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SUMS OF AT LEAST 9 ORDINALS

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1 *Introduction** Ordinal addition is noncommutative. For each positive integer n, let m_n be the maximum number of distinct values that can be assumed by a sum of n nonzero ordinals in all $n!$ permutations of the summands. Then for $n \geq 3$, $m_n < n!$; furthermore,

$$\lim_{n\to\infty} \frac{m_n}{n!} = 0.$$

Formulas for m_n were given by Erdös, [1], and Wakulicz, [3] and [5]; from these formulas it is readily established that for $n \geq 10$, $n \neq 14$,

$$m_n = 3^{4(k-(l\dot{-}1))-3(1\dot{-}l)}11^{1\dot{-}l}193^{l\dot{-}1},$$

where $n = 5k + l$ for k, l nonnegative integers with $l \leq 4$, and where for nonnegative integers r and s,

$$r \dot{-} s = \begin{cases} r - s \text{ for } r \geq s, \\ 0 \text{ for } r < s. \end{cases}$$

For positive integers n and k let Σ_n be the symmetric group on n letters and let E_n be the set of all k for which there exist n (not necessarily distinct) nonzero ordinals $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that

$$\sum_{i=1}^{n} \alpha_{\phi(i)}$$

takes on exactly k values as ϕ ranges over Σ_n. It is known that $E_n = \{1, 2, \ldots, m_n\}$ for $n = 1, 2, 3, 4, 6, 7$, and 8 ([2], [4], and [6]), but that $E_5 = \{1, 2, \ldots, m_5\} - \{30\}$ ([5]). In this paper we show that E_n is properly included in $\{1, 2, \ldots, m_n\}$ for all $n \geq 9$.

For every ordinal number $\alpha > 0$, let

(1) $$\alpha = \omega^{\lambda_1}a_1 + \omega^{\lambda_2}a_2 + \ldots + \omega^{\lambda_r}a_r$$

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be the (*Cantor*) *normal form of* α; here $r, a_1, a_2, \ldots, a_r$ are positive integers and $\lambda_1 > \lambda_2 > \ldots > \lambda_k \geq 0$ are ordinals. In (1), λ_1 is called the *degree of* α (written, $\deg \alpha$) and a_1, the *leading coefficient of* α. If $\deg \alpha = \lambda_1$, we call α a λ_1-*ordinal*. By the *remainder of* α we shall always mean $\omega^{\lambda_2} a_2 + \ldots + \omega^{\lambda_r} a_r$ (or zero, if $r = 1$).

Unless otherwise specified, variables will range over (nonnegative) integers.

2 *The Main Result* Let $n \geq 1$. The positive integers $k_1, k_2, \ldots, k_s$ are said to form an S_n-*system* if

$$\sum_{i=1}^{s} k_i = n,$$

and whenever $\sum_{i=1}^{t} l_i = n$, then $l_1 \prod_{i=2}^{t} (1 + l_i\, 2^{l_i - 1}) \leq k_1 \prod_{i=2}^{s} (1 + k_i\, 2^{k_i - 1})$.

In any S_n-system for $n \geq 3$, k_1 must be 1, and for $n \geq 9$ and $2 \leq j \leq s$, k_j is 4, 5, or 6 ([3], pp. 256-260); thus for $n \geq 9$, $s \geq 3$.

Lemma 1. *For* $n \geq 9$, *let* $1, k_2, \ldots, k_s$ *form an* S_n-*system and let* $\sum_{i=1}^{t} l_i = n$, *where* $l_1, l_2, \ldots, l_t$ *are positive integers that do not form an* S_n-*system.*

$$\textit{Then } l_1 \prod_{i=2}^{t} (1 + l_i\, 2^{l_i - 1}) \leq \prod_{i=2}^{s} (1 + k_i\, 2^{k_i - 1}) - 36.$$

We omit the routine proof of Lemma 1, which utilizes the calculations of [3], pp. 256-260 and [5], p. 239. We now show that for $n \geq 9$, E_n is properly included in $\{1, 2, \ldots, m_n\}$. To this end we define $s_n = \max(E_n - \{m_n\})$, $n \geq 1$. It suffices to show that $s_n < m_n - 1$ for $n \geq 9$.

Theorem 1. *For all* $n \geq 9$, E_n *is properly included in* $\{1, 2, \ldots, m_n\}$. *In fact,*

(a) *for* $n = 9, 10, 14, 15,$ *and* 20, $s_n = m_n - 4$;
(b) *for* $n = 11, 12, 16, 17, 18,$ *and for all* $n \geq 21$, $s_n = m_n - 8$;
(c) *for* $n = 13$ *and* 19, $s_n = m_n - 16$.

