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A PROOF OF THE LÖWENHEIM-SKOLEM THEOREM

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A modification of the proof procedure described in the appendix to Quine's *Methods of Logic* (revised edition) yields a simple proof[1] of a strong version of the Löwenheim-Skolem Theorem:

Any interpretation has an elementarily equivalent subinterpretation whose universe is countable.

(**I** is a subinterpretation of **J** if the universe of **I** is a subset of that of **J**, **I** assigns any (free) variable the same denotation as **J**, and **I** assigns any predicate letter the restriction to its own universe of whatever **J** assigns it. Two interpretations are elementarily equivalent if the same schemata are true under both.)

Proof. Let **J** be an interpretation, S_1, S_2, S_3, . . . an enumeration of all and only the prenex schemata true under **J**, and v_1, v_2, v_3, . . . an infinite non-repeating sequence of variables[2] none of which occurs in any S_i. We suppose all bound variables in the S_is to have been relettered so that no variable occurs bound in some S_i and free in some S_j. We further suppose S_1 to begin with an existential quantifier. We form an (infinite) list of schemata by writing down S_1, then applying EI as many times as we can, then applying UI as many times as we can, then writing down S_2, then applying EI as many times as we can, then applying UI as many times as we can, then writing down S_3, then applying . . . . In forming the list, we never

1. Our proof of the strong version stated above proceeds by a route somewhat different from that taken by the classical proofs, [2] and [3], in that we avoid the use of "Skolem normal forms" for satisfiability, and do not (apparently) close a countable (= finite or countably infinite) set under all Skolem functions for all prenex sentences true under the original interpretation. The method we have used can be extended to give a proof of the sill stronger version of the theorem stated in [3]: Any interpretation has an elementary subinterpretation whose universe is countable.

2. Each v_i is alphabetically later than any variable in any S_j; if $n > m$, v_n is alphabetically later than v_m. *Cf.* [1], p. 164.

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apply EI to a schema more than once, and whenever we apply EI to a schema, the instantial variable is always the earliest unused v_i; whenever we apply UI, the instantial variable is always a variable that already occurs free in a schema on the list, but we never repeat a line on the list through a use of UI.

We expand **J** to **L** by assigning the v_is denotations as follows: Suppose d_k to have been assigned to v_k, for all $k \leq n$. Further suppose that any formula on the list above the conclusion C of the application of EI that introduces v_{n+1} is true under the expansion of **J** that assigns d_k to v_k, for all $k \leq n$. We assign v_{n+1} a denotation d_{n+1} in the universe of **J** such that C is true under the expansion of **J** that assigns d_k to v_k, for all $k \leq n + 1$. Since the premiss of the application of EI is true under the expansion of **J** that assigns d_k to v_k, for all $k \leq n$, a suitable choice of d_{n+1} will always be possible. Since a schema will always have the same truth value under two interpretations that are alike, except that one of them assigns denotations to variables not occurring in the schema, all schemata on the list are true under **L**. Let **K** be the subinterpretation of **L** whose (countable) universe is the set of denotations that **L** assigns to variables.

Suppose there were a schema on the list false under **K**. Let S be a shortest such. S can't be universal, as each instance of it whose instantial variable is a variable that occurs free in some schema on the list appears on the list. Since all these instances are shorter than S, they are all true under **K**. But as the universe of **K** contains just the denotations of the variables that occur free in some schema on the list, S would then be true under **K**. S can't be existential, as an instance of it, shorter and hence true under **K**, appears on the list. S would then be true under **K**, as any instance implies it. But S can't be an atomic schema or a truth-functional combination of atomic schemata, as such schemata always have the same truth-values under any subinterpretation of an interpretation as under the interpretation. Hence every schema on the list is true under **K**.

If **I** is the "contraction" of **K** that assigns to each predicate letter or variable to which **J** assigns something (and only to these) whatever **K** assigns it, then **I** is a subinterpretation of **J** whose universe is countable. If S is a schema true under **J**, a prenex equivalent of it containing no v_is will occur on the list, and hence be true under **K**, and hence true under **I**. Thus any schema true under **J** is true under **I** and the theorem is proved.

REFERENCES

[1] Quine, W. V. O., *Methods of Logic*, revised ed., New York (1959).

[2] Skolem, Th., "Logisch-kombinatorische Untersuchungen über die Erfüllbarkeit oder Beweisbarkeit mathematischer Sätze nebst einem Theorie uber dichte Menger," *Videnskapsselskapets skrifter, I. Matematisknaturvidenskabelig klasse*, no. 3 (1920), of which a translation of section one appears in J. van Heijenoort, *From Frege to Gödel*, Cambridge (1967).

[3] Tarski, A., and R. L. Vaught, "Arithmetical Extensions of Relational Systems," *Compositio Mathematica*, vol. 13 (1957), pp. 81-102.

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