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EQUATIONAL LOGIC

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§1. *General Introduction* The present section is by A. N. Prior, while those which follow are by C. A. Meredith, except where otherwise indicated. The second section, which is of date 1956, carries further a result which Łukasiewicz published in 1952,[1] namely that if F, T and N be intuitionist implication, conjunction and negation, and Cpq be defined as $NTpNq$, the classical axioms $CCpqCCqrCpr$, $CCNppp$ and $CpCNpq$ are provable from the intuitionist basis, and the rule of C-detachment (to infer $\vdash\beta$ from $\vdash C\alpha\beta$, i.e. $\vdash NT\alpha N\beta$, and $\vdash\alpha$) is provable for formulae in C and N. Meredith's improvement on this is that if Cpq be defined as $FFFqppq$, the C-classical axioms $CCpqCCqrCpr$, $CCCpqpp$, $CpCqp$ are provable from the intuitionist base, and detachment for this C (which unlike Łukasiewicz's is stronger than F) is provable without restriction.

The proof which Meredith sketches in his note is not, however, a simple substitution-and-detachment deduction, but the derivation of classical *equations* for the defined functor (which he writes as G) from intuitionist equations for the undefined (which he writes as C).

An early example of an equational axiomatisation of the full propositional calculus is that given by W. E. Johnson in his articles of 1892.[2] Johnson's undefined functors are conjunction, represented by juxtaposition, and negation, represented by a superimposed bar. His axioms are the five equations

1. $xy = yx$
2. $(xy)z = x(yz)$
3. $xx = x$
4. $\bar{\bar{x}} = x$
5. $\bar{x} = \overline{xy}\,\overline{x\bar{y}}$

It might be argued that these involve not only conjunction and negation but also equivalence as a primitive, but the "=" sign is to be thought of rather as on the same level as the assertion sign in ordinary substitution-and-detachment systems. The sole rules Johnson uses are substitution for

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variables and inter-changeability, in any context, of equated items. Of proved equations one of the most important is the "Rule of Constancy", $a\bar{a} = c\bar{c}$, provable thus:

$$a\bar{a} = \bar{\bar{a}}\bar{a}\ (4) = \overline{\bar{\bar{a}}c}\ \overline{\bar{\bar{a}}\bar{c}}\ \overline{ac}\ \overline{a\bar{c}}\ (5) = \overline{\bar{\bar{c}}a}\ \overline{\bar{\bar{c}}\bar{a}}\ \overline{ca}\ \overline{\bar{c}a}\ (1,2) = \bar{\bar{c}}\bar{c}\ (5) = c\bar{c}\ (4).$$

This means that "any conjunction of contradictories has the same propositional value", in the sense that if we have any equation of the form $\alpha = \beta\bar{\beta}$ we may substitute for variables in α and leave $\beta\bar{\beta}$ quite alone, even if it happens to contain the variables substituted for in α. Representing this constant by ϕ, which Johnson reads as "Falsism", its negation may be called "Truism" and written τ. For Falsism, Johnson established the "Rule of Nonsignificance", $a\phi = \phi$ (for $a\phi = aa\bar{a} = a\bar{a}$, by 3, $= \phi$); and for Truism the "Rule of Insignificance", $a\tau = a$.[3]

The relation of this type of system to the ordinary substitution-and-detachment type may be seen more easily with some of Meredith's systems. In the note which follows, on the modelling of two-valued implication within intuitionist, Meredith gives for two-valued C the equations $CpCqr = CqCpr$, $CCpqp = p$, $CCpqq = CCqpp$. In 1957 he gave the shorter basis

1. $CCpqp \quad = p$
2. $CCpqCrq = CCqpCrp$

If we define $\alpha = \beta$ within an ordinary system as the pair of asserted implications $\vdash C\alpha\beta$, $\vdash C\beta\alpha$, we have 1 by Peirce's law, $CCCpqpp$, and Simp, $CpCqp$, and 2 more remotely. The rule that given $\vdash C\alpha\beta$ and $\vdash C\beta\alpha$ we may interchange α and β in all implicational contexts is easily established in the common systems, so they plainly contain the equational ones. Conversely from 1 and 2 we may prove a "Rule of Constancy" $Cpp = Cqq$, define the True (1) as $C\alpha\alpha$, $\vdash\alpha$ as $\alpha = 1$, and prove the Tarski-Bernays axioms $\vdash CCCpqpp$, $\vdash CpCqp$, $\vdash CCpqCCqrCpr$ and the rule of detachment $\vdash\alpha$, $\vdash C\alpha\beta \rightarrow \vdash\beta$. In detail, we have

3. $Cpq = CCCpqpCpq$ (1) $= CpCpq$ (1)
4. $CCpqq = CCpqCCpqq$ (3) $= CCqpCCpqp$ (2) $= CCqpp$ (1)
5. $CCpqCpq = CCpqCCpqCpq$ (3) $= CCpCpqCCpqCpq$ (3) $=$
   $CCCpqpCCpqp$ (2) $= Cpp$ (1)
6. $CCCpqqCCpqq = CCpqCpq$ (5) $= Cpp$ (5)
7. $Cqq = CCCqppCCqpp$ (6) $= CCCpqqCCpqq$ (4) $= Cpp$ (6) $= 1$ (Def.)
8. $CpCqp = CC1pCqp$ (1) (7) $= CCp1Cq1$ (2) $= CCpCppCqCqq$ (7) $=$
   $CCppCqq$ (3) $= C11 = 1$ (7)
9. $1 = CCpqCCrqCpq$ (8) $= CCpqCCqrCpr$ (2)
10. $\alpha = 1$, $C\alpha\beta = 1 \rightarrow C1\beta = 1 \rightarrow \beta = 1$ (1,7)