Proof. For any $n \geq 1$, let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be any ordinals of which

$$(2) \quad \begin{cases} k_i \text{ are of degree } \lambda_i,\ i = 1, 2, \ldots, s, \\ \text{where } \lambda_1 > \lambda_2 > \ldots > \lambda_s, \text{ and } \sum_{i=1}^{s} k_i = n. \end{cases}$$

Erdös, [1] and Wakulicz, [3] showed that $\alpha_1, \alpha_2, \ldots, \alpha_n$ yield at most

$$k_1 \prod_{i=2}^{s} (1 + k_i\, 2^{k_i - 1})$$

distinct sums in all $n!$ permutations of these summands and that, in fact, there exist ordinals $\alpha_1, \alpha_2, \ldots, \alpha_n$ that yield m_n sums; here $k_1, k_2, \ldots, k_s$, of (2), form an S_n-system. Furthermore, for $n \geq 9$, it follows from Lemma 1 that for the purpose of our theorem we need only consider ordinals $\alpha_1, \alpha_2, \ldots, \alpha_n$, subject to (2), for which $k_1, k_2, \ldots, k_s$ form an S_n-system.

Let $n \geq 9$. Let

$$\alpha_1 \qquad = \omega^{\lambda_1} \mathscr{a}_{1,1} + a_{1,1} + \omega^{\lambda_2} \mathscr{a}_{1,2} + a_{1,2} + \ldots + \omega^{\lambda_s} \mathscr{a}_{1,s} + a_{1,s}$$

$$\begin{array}{lll}
\alpha_2 & = & \omega^{\lambda_2}\mathcal{a}_{2,2} + a_{2,2} + \ldots + \omega^{\lambda_s}\mathcal{a}_{2,s} + a_{2,s} \\
\alpha_3 & = & \omega^{\lambda_2}\mathcal{a}_{3,2} + a_{3,2} + \ldots + \omega^{\lambda_s}\mathcal{a}_{3,s} + a_{3,s} \\
\vdots & & \\
\alpha_{k_1+k_2} & = & \omega^{\lambda_2}\mathcal{a}_{k_1+k_2,2} + a_{k_1+k_2,2} + \ldots + \\
 & & \omega^{\lambda_s}\mathcal{a}_{k_1+k_2,s} + a_{k_1+k_2,s} \\
\vdots & & \\
\alpha_{k_1+k_2+\cdots+k_{s-1}+1} & = & \omega^{\lambda_s}\mathcal{a}_{k_1+k_2+\cdots+k_{s-1}+1,s} + a_{k_1+k_2+\cdots+k_{s-1}+1,s} \\
\alpha_{k_1+k_2+\cdots+k_{s-1}+2} & = & \omega^{\lambda_s}\mathcal{a}_{k_1+k_2+\cdots+k_{s-1}+2,s} + a_{k_1+k_2+\cdots+k_{s-1}+2,s} \\
\vdots & & \\
\alpha_n & = & \omega^{\lambda_s}\mathcal{a}_{n,s} \qquad\qquad + a_{n,s},
\end{array}$$

where $\mathcal{a}_{1,1}$ and $\mathcal{a}_{i,j}$ are positive integers for

$$\sum_{\mu=1}^{j-1} k_\mu < i \leq \sum_{\mu=1}^{j} k_\mu, \ 2 \leq j \leq s,$$

and $\mathcal{a}_{i,j}$ are nonnegative integers otherwise; $a_{i,j}$ is the sum of those terms of α_i which are of degree d, where $\lambda_{j+1} < d < \lambda_j$, $1 \leq i \leq n$ and $i \leq j < s$, and $a_{i,s}$ is the sum of those terms of α_i which are of degree $<\lambda_s$, $1 \leq i \leq n$.

Consider all ordinals of the form

$$\begin{array}{ll}
\beta_1 = \alpha_1 & \\
\beta_2 = \omega^{\lambda_2}\mathcal{a}_{(2)} + a_{i_2,2} + \omega^{\lambda_3}\mathcal{a}_{i_2,3} + a_{i_2,3} + \omega^{\lambda_4}\mathcal{a}_{i_2,4} + a_{i_2,4} + \ldots + \omega^{\lambda_s}\mathcal{a}_{i_2,s} + a_{i_2,s} & \\
\beta_3 = \qquad \omega^{\lambda_3}\mathcal{a}_{(3)} + a_{i_3,3} + \omega^{\lambda_4}\mathcal{a}_{i_3,4} + a_{i_3,4} + \ldots + \omega^{\lambda_s}\mathcal{a}_{i_3,s} + a_{i_3,s} & \\
\vdots & \\
\beta_s = \qquad \omega^{\lambda_s}\mathcal{a}_{(s)} + a_{i_s,s}, &
\end{array}$$

where for each j, $2 \leq j \leq s$, we have

$$\sum_{\mu=1}^{j} k_\mu < i_j \leq \sum_{\mu=1}^{j} k_\mu \tag{3}$$

and

$$\left\{\begin{array}{l} \mathcal{a}_{(j)} = \mathcal{a}_{i_j,j} + \sum \mathcal{a}_{i,j}(i \in J_j), \text{ where } J_j; \text{ is some subset (possible empty) of} \\ \left\{1 + \sum_{\mu=1}^{j-1} k_\mu, 2 + \sum_{\mu=1}^{j-1} k_\mu, \ldots, \sum_{\mu=1}^{j} k_\mu\right\} - \{i_j\}. \end{array}\right. \tag{4}$$

