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THE PROPOSITIONAL CALCULUS MC AND ITS MODAL ANALOG

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In [5], Łukasiewicz sets down a system for which the matrix

p	Np
0	1
$\frac{1}{2}$	1
1	0

C	0	$\frac{1}{2}$	1
0	0	$\frac{1}{2}$	1
$\frac{1}{2}$	0	0	1
1	0	0	0

K	0	$\frac{1}{2}$	1
0	0	$\frac{1}{2}$	1
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1
1	1	1	1

A	0	$\frac{1}{2}$	1
0	0	0	0
$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$
1	0	$\frac{1}{2}$	1

(with 0 as designated value) is characteristic. This system is formed by adding to the intuitionist propositional calculus (**IC**) the axiom

$$CCNpqCCCqpqq \qquad (1);$$

he notes that Apq may be defined in this system by the formula

$$KCCpqqCCqpp \qquad (2).$$

This definition is, of course, "characteristic" of Dummett's system **LC** [1] in the sense that its addition to **IC** yields **LC**. In the present section of this paper, we shall propose a definition of Apq "stronger" than that above, and will show that it is characteristic of a system—which we call **MC**—equivalent to that of Łukasiewicz [5]. In the latter part of this paper we shall investigate the Lewis-modal system analogous to **MC**.

We shall call **MC** the system formulable by adding to **IC** the definition

$$Apq \textit{ for } KCNpqCCqpp \qquad (3).$$

Alternate formulations are available; if we add to **IC** the axiom

$$ACpqCNNqp \qquad (4)$$

or

$$ACpqCCNqpp \qquad (5)$$

or, finally

$$ANpAqCqp \qquad (6)$$

the result will be **MC**. For the moment, let us call **IC** + (4) **MC'**, and

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IC + (5) MC''. It may be shown without much trouble that (6) implies (4) in IC and that (4) implies (6) in the system KC. By

IC	*CCNNqpCqp*	(7)
IC	*CCpqCArpArq*	(8)
(4), (7), (8)	*ACpqCqp*	(9)

MC' contains LC, and so KC; IC + (4), then, is equivalent to IC + (6). And further:

KC	*CCNNqpANqp*	(10)
IC	*CANqpCCNqpp*	(11)
(10), (11), syl	*CCNNqpCCNqpp*	(12)
(4), (8), (12)	*ACpqCCNqpp*	(13)

Since (13) is formula (5), the system MC' contains MC''; conversely:

IC	*CCCNqppCNNqp*	(14)
(5), (8), (14)	*ACpqCNNqp*	(15)

With (15) provable in MC'', MC' is included in MC'', and the two systems are equivalent. Let us now assume system MC'':

Hyp	*CNqp*	(16)
Hyp	*CCpqq*	(17)
IC	*CCNqpCCCNqppp*	(18)
(16), (18)	*CCCNqppp*	(19)
(5), (8), (19)	*ACpqp*	(20)
IC, (20), (8), (17)	*Aqp*	(21)

From the hypotheses (16) and (17), then, we are able to prove (21) in MC''; thus, by the deduction theorem we have

CCNqpCCCqppAqp (22)

as a theorem of MC''; MC'' thus contains MC, since *CApqKCNpqCCqpp* holds even in IC. Now let us assume the system MC:

Df. *A* in MC	*CCNpqCCCqppApq*	(23)
IC	*CNCpqCCNqpp*	(24)
IC (by syl-simp, Hilbert)	*CCCCNqppCpqCpq*	(25)
(23), *p/Cpq*, *q/CCNqpp*	*CCNCpqCCNqppCCCCCNqppCpqCpqACpq CCNqpp*	(26)
(26), (24), (25)	*ACpqCCNqpp*	(27)

Since (27) is the special axiom of MC'', MC contains MC'', and we have the three formulations, MC, MC', and MC'' as equivalent.

That MC is included in the system of Łukasiewicz [5] may be seen by noting that all MC theses will be validated by the earlier-stated three-valued matrix; that the system of [5] is included in MC is shown as follows:

IC	*CCCqpqCCqpCCNpqq*	(28)
IC	*CApqCCqrCCprr*	(29)
(29), *p/Cqp q/CCNpqq r/CCNpqq*	*CACqpCCNpqqCCCCNpqqCCNpqqCCCqp CCNpqqCCNpqq*	(30)

(30), (5), *Cpp*	*CCCqpCCNpqqCCNpqq*	(31)
(28), (31), **IC**	*CCCqpqCCNpqq*	(32)
(32), **IC**	*CCNpqCCCqpqq*	(33)

But (33) is the axiom of the system of [5], which is then equivalent to **MC**.

The Modal Analog of **MC** In [1], Dummett and Lemmon introduce the modal systems S4.2 and S4.3 as analogs, respectively, of the systems **KC** and **LC**; these systems are related in the same way that S4 is related to **IC** as was shown by McKinsey and Tarski [3]. The translation of [3] by which these systems are related requires that in a formula of **IC** or one of its extensions we replace each variable—*p*, *q*, . . .—by *Lp* , *Lq*, . . .; that we replace each sign of implication *C* by *LC*. The resulting formula will be a theorem of the respective modal analog if and only if the original formula was a theorem of **IC** or the one of its extensions under consideration.