$\vdash CCCpqpp$ follows obviously by applying Ax.1 to $Cpp = 1$.[4]

In a similar way Johnson's five equations may be shown to be equivalent to a sufficient set of detachment-axioms in K and N. Like Peirce's and Frege's earlier postulate-sets for propositional calculus, and Russell and Whitehead's later one, Johnson's set of equations contains one, namely the third, which is derivable from the rest.[5]

For intuitionist implication Meredith's equational axioms are 1.$CpCqr = CCpqCpr$, 2.$CCppq = q$, 3.$CCCCpqqpp = CCCCqppqq$. The third of these is needed to prove the "Rule of Constancy", thus: $Cpp = CCppCpp$ (2) $= CCCqqCppCpp$ (2) $= CCCCqCppCppqq$ (3) $= CCCCCqpCqpCppqq$ (1) $= CCCppqq$ (2) $= Cqq$ (2). Given this, it is easy to prove $\vdash CpCqp$ and $\vdash CCpCqrCCpqCpr$ (Łukasiewicz's axioms for intuitionist implication) from 1 and 2, and detachment from 2. All this, indeed, could be obtained if instead of the cumbersome 3 we laid down the "Rule of Constancy" axiomatically; but 3 enables us also to represent any asserted two-way implication as an equation. For given $C\alpha\beta = C\beta\alpha = 1$ (call it 4) and the above axioms, we have

5. $CC\alpha\beta\beta = C1\beta = \beta$ (4,2)
6. $CC\beta\alpha\alpha = C1\alpha = \alpha$ (4,2)
7. $\alpha = CC\beta\alpha\alpha$ (6) $= CCCC\alpha\beta\beta\alpha\alpha$ (5) $= CCCC\beta\alpha\alpha\beta\beta$ (3) $= CC\alpha\beta\beta$ (6) $= \beta$ (5).

In the 2-valued equations, where $CCpqq = CCqpp$ is an axiom or provable, $\alpha = \beta$ follows from 5 and 6 above more directly.

In his note below, Meredith sketches the derivation of the classical equations in his G from his intuitionist ones in C, touching on the way upon other aspects of intuitionist logic which require no explanatory comment here.

A little more should be said, however, about the relation of Meredith's model to Łukasiewicz's. With Łukasiewicz's $NT\alpha N\beta$, detachment is only provable where β itself begins with N (or—this being just a special case of beginning with N—with Łukasiewicz's defined implication itself). With Meredith's G, detachment follows easily and without restriction from the fact that G is stronger than the intuitionist C, i.e. $CGpqCpq$ ($CCCCqppqCpq$) is intuitionistically valid. On the other hand, Łukasiewicz obtains a model of classical implication *and negation* within intuitionist, simply taking over intuitionist negation and combining it with his defined implication. Nothing of this sort is possible with Meredith's G; not only is $GNNpp$, for example, not provable with N for intuitionist negation (if it were, $CNNpp$, with intuitionist implication, would immediately follow by $CGpqCpq$), but we cannot even *introduce* a negation N or a constant falsehood 0 for which all the classical laws hold with G. We cannot, e.g., introduce a constant 0 with $G0p$ as a law, as this would expand to $CCCp00p$ (i.e. $CNNpp$ if we define Np as $Cp0$), which when subjoined to intuitionist implication gives all classical implicational laws, e.g. $CCCpqpp$.[6] Nor could we introduce an undefined N with $GpGNpq$ as a law, since from this and Czz we would get $GNCzzp$, i.e. $CCCpNCzzNCzzp$, from which $CCCpqpp$ is derivable by the same steps as from $CCCp00p$. In fact with any characteristic matrix for the intuitionist implication it will be found that in the derived matrix for G there is no element k such that Gkp has a designated value for all values of p. The same is true of some matrices which are not quite characteristic for intuitionist implication but do verify all its laws and falsify many classical ones, e.g. the well-known "H3",

C	1	2	0
* 1	1	2	0
2	1	1	0
0	1	1	1

With this for C, G works out as

G	1	2	0
* 1	1	2	0
2	1	1	0
0	1	2	1

which verifies all classical implicational laws but not $G0p$ (nor $G1p$ nor $G2p$).[7]

Returning to equational versions of propositional logic, it is easy to set these in a wider body of abstract algebra; and the same—though this is perhaps less obvious—may be done with substitution-and-detachment versions. We get glimpses of this wider mathematical context in sections 3 and 4. The letter which forms section 3 is in part Meredith's comment on a conversation with Professor H. G. Forder that I reported to him in 1957. Forder had suggested that a perfect illustration of "pure mathematics" would be to take an equation of uninterpreted symbol-strings or "words", e.g. $abcab = c$, and see what would follow from it. A little more must be said, of course. Follow by what rules? Interchangeability of equated words, or some equivalent rule, would be needed (Forder used "If $\alpha = \beta$, $\chi\alpha = \chi\beta$"), and substitutability of "words" for single letters, the same substitution throughout an equation; "words" being those symbol-sequences, and only those symbol-sequences, given by the following recursion: single letters are words, and if α and β are words so is $(\alpha\beta)$. The number of brackets may be lessened by a convention of association to the left, so that the given formula $abcab = c$, for example, is short for $((((ab)c)a)b) = c$. Or if one drops all right-hand brackets and replaces left-hand ones by a letter, one gets Łukasiewicz-type formulae, Forder's being $CCCCabcab = c$. Not that this law holds for any sort of implication; but interpretation can come afterwards, if at all. In his letter Meredith noted that using the variables for positive whole numbers, Forder's (ab) could be read as, $a + b$, his axiom yielding all equations actually holding with this interpretation.[8] Meredith also gives in this letter an equivalent detachment-axiom for this functor, "theses" being formulae which, under this interpretation, $= 0$ for all values of their variables. (This is a result of some years earlier, and we give in section 4 Meredith's 1950-52 derivation of this single detachment-axiom from two more straightforward ones.) Further, the function, $p + q$ is used to define the implication of Łukasiewicz's infinite-valued system[9] in terms of its alternation.