Then for any $\phi \in \Sigma_n$,

$$\sum_{i=1}^{n} \alpha_{\phi(i)} \text{ is of the form } \sum_{i=1}^{s} \beta_{\psi(i)},$$

where $\psi \in \Sigma_s$. It follows that either

$$\sum_{i=1}^{n} \alpha_{\phi(i)} = \beta_1 \text{ or else } \sum_{i=1}^{n} \alpha_{\phi(i)} \text{ is of the form}$$

$$\beta_1 + \sum_{\mu=1}^{t} \beta_{j_\mu}, \tag{5}$$

where $2 \leq j_2 < j_3 < \ldots < j_t \leq s$. Let $f(\mu) = k_{j_\mu}$, $\mu = 2, 3, \ldots, t$; then there are at most

$$\prod_{\mu=2}^{t} f(\mu)2^{f(\mu)-1}$$

possibilities corresponding to the index set

(6) $$\{j_1, j_2, \ldots, j_t\} = \{1, j_2, \ldots, j_t\}, \; t \geq 2$$

(and exactly one possibility for the former case).

Suppose there were less than $k_j 2^{k_j - 1}$ possibilities for some β_j, $2 \leq j \leq s$. Then corresponding to each index set (6) containing j, at least $k_j 2^{k_j-1}$ sums (5) would be eliminated. We shall show that $s_n > m_n - k_j 2^{k_j-1} \geq m_n - 32$ for each $j = 2, 3, \ldots, s$; thus we need only consider ordinals $\alpha_1, \alpha_2, \ldots, \alpha_n$ for which

(7) $\begin{cases} \text{there are precisely } k_j 2^{k_j-1} \text{ distinct sums for} \\ \text{each of the corresponding } \beta_j,\ 2 \leq j \leq s. \end{cases}$

(7) implies that there are precisely $1 + k_j 2^{k_j-1}$ distinct sums of the form $\beta_1 + \beta_j$ for each j, $2 \leq j \leq s$. Hence we need only be concerned with index sets (6) in which $t \geq 3$. We note further that considerations along the lines of [3], pp. 261-263, show that (7) implies that for each such j, if $u, v (\neq u)$ satisfy

$$\sum_{\mu=1}^{j-1} k_\mu < u \leq \sum_{\mu=1}^{j} k_\mu, \; \sum_{\mu=1}^{j-1} k_\mu < v \leq \sum_{\mu=1}^{j} k_\mu,$$

then α_u and α_v have distinct leading coefficients and distinct remainders. Consider a sum of the form (5) in which $t \geq 3$, and let $\Gamma = \{j_2, j_3, \ldots, j_t\}$ where $\{1\} \cup \Gamma$ is the corresponding index set (6). For $\gamma (= j_k) \in \Gamma$, let $\gamma' = j_{k-1}$ and let $\gamma^+ = j_{k+1}$ in case $k < t$. In the notation of (3) and (4) any sum of the form (5) can be written as

(8) $$\omega^{\lambda_1} \mathcal{A}_{1,1} + e_{1,1} + \Sigma(\omega^{\lambda_\gamma}(\mathcal{A}_{i_{\gamma'},\gamma} + \Sigma \mathcal{A}_{i,\gamma}(i \in J_\gamma) + \mathcal{A}_{i_\gamma,\gamma}) + e_{i_\gamma,\gamma}) \; (\gamma \in \Gamma),$$

in which $i_1 = 1$ and for each $\gamma \in \Gamma$, $e_{i_\gamma,\gamma}$ consists of all terms of α_{i_γ} that are of degree less than λ_γ and (in case $\gamma = j_k$ with $k < t$) greater than λ_{γ^+}. There is one possibility for $\langle \mathcal{A}_{1,1}, e_{1,1}, \mathcal{A}_{1,j_2} \rangle$, and by (7), there are $k_\gamma 2^{k_\gamma - 1}$ possibilities for