Clearly, we can do the same for **MC**; if we use (4) as our axiom for **MC**, the modal logic in which we are interested will be the one formulable by subjoining to S4 the axiom

$$ALCLpLqLCLNLNLqLp \qquad (34).$$

We shall call the system thus formed S4.3.2. (34) would do as an axiom for S4.3.2, but it is possible to formulate the system more neatly; the axiom we suggest is

$$ALCLpqCMLqp \qquad (35).$$

We now will show that S4 plus (35) is equivalent to S4 plus (34). Assuming first of all formula (35):

(35), *p/Lp*, *q/Lq*, S4	*ALCLpLqCMMLqLLp*	(36)
S4	*CCMpLqLCpq*	(37)
(36), (37)	*ALCLpLqLCMLqLp*	(38)
(38), S4, Df. *M*	*ALCLpLqLCLNLNLqLp*	(39)

S4 plus (35), then, contains S4.3.2; let us now assume S4.3.2, that is, S4 plus (34):

S4	*CLqLMLq*	(40)
(34), (40), S4	*ALCLpLqLCLqLp*	(41);

formula (41) plus S4 yields S4.3. S4.3.2 thus contains S4.3, and so S4.2:

(34), S4.2, Df. *M*	*ALCLpLqLCMLqLp*	(42)
(42), S4	*ALCLpqCMLqp*	(43).

Since (43) is formula (35), S4 plus (35) is equivalent to S4.3.2.

In [4], Sobociński introduces the system S4.4, which is formulated by subjoining to S4 the axiom

$$CpCMLpLp \qquad (44).$$

On p. 307 of [4], as formula *Z7*, Sobociński has

LCNpCMKLqrLCLpq (45).

(I have here corrected an obvious typographical error in *Z7*: there should be an *L*, as in (45) as the fifth last character of the formula) (45) is, as is shown in [4], a theorem of S4.4.

(45), S4	*CMKLqrCNpLCLpq*	(46)
S4	*LCLqCrKLqr*	(47)
(47), S4	*CMLqMCrKLqr*	(48)
(48), S4	*CMLqCLrMKLqr*	(49)
(49), (46), PC	*CMLqCLrCNpLCLpq*	(50)
(50), PC	*CLrCKMLqNpLCLpq*	(51)
(51), *r/Cpp*, S4	*ACMLqpLCLpq*	(52).

Since (52), then, is a theorem of S4.4, S4.3.2 is contained in S4.4.

In [2] it is shown that the formula

CLCLCpLpLpCMLpLp (53)

is independent of S4.3; in [4] Sobociński shows that S4.4 contains (53). The system S4.3.1--which is the result of adding (53) to S4.3--is then contained in S4.4, but is not contained in S4.3; furthermore, Sobociński supplies us with a matrix which shows that S4.3.1 is *properly* contained in S4.4. Since (53) is an S4.4 thesis, and since we have already shown that S4.3.2 is contained in S4.4, the system which is the result of adding (53) to S4.3.2 will also be contained in S4.4. Let us now assume S4.3.2 and (53):

(35), PC	*CNpCMLqLCLpq*	(54)
(54), *p/CpLp*, *q/Lp*, S4	*CNCpLpCMLpLCLCpLpLp*	(55)
(53), (55), PC	*CNCpLpCMLpCMLpLp*	(56)
(56), PC	*ACpLpCMLpLp*	(57)
(57), PC	*CpCMLpLp*	(58).

Formula (58) is the axiom for S4.4; the result of adding (53) to S4.3.2, then, is precisely S4.4.

Sobociński uses the following matrix:

p	1*	2	3	4	5	6	7	8
Lp	1	6	8	8	5	6	8	8
Mp	1	1	3	4	1	1	3	8

to show that S4.3.1 is properly contained in S4.4; as he points out, it validates S4.3 and (53), but fails to validate S4.4. This matrix will also show that S4.3.2 is contained neither in S4.3 nor in S4.3.1; when p=5 and q=2 or 6, *ALCLpqCMLqp* takes the value 5; since we earlier showed that S4.3 is contained in S4.3.2, this means that it is properly so contained. On the other hand, the matrix

p	1*	2	3	4	5	6	7	8
Lp	1	8	7	8	7	8	7	8
Mp	1	2	1	2	1	2	1	8

validates S4.3.2, but fails to do so for both (53) and *CpCMLpLp*--at, say,

p=3 for both formulas. Thus S4.3.2 is properly contained in S4.4, and is neither contained in nor does it contain S4.3.1.

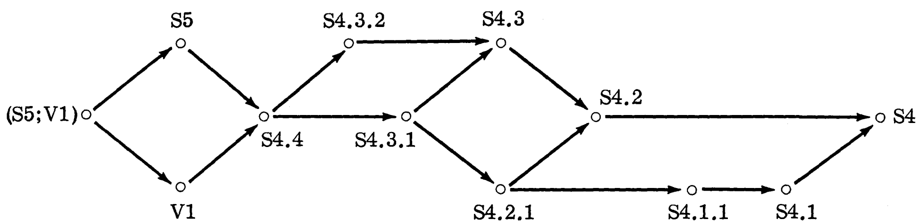
We may summarize the discussion of S4.3.2 and its position among the modal systems between S4 and S5 by reproducing an updated version of the diagram of relations between these systems appearing in [4]; first of all, note the formulas:

$$CMLpLMp \quad (59)$$
$$CLCLCpLppCMLpp \quad (60)$$
$$ALpALCpqLCpNq \quad (61);$$

the systems appearing in the diagram below are formulated as follows:

(S5;V1)	= S5 + (61)	S4.3	= S4 + (41)
S4.4	= S4 + (44)	S4.2.1	= S4.2 + (53)
	= S4.3.2 + (53)	S4.2	= S4 + (59)
V1	= S4 + (61)	S4.1.1	= S4 + (53)
S4.3.2	= S4 + (35)	S4.1	= S4 + (60)
S4.3.1	= S4.3 + (53)		

The diagram of relationships, including S4.3.2, the analog of **MC**, is as follows.



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