The 1957 letter also contains postulates for various 2-valued functors, but none for Johnson's pair K and N. This gap is filled in the note given as section 5 (the constant 0 is also used as a primitive here, but is easily eliminable). This note also illustrates a method, sketched in the 1957 letter, of setting out equational derivations analogously to the use of **D** in setting out detachments.

In the note which forms Section 6, *C* is used neither for two-valued nor for intuitionist implication, but for the "strict" implication of Lewis's S5. Meredith found in 1956 that *CCpqCCqrCpr*, *Cpp* and *CCCCpqrqCpq* formed a sufficient basis for pure S5 strict implication. Lemmon at the same time conjectured that *CCpqCCqrCpr*, *CCpCpqCpq* and *CCrpCqq* formed a sufficient basis for pure S3 strict implication, and the same with *CCrpCqq* strengthened to *CpCqq* for S4, and established that the following common set of additions in *C*, *N* and *K*:

1. *CKpqp*	5. *CpNNp*
2. *CKpqq*	6. *CNNpp*
3. *CCpqCCprCpKqr*	7. *CKpNKpNqq*
4. *CCpqCNqNp*	

yielded full S2, S4 and S5 when combined with the proposed implicational fragments[10]. Hacking later showed certain postulates to suffice for the strict implicational fragments of S2, S3, S4, S5 and T (the S3 and S4 ones being equivalent to Lemmon's) and gave common completions in *C*, *N*, *K* and *A* (the *C*-*N* additions sufficing for full *C*-*N* and the *C*-*K*-*A* ones for full *C*-*K*-*A*, a stratification not possible with Lemmon's basis).[11] In 1963, using *F* for strict implication and *C* for material, Meredith gave a basis for full S5 in *F*, *C* and a constant impossibility 0 (the *F*-*C* and *F*-0 additions sufficing for full *F*-*C* and *F*-0)[12]. It may be noted that if *Np* is defined not as *Cp*0 but as *Fp*0 it will assert not merely *p*'s falsehood but its impossibility, and gives the *N* of what Hacking calls the "Lewy systems". And in Section 6 below, which is of date 1959, Meredith gives an equational basis for *C*-*K*-*A*-0-1 (*C* strict).

His immediate problem (originally suggested to him by Professor E. Furlong) is to introduce a form of negation for which the law of contradiction does not hold. His starting-point is a 4-axiom equational logic with *K*, *A*, 1 and 0 as primitives, and into this is introduced an implication *C* defined by the conditions that $Cpq = 1$ if and only if $Kpq = p$, and otherwise $Cpq = 0$. This is equivalent to a definition of "formal" implication given in Johnson's *Logical Calculus* articles. Johnson defines material implication in the ordinary way as $\overline{x\bar{y}}$, but α "formally" implies β, he says, if and only if it is equivalent to a conjunction in which β is a conjunct, i.e. if and only if we can establish something of the form

$$\alpha = \ldots \beta \qquad (1)$$

Meredith's variant is

$$\alpha = \alpha\beta \qquad (2)$$

It is clear that if we have (2) we have (1), (2) being the case of (1) in which the gap is filled by α, and conversely if we have (1) we have (2), for with $xx = x$ (1) gives

$$\alpha = \ldots \beta\beta$$

and replacement of $\ldots \beta$ in this by α, in virtue of (1), gives (2). And if we have $\vdash\alpha$ for $\alpha = 1$, the asserted formulae in C which are given by Meredith's conditions and his equations for K, A, 0 and 1 are precisely the formulae of S5 (and the formulae in C, K, A, 0 and 1 are precisely the C-K-A-0-1 portion - C strict - of S5).

To illustrate rather than prove Meredith's last point, we may disprove the C-material thesis $CCCpqpp$ and prove the C-strict $CCCCpqrqCpq$ for this equational C. To disprove $CCpqpp$, let p take a value between 0 and 1 (there being nothing in the given basis to prevent this), and q be 0. Then $Kpq = Kp0 = 0 \neq p$, so $Cpq = 0$, and $CCpqp = C0p = 1$ (for $0 = K0p$); and $KCCpqpp = K1p = p \neq 1 \neq CCpqp$, i.e. $CCCpqpp = 0$. For $CCCCpqrqCpq$ consider the two possibilities $Cpq = 1$ and $Cpq = 0$. If $Cpq = 1$, $CCCpqrq = KCCCpqrqCpq$ ($p = Kp1$), i.e. $CCCCpqrqCpq = 1$. If, on the other hand, $Cpq = 0$, then (i) $C1q = 0$ (for if $C1q = 1$, i.e. $1 = K1q$, then $q = 1$, since $K1q = q$; and so $Cpq = 1$, since $Cp1 = 1$, by $1 = Kp1$; but $Cpq = 0$, *hyp.*). Hence: (ii) $C1q = CK1q0$ ($0 = Kp0$), so $CC0rq = KCC0rq0$ ($C0r = 1$), so $CCCpqrq = KCCCpqrqCpq$ ($Cpq = 0$, *hyp.*), i.e. $CCCCpqrqCpq = 1$.

