also [47].

16. *Cf.* [32], [37], and [38].

17. *Cf.* [1], [2], [19], and [40].

18. *Cf.* [46], p. 53.

SECTION I

1. For a description of the systems S1 and S2, *cf.* [31], pp. 122-178, 492-502. S1° and S2° are proper subsystems of S1 and S2 respectively. They were introduced in [14]. For a detailed account *cf.* [15], pp. 43-78. Also *cf.* [46].

2. *Cf.* [27], p. 180

3. *Cf.* [29] and [30]. Lemmon does use "essential techniques" to extend his results from, say, E2 to S2. But if our main interest is decidability the same results can be obtained in a straightforward manner at the expense of elegance.

4. *Cf.* [29], p. 47.

5. *Cf.* [30], p. 200, Theorem 20(ii).

6. *Cf.* [42], pp. 55-56. Note, however, that we use proper axioms, not axiom schemes as in [42].

SECTION II

1. *Cf.* [34], pp. 145-146, Definition 1.1, also cf. [35].

2. *Cf.* [33].

3. *Cf.* [36].

4. *Cf.* [34].

5. *Cf.* [34], pp. 147-150.

6. For a study of Boolean algebra with operators with a purely algebraic motivation *cf.* [23].

7. *Cf.* [28] where extension algebras are introduced to study the system T. *Cf.* [29] and [30] where there is an abundance of such algebras.

8. *Cf.* [29], p. 49, Definition 1.

9. *Cf.* [29], p. 53, Theorem 9.

10. *Cf.* [29] and [30].

11. Definitions II.3-II.18 are obvious adaptations of definitions found in the literature. See especially [29] and [33].

12. The algebra described here is derived from the matrix of group V, p. 494, [31]. Since we are concerned with algebras and not matrices the designated elements play no role.

13. *Compare the definition of epistemic algebras in [29], p. 54.*

14. The algebra described here is derived from the matrix K_4^+ of [29], p. 64. The remark made in note 12 above also applies here.

15. *Cf.* [29], p. 50, Theorem 7.

16. *Cf.* [33], p. 122.

17. This matrix is that of group V, p. 494, [31].

SECTION III

1. *Cf.* [8].

SECTION V

1. Compare Theorem V.2 with Theorem 3, p. 120 [33].

2. The proof that follows bears a certain resemblance to the deduction in [45] where it is proved that $\{S1, \Diamond(p \wedge \sim p) \dashv \Diamond p\} \rightleftarrows \{S2\}$. Note, however, that the deduction in [45] is logistical whereas here it is algebraic.

3. *Cf.* [46].

4. A GENERAL NOTE ON SECTION IV AND SECTION V

Of the systems treated in the work—T1°, T1, S1°, S1, S2, S4°—the Second Completeness Theorem can be somewhat strengthened for T1, S1, and S2. First note that the algebras related to the system are weak epistemic algebras; and for such algebras we have $x \leqq \mathrm{P}x$. Hence $1 \leqq \mathrm{P}1$. Thus $\mathrm{P}1 = 1$. So in the Finite Embedding Theorem for weak epistemic algebras we need not stipulate that P0 and P1 be included in the construction of M_1. They were included because we wanted to be sure that every element of M_1 was covered by some non-empty subset of M_1; but here it is easy to see that if $x \in M_1$, then x is covered by $\{1\}$ and, of course, $\{1\} \subseteq M_1$. So for T1 we can replace $2^{2^{r+2}}$ by 2^{2^r}. But for the systems S1 and S2 although it is true that every element of M_1 is covered by $\{1\}$ it is necessary to include P0 in the construction of M_1 for other purposes. Consequently, for S1 and S2, we can replace $2^{2^{r+2}}$ by $2^{2^{r+1}}$

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