The apparatus so far used, even though it includes 0, is not sufficient to give us negation (though $Cp0$ gives p's impossibility). And if we introduce N as a new primitive with the axiom $Apq = NKNpNq$ (de Morgan), this gives us many characteristic laws, but not $ApNp = 1$ or $KpNp = 0$. These equations hold when p is itself 0 or 1 but are not provable for other values of p. Meredith does, however, obtain a classical negation by introducing the notion of a "world", in the sense of a proposition w such that for all p either $Kpw = 0$ or $Kpw = w$[13]. If we define N_0p as the disjunction of all worlds such that $Kpw = 0$, all classical laws for N are now deducible in N_0. The definition of "world", it may be noted, strictly speaking covers 0 itself; we can generally ignore this case, but it is needed to give a sense to N_01, and sometimes to other negations.

These conditions for 0, 1, w, N_0 are met by some curious examples. Suppose, for instance, we have a finite set of "atomic" propositions a, b, c, etc., and all non-atomic propositions are constructed out of these by A and K. All the laws for 0, e.g. the "Rule of Non-Significance" $Kp0 = 0$, will then be met by the logical product of the whole set of atoms, a, b, c, etc. This is so to speak the most that can be said with this apparatus, so that anything we conjoin with it will add nothing to the total content. (The importance of finding "an expression for *HOW MUCH a proposition says*", a "measure of amount-that-is-said", and the fact that by all the obvious criteria "the proposition that says the most" turns out to be "contradiction", seem to have impressed Wittgenstein when he was writing the *Tractatus*.[14] And the interest of the present model is that it takes over from ordinary contradiction *nothing but* this "saying-the-maximum", making $Kp0 = 0$ simply a special case of $KpKpq = Kpq$). The laws for tautology or 1, e.g. the "Rule of Insignificance" $Kp1 = p$, will similarly be met by the logical *sum* of all the atoms. Further, any logical product of all the atoms but one will have the specified properties of a "world"—conjoin with it the one non-contained atom, or anything implying that, and we obtain the totality 0, and anything *not* implying that atom will be implied by the conjunction of all the others. Every atom will then be contained as a conjunct in all "worlds" but one, and that one will be its "negation", i.e. it will be the one w (and so the disjunction of "all" the w's) such that, for that atom a, Kaw = the totality 0. A logical product of n atoms will have for its "negation" the logical sum of the n worlds from which these atoms are respectively absent. The ordinary A-K equations will suffice to prove KpN_0p for every p constructible in this system; and we have $ApN_0p = 1$ if we add a postulate equating each atom with the disjuction of its worlds.

§2. *A Model of C-Classical in C-Positive*

(1) Equational axioms for positive implication:

1. $CpCqr = CCpqCpr$
2. $CCppq = q$
3. $CCCCpqqpp = CCCCqppqq$

(2) For 2-valued:

1. $CpCqr = CqCpr$
2. $CCpqp = p$
3. $CCpqq = CCqpp$

(3) Adding K to positive logic causes little disturbance, since:

4.1. $CKpqr = CpCqr$
4.2. $CpKqr = KCpqCpr$

(4.1 is a sufficient equational axiom, since from it $CpCqKpq = 1$, $CKpqp = 1$, $CKpqq = 1$)

(4) (a) $CCpqp = KCqpCCpqq$,
for $CCCpqpCqp$ ($CCCpqrCqr$: T1)
and $CCCpqpCCpqq$ (T2)
and $CCqpCCCpqqCCpqp$ ($CCqrCCpqCpr$: T3)

(b) $CCCpqqp = KCqpCCCpqpp$,
for $CCCCpqqpCqp$ (by T1)
and $CCCCpqqpCCCpqpp$ ($CCCCpqqrCCCpqpr$, by T2 and Syll : T4)
and $CCqpCCCCpqppCCCpqqp$ ($CCqrCCCprsCCpqs$: T5)

(c) $CCCCpqqpp = CCpqCCCCpqppp$
$= CCqpCCpqp$ ($Cqp = CCCqppp$)
$= CCqpCCpqq$ ($CpKpq = Cpq$, and (a))
$= CCpqCCpqp$ ($CpCqr = CqCpr$)
$= CCCCqppqq$

(5) We now find 14 meaningfully distinct expressions in p, q in Positive (implicational) Logic. The addition of K gives only 4 more, so:

There are 18 *meaningfully distinct expressions in* p, q *in Intuitionist* $\{C\text{-}K\}$ *logic.*

The matrix of these expressions can be expressed as the product $H_3 \times H_3 \times H_2$, where H_3:

C	1	2	3
1	1	2	3
2	1	1	3
3	1	1	1

K	1	2	3
1	1	2	3
2	2	2	3
3	3	3	3

and H_2:

C	1	2
1	1	2
2	1	1

K	1	2
1	1	2
2	2	2

The 18 values are:

(1) $Cpp = 111 = Cqq$

(p_1)	$p = 322$	(q_1)	$q = 232$
(p_2)	$Cqp = 311$	(q_2)	$Cpq = 131$
(p_3)	$CCqpp = 122$	(q_3)	$CCpqq = 212$
(p_4)	$CCpqp = 312$	(q_4)	$CCqpq = 132$
(p_5)	$CCCpqqp = 321$	(q_5)	$CCCqppq = 231$
(p_6)	$CCCpqpp = 121$	(q_6)	$CCCqpqq = 211$

(7) $CCCCpqqpp = 112 = CCCCqppqq$

(K) $Kpq = 332$

(E) $Epq = KCpqCqp = 331$

(V) $Vpq = KCCpqqCCqpp = 222$

(U) $Upq = KCCCpqppCCCqpqq = 221$

With H_3:

	p	Cqp	$CCqpp$	$CCpqp$	$CCCpqpp$	$CCCCpqqpp$
1	1 1	1 1 1	1 1 1	1 1 1	1 1 1	1 1 1
2	2 ← 2	2 1 ← 1	1 2 ← 2	2 2 ← 1	1 1 ← 1	1 2 ← 1
3	3 ↗ 3	3 3 ↗ 1	1 1 ↗ 3	3 3 ↗ 3	1 1 ↗ 1	1 1 ↗ 3

Kpq	Epq	Vpq	Upq
1 2 3	1 2 3	1 1 1	1 1 1
2 2 ← 3	2 1 ← 3	1 2 ← 2	1 1 ← 2
3 3 ↗ 3	3 3 ↗ 1	1 2 ↗ 3	1 2 ↗ 1

(Note by A.N.P.: By saying that these 18 expressions have an $H_3 \times H_3 \times H_2$ matrix, Meredith means that if we apply C or K to any pair of expressions drawn from these 18 we obtain an expression equivalent to one of the 18, and the table showing which it is in each case has the structure indicated.

The C-table can be so drawn up as to consist of four 9 X 9 parts, three the same and one different, arranged thus:

a	b
a	a

and each 9 X 9 part has nine 3 X 3 parts arranged thus:

a	b	c
a	a	c
a	a	a

each 3 X 3 part having the same arrangement. One of the 9 X 9 parts would be this:

C	1	q_6	p_2	p_6	U	p_5	q_2	q_5	E
1	1	q_6	p_2	p_6	U	p_5	q_2	q_5	E
q_6	1	1	p_2	p_6	p_6	p_5	q_2	q_2	E
p_2	1	1	1	p_6	p_6	p_6	q_2	q_2	q_2
p_6	1	q_6	p_2	1	q_6	p_2	q_2	q_2	E
U	1	1	p_2	1	1	p_2	q_2	q_2	E
p_5	1	1	1	1	1	1	q_2	q_2	q_2
q_2	1	q_6	p_2	1	q_6	p_2	1	q_6	p_2
q_5	1	1	p_2	1	1	p_2	1	1	p_2
E	1	1	1	1	1	1	1	1	1

There will be an analogous table for K.)

(6) (a) $Gpq = CCCqppq$ (gives equations for 2-valued)

(b) $GGpqq = CCCqGpqGpqq$
$= CGpqq$ $(CqGpq = 1)$
$= CCCCqppqq$
$= CCCCpqqpp$
$= GGqpp$

(c) $GGpqp = CCCpGpqGpqp$
$= CCpqGpqp$ $(CpCCCqppq = Cpq)$
$= CCCpqCCCqppqp$
$= CCCCqpqqp$ $(Cq_2q_5 = q_6)$
$= p$ $(Cq_6p_1 = p_1)$

(d) $GpGqr = CCCCCCrqqrppCCCrqqr$
$= CCCrppCCCrqqr$ $(CfCpqCpr = CfqCpr)$
$= CCCrqqCCCrppr$
$= GqGpr$

(e) $\vdash Gpp$ $(\vdash CCCpppp)$

By contrast with Łukasiewicz's functor (which was virtually $NNCpq$) Gpq is stronger than Cpq.

§3. *Equational Miscellany* (Letter of December 13, 1957).

(1) Equational axioms:

$\{C,0\}$ 1. $CCp0Cqr = CCrpCqp$, 2. $CCpqp = p$, or in place of 2, 2a. $CCppq = q$.

$\{C,N\}$ 1. $CpCqp = CCrpCqp$, 2. $CCpqp = p$, or in place of 2, 2b. $CCrCppq = q$. *Not* 2a ($C = E$, N = Verum).

$\{A,N\}$ 1. $ApAqr = CCrpAqp$ and 2 or 2a as before ($C = AN$).

$\{A,K,N,0,1\}$ 1. $ApKqr = KArpAqp$, 2. $ApKqp = p$ ($\{A - K\}$ complete), 3. $ApNp = 1$, 4. $KpNp = 0$.

$\{E\}$ There are many single axioms, e.g. $EEEpqrEpr = q$ (elementary precaution: the single letter must occur neither first nor last on the other side, e.g. $CCpqp = p$ is satisfied either by $Cpq = p$ or $Cqp = q$).

$\{L\}$ Lpq being, $p + q$ (Abelian groups). $LLLLpqrq = r$ (this is H.G.F.'s axiom). (With zero designated and L-detachment $LLpqLLLrqpr$ is axiom.)

(2) Formalization of equational reasoning in systems not involving variable functors:

(i) Primitive form: Euclide $\alpha = \beta$, $\alpha = \gamma \rightarrow \beta = \gamma$. If the numberings of the three equations were n_1, n_2, n_3 I will write $n_1 \times n_2 - n_3$.

(ii) If n_1 is $\alpha_1 = \beta_1$ and n_2 is $\alpha_2 = \beta_2$, Cn_1n_2 denotes the equation $C\alpha_1\alpha_2 = C\beta_1\beta_2$, the two equations being supposed to have no variables in common (e.g. for $\{C\text{-}N\}$ above $C22$ is $CCCpqpCCrsr = Cqs$). Similarly of course for any other functors, instead of C, i denoting the equation $p = p$, Cn_1i denotes $C\alpha_1x = C\beta_1x$ where x is a new variable.

(iii) Suppose α_1, α_2 are conformed to α with maximum generality and that $S_1\alpha_1 = S_2\alpha_2 = \alpha$ (Identity =), then $n_1 \times n_2$ is the equation $S_1\beta_1 = S_2\beta_2$ (S_1, S_2 denote substitutions). Thus for $\{C\text{-}N\}$ axioms 1, 2:

$2 \times C2i$	- 3.	$CpCpq = Cpq$
$1 \times Ci2$	- 4.	$CCrpCCrqp = CNpr$
3×4	- 5.	$CCrpp = CNpr$
5×2	- 6.	$CNpp = p$
$5 \times C6i$	- 7.	$CNpNp = Cpp$
$1 \times i$	- 8.	$CCrpCqp = CNpCqr$
$Ci8 \times 1$	- 9.	$CNsCNpCqr = CCCqpsCCrps$
9×3	- 10.	$CCCqppCCrpp = CNpCqr$

(Note: even if i is not an axiom, nor deducible by the normal mode of reasoning, we can get it by this formal process, since $i \times i$ is i. Thereby this useless equation is relegated to its proper position as a compound in the deductive process.) For abbreviated record I use εmn instead of $m \times n$. Thus $\varepsilon 3\varepsilon 1Ci2 = 5$.

(3) More of $\{L\}$: Let $p, q \ldots$ be integers (+ 0 -) or indeed real. Extend designated values to ≥ 0. Let $Apq = \max(pq)$, $Cpq = ApLpq$. Then $Apq = CLpqq$. $\{C,L,p\}$ with C-detachment is easily axiomatisable and is saturated. $\{C\}$-pure is $L\chi_0$. Negation is impossible (strictly $C0p$ leads to a contradiction). $Kpq = \min(p,q) = LCLpqLpqq$.

Non-commutative groups can be dealt with in the same way as L but I haven't got single axioms—either with L detachment or equationally. We might of course take $Rpq = p - q$ (here an A leak is impossible. 0 alone is designated). R-detachment axiom $RRpqRRprRqr$.

§4. *Abelian Groups*

	1.	$CCpqCCqrCpr$
	2.	$CCCpqqp$
D11 =	3.	$CCCqrCprsCCpqs$
D31 = D32 =	4.	$CCpqCCCprsCCqrs$
D3D42 =	5.	$CCsCCpqqCCprCsr$
DD452 =	6.	$CCCCprCsrCCpqqs$
D66 =	7.	$CpCCCCprqqr$
D57 =	8.	$CCCCpqqrCpr$
D83 =	9.	$CCqrCCpqCpr$
D19 =	10.	$CCCpqCprsCCqrs$
DD10.10.2 =	11.	$CCpqCCCrqpr$

	1.	$CCqrCCpqCpr$
	2.	$CCCpqqp$
D12 =	3.	$CCrCCpqqCrp$
D13 =	4.	$CCsCrCCpqqCsCrp$
D41 =	5.	$CCpqCCCrqpr$

(5 = DD1D121 = λxDD12D1x = $\lambda x\lambda y$D2DD1xy = $\lambda x\lambda y$D2λzDxDyz = D$\lambda a\lambda b\lambda cDa\lambda dDbDcd$2. If a = -$CCCpqrCCsqCCpsr$, 5 =Da2)

§5. *{K-N} Equational*

(Note by A.N.P.: In the form given, i.e. with 0 as well as K and N, these four equations have the same total length—29 letters—as Huntington's 1933 three, viz. $Kpq = Kqp$, $KKpqr = KpKqr$, $p = KNKNpNqNKNpq$. With 0 in the last axiom replaced by $KqNq$, to give pure $\{K\text{-}N\}$, they are three letters longer; but Meredith's first two, unlike Huntington's, suffice for pure $\{K\}$. His opening deductions in effect show this, since given his ninth equation, $Kqp = Kqp$, we can get $KKpqr = KpKqr$ from the second axiom, and so can put any K-sequence into a standard order and grouping—say the variables in alphabetical order and all the K's at the beginning—and eliminate repetitions by Ax.1. Equations thus standardised will either have exactly the same variables on both sides, in which case they will be provable as substitutions in $p = p$, or not, in which case the standard matrix for K will refute them, if we put 1 for all variables on one side and 0 for one of the different variables on the other. The importance of equations of the form $K\alpha\beta = \alpha$, at the end, will become clear in section §6.)

1. $Kpp = p$
2. $KpKqr = KqKrp$
3. $KpNKpNq = Kpq$
4. $KpNp = 0$

$\varepsilon 2Ki1$ - 5. $KpKpq = Kqp$
$\varepsilon 25$ - 6. $KpKqp = Kqp$
$\varepsilon 21$ - 7. $KpKqKpq = Kpq$
$\varepsilon Ki67$ - 8. $KpKpq = Kpq$
$\varepsilon 58$ - 9. $Kpq = Kqp$
$\varepsilon 3KiN3$ - 10. $KpKpNq = KpNKpq$
$\varepsilon(10)5$ - 11. $KpNKpq = KNqp$
$\varepsilon 3(11)$ - 12. $Kpq = KNNqp$
$\varepsilon(12)1$ - 13. $KNNpp = p$
$\varepsilon(12)i$ - 14. $KNNqp = Kpq$
$\varepsilon(14)1$ - 15. $KNNpp = NNp$
$\varepsilon(15)(13)$ - 16. $NNp = p$
$\varepsilon 3KiN4$ - 17. $Kpp = KpN0$
$\varepsilon(17)1$ - 18. $KpN0 = p$

? $KpNKqNp = p$:

$\varepsilon KiN(18)3$ - 19. $KpNp = Kp0$
$\varepsilon(19)4$ - 20. $Kp0 = 0$
$\varepsilon(2)Ki4$ - 21. $KpKNpq = Kq0$
$\varepsilon(21)Ki9$ - 22 $Kq0 = KpKqNp$
$\varepsilon(22)(20)$ - 23. $KpKqNp = 0$
($KpNKpNKqNp = KpKqNp = 0$)
($p = KpN0 = KpNKpNKpNKqNp =$
$KpKpNKqNp = KpNKqNp$)
$\varepsilon 3i$ - 24. $Kpq = KpNKpNq$
$\varepsilon(24)(23)$ - 25. $KpNKpNKqNp = 0$
$\varepsilon KiN(25)3$ - 26. $KpN0 = KpKpNKqNp$
$\varepsilon(26)(18)$ - 27. $KpKpNKqNp = p$
$\varepsilon(8)(27)$ - 28. $KpNKqNp = p$

$\alpha = \beta \sim K\alpha N\beta = 0,\ K\beta N\alpha = 0$

$K\alpha N\beta = 0 \rightarrow K\alpha NK\alpha N\beta = \alpha$
$\rightarrow K\alpha\beta = \alpha$

§6. *Negation without the Law of Contradiction.*

(1) Take $\{K, A, p\}$ normal, equationally, e.g.:

$KpApq = p$ A1
$KpAqr = AKrpKqp$ A2

(2) 1 and 0 are easy to add for a finite set of elements. Otherwise I will add:

$Kp1 = p$ A3
$Ap0 = p$ A4

(3) (i) $Cpq = 1$ if and only if $Kpq = p$ (or equally $Apq = q$). This gives the rule:

$$Cpq = 1,\ Cqr = 1, \rightarrow\ Cpr = 1$$

Also: $CpApq$, $CAppp$, etc.

(ii) *Otherwise* $Cpq = 0$ (if $Kpq = p$, $Cpq = 1$; if $Kpq \neq p$, $Cpq = 0$):

$$\begin{array}{l} CCpqCCqrCpr \\ Cpp \\ CCCCpqrqCpq \\ C0p \end{array}$$

(4) If to A1, A2 we add:

$$Apq = NKNpNq \qquad \text{B1}$$

we can prove $NNp = p$, $N1 = 0$, $N0 = 1$. Further if $p = Kpq$, then $Np = NKpq = ANpNq$. Thus if $Cpq = 1$, $CNqNp = 1$. By the two-valuedness of Cpq, $Cpq = CNqNp$.

But no single-N thesis can be proved. There is nothing contradictory in assuming an $\alpha \neq 0,1$ such that $N\alpha = \alpha$. α is not a logical necessity but may be true: $C1\alpha = 0 = C\alpha 0$

(5) Construction of a $\{K, A, N, p\}$ from two $\{K, A, p\}$ systems:

p is the ordered pair (p_1, p_2); $Kpq = (Kp_1q_1, Ap_2q_2)$. (In this definition of the new K for pairs I would use a new letter); and $N(p_1, p_2) = (p_2, p_1)$. Then:

$$\begin{aligned} KNpNq &= (Kp_2q_2, Ap_1q_1) \\ NKNpNq &= (Ap_1q_1, Kp_2q_2) \end{aligned}$$

We may define Apq as $NKNpNq$; the new 1 as (1, 0); and the new 0 as (0, 1). (If $p_1 = p_2$, $p = Np$) Then:

$KpNp = (Kp_1p_2, Ap_1p_2)$ (This is 0 only if $Kp_1p_2 = 0$, $Ap_1p_2 = 1$).

If $\alpha = (1, 1)$, $Kp\alpha = (p_1, 1)$, $Ap\alpha = (1, p_2)$. So: $AKp\alpha Ap\alpha$

In a closed $\{K, A, p\}$ set of propositions a (possible) world w is a p such that for all q, either $Kqp = 0$ or $Kpq = p$. If a set is constructible from its worlds, i.e. for all p, p is the logical sum of all w such that $Kpw = w$, and $\phi(w)$ is a function of the w's on w's such that $\phi(\phi(w)) = w$, and we extend the range of ϕ to all p's by $\phi(p)$ = logical sum of all $\phi(w)$ such that $Kpw = w$, we will have $\phi Apq = A\phi p\phi q$, $\phi Kpq = K\phi p\phi q$, $\phi\phi p = p$, $\phi 0 = 0$, and $\phi 1 = 1$.

If we define an N_0p by N_0p = logical sum of all w such that $Kpw = 0$, we have $\phi N_0p = N_0\phi p$, and we have also the standard:

$$\begin{array}{ll} (1) & KpN_0p = 0 \quad ApN_0p = 1 \\ (2) & N_0KN_0pN_0q = Apq \end{array}$$

Now let $Np = \phi N_0p$. Then:

$$\begin{aligned} NKNpNq &= \phi N_0K\phi N_0p\phi N_0q \\ &= \phi\phi N_0KN_0pN_0q \\ &= \phi\phi Apq \\ &= Apq \end{aligned}$$

so that de Morgan is preserved.

But $KpNp = KpN_0\phi p$ is not, in general, constant. If for any α, $\phi\alpha = N_0\alpha$, $K\alpha N\alpha = \alpha$. For such α to occur ϕ must satisfy:

$$\phi w \neq w \text{ for all } w$$

If there are $2n$ worlds:

$$w_1\, w_2, \ldots, w_n, \phi w_1, \ldots, \phi w_n$$

then there are 2^n such α's.

Examples: But say we take just three worlds $w_1\, w_2\, w_3$ with $w_2 = \phi w_3$,

$p =$	0	w_1	w_2	w_3	Aw_2w_3	Aw_3w_1	Aw_1w_2	1
		(p_1)	(p_2)	(p_3)	(p_4)	(p_5)	(p_6)	
$Np =$	1	p_4	p_6	p_5	p_1	p_3	p_2	0
$KpNp =$	0	0	p_2	p_3	0	p_3	p_2	0

Suppose we have 3 "atomics", without negation. They generate a set of 18 propositions:

$$\begin{aligned}
0 &= KKpqr \\
1 &= AApqr \\
w_1 &= Kqr \quad w_2 = Krp \quad w_3 = Kpq \\
v_1 &= Aw_3w_2 = KpAqr \text{ etc. for } v_2 \text{ and } v_3. \\
x &= KApqKAqrApr = AKpqAKqrKpr \\
&p \\
&q \\
&r \\
\bar{v}_1 &= ApKqr \text{ etc. for } \bar{v}_2 \text{ and } \bar{v}_3. \\
\bar{w}_1 &= Aqr \text{ etc. for } \bar{w}_2 \text{ and } \bar{w}_3.
\end{aligned}$$

Now $Np = p$, $Nq = q$, $Nr = r$, gives $Nx = x$, $Nw_1 = w_1$, $Nv_1 = v_1$ etc. (Note A.N.P.: It is instructive to compare Meredith's logic without $NKpNp$ with the $\{K\text{-}A\}$ system discussed in C. L. Hamblin's "One-valued Logic", *Philosophical Quarterly* (1967), pp. 38-45).

NOTES

1. J. Łukasiewicz, "On the Intuitionist Theory of Deduction," *Indagationes Mathematicae*, Vol. 14, No. 3, 1952.
2. W. E. Johnson, "The Logical Calculus," *Mind* 1892, pp. 3-30, 235-250, 340-351. Compare the equivalent systems in E. V. Huntington's, "New Sets of Independent Postulates for the Algebra of Logic," *Transactions of the American Mathematical Society*, Vol. 35 (1933), pp. 274-304.
3. Cf. Wittgenstein, *Tractatus*, 4. 461.
4. The completeness of Meredith's other (3-equation) base is shown in a different way in J. A. Kalman's "Equational Completeness and Families of Sets closed under Subtraction," *Koninkl. Nederl. Akademie van Wetenschappen*, Proceedings Series A, 63, No. 4, and *Indagationes Mathematicae*, 22, No. 4, 1960.

5. This was pointed out by J. A. Kalman in a letter of October 1959. Kalman also drew my attention to the fact that E. V. Huntington has derived both the third and the fourth from Johnson's first two plus the fifth modified to $x = \overline{xy}\,\overline{xy}$. See his "Boolean Algebra: A Correction," *Transactions of the American Mathematical Society*, Vol. 35 (1933), pp. 557-8.

6. For this derivation see A. Church, *Introduction to Mathematical Logic*, pp. 84-6, Exercise 12.6 and references.

7. For a connected observation by Łukasiewicz on the same matrix, see C. Lejewski, "On Implicational Definitions," *Studia Logica* VIII (1958), pp. 202-3.

8. For a much more general result of this sort see Graham Higman and B. H. Neumann, "Groups as Groupoids with one Law," *Publicationes Mathematicae*, Debrecen, 2 (1952), pp. 215-221.

9. For details of this see Tarski's *Logic, Semantics, Metamathematics*, Paper IV.

10. E. J. Lemmon, C. A. Meredith, A. N. Prior and I. Thomas, "Calculi of Pure Strict Implication" (mimeograph, University of Canterbury, Christchurch, New Zealand).

11. Ian Hacking, "What is Strict Implication," *The Journal of Symbolic Logic*, vol. 28, no. 1 (March 1963), pp. 51-71.

12. C. A. Meredith and A. N. Prior, "Investigations into Implicational S5," *Zeitschrift für Mathematik und Logik*, vol. 10 (1964), pp. 203-220.

13. Cf. C. A. Meredith and A. N. Prior, "Modal logic with functional variables and a contingent constant," *Notre Dame Journal of Formal Logic*, vol. 6 (1965), pp. 99-109.

14. See his *Notebooks 1914-1916*, edited and translated by G. E. M. Anscombe (Blackwells, 1961), p. 54. Miss Anscombe drew my attention to this parallel.

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