(9) $$\langle \Sigma \mathcal{A}_{i,\gamma}(i \in J_\gamma) + \mathcal{A}_{i_\gamma,\gamma}, e_{i_\gamma,\gamma}, \mathcal{A}_{i_\gamma,\gamma^+} \rangle$$

for each $\gamma \in \Gamma - \{j_t\}$ as well as for

(10) $$\langle \Sigma \mathcal{A}_{i,\gamma}(i \in J_\gamma) + \mathcal{A}_{i_\gamma,\gamma}, e_{i_\gamma,\gamma} \rangle$$

for $\gamma = j_t$. Consider all ordered $(t - 1)$-tuples whose ith coordinate is an ordered triple of the form (9) corresponding to $\gamma = j_{i+1}$, $i = 1, 2, \ldots, t - 2$, and whose $(t - 1)$st coordinate is an ordered pair of the form (10). Two such distinct ordered $(t - 1)$-tuples can yield the same sum (8) if and only if for some $\gamma \in \Gamma - \{j_t\}$, for $u, v (\neq u)$ such that

$$\sum_{\mu=1}^{\gamma-1} k_\mu < u \leq \sum_{\mu=1}^{\gamma} k_\mu, \; \sum_{\mu=1}^{\gamma-1} k_\mu < v \leq \sum_{\mu=1}^{\gamma} k_\mu, \; \sum_{\mu=1}^{\gamma^+ - 1} k_\mu < z \leq \sum_{\mu=1}^{\gamma^+} k_\mu$$

and for distinct subsets J and K of

$$\left\{1 + \sum_{\mu=1}^{\gamma^+-1} k_\mu, \; 2 + \sum_{\mu=1}^{\gamma^+-1} k_\mu, \ldots, \sum_{\mu=1}^{\gamma^+} k_\mu\right\} - \{z\},$$

$$e_{u,\gamma} = e_{v,\gamma},\ \mathcal{A}_{u,\gamma^+} \neq \mathcal{A}_{v,\gamma^+}, \text{ as well as} \tag{11}$$

$$\mathcal{A}_{u,\gamma^+} + \Sigma \mathcal{A}_{i,\gamma^+}(i \in J) = \mathcal{A}_{v,\gamma^+} + \Sigma \mathcal{A}_{i,\gamma^+}(i \in K). \tag{12}$$

If these conditions are met, then corresponding to each subset L of

$$\left\{1 + \textstyle\sum_{\mu=1}^{\gamma-1} k_\mu,\ 2 + \sum_{\mu=1}^{\gamma-1} k_\mu,\ \ldots,\ \sum_{\mu=1}^{\gamma} k_\mu\right\} - \{u,\ v\},$$

as well as to each r such that

$$\begin{cases} r = 1 \text{ if } \gamma' = 1, \\ \sum_{\mu=1}^{\gamma'-1} k_\mu < r \leq \sum_{\mu=1}^{\gamma'} k_\mu \text{ if } \gamma' > 1, \end{cases} \tag{13}$$

we have

$$\begin{aligned} &\omega^{\lambda_\gamma}(\mathcal{A}_{r,\gamma} + \Sigma \mathcal{A}_{i,\gamma}(i \in L \cup \{v\}) + \mathcal{A}_{u,\gamma}) + e_{u,\gamma} \\ &\qquad\qquad + \omega^{\lambda_{\gamma^+}}(\mathcal{A}_{u,\gamma^+} + \Sigma \mathcal{A}_{i,\gamma^+}(i \in J) + \mathcal{A}_{z,\gamma^+}) + e_{z,\gamma^+} \\ &= \omega^{\lambda_\gamma}(\mathcal{A}_{r,\gamma} + \Sigma \mathcal{A}_{i,\gamma}(i \in L \cup \{u\}) + \mathcal{A}_{v,\gamma}) + e_{v,\gamma} \\ &\qquad\qquad + \omega^{\lambda_{\gamma^+}}(\mathcal{A}_{v,\gamma^+} + \Sigma \mathcal{A}_{i,\gamma^+}(i \in K) + \mathcal{A}_{z,\gamma^+}) + e_{z,\gamma^+}. \end{aligned}$$

Consequently, at least $k_{\gamma'} 2^{k_\gamma - 2}$ of the sums (8) are repeated.

Moreover, if δ, ε, ζ are such that $1 \leq \delta < \varepsilon < \zeta \leq s$, then for any index set (6) containing $\delta = j_i$, $\varepsilon = j_{i+1}$, and $\zeta = j_{i+2}$, $1 \leq i \leq t - 2$, the above argument applies with δ, ε, ζ replacing γ', γ, γ^+, respectively. Thus for δ, ε, ζ as indicated, as many as

$$k_\delta 2^{\max\{0,\delta-2\} + k_\varepsilon - 2 + s - \zeta}$$

of the sums (8) might be eliminated.

We now show that for $\delta = 1$, $\varepsilon = 2$, and $\zeta = s$, there are, in fact, ordinals, $\alpha_1, \alpha_2, \ldots, \alpha_n$, subject to (2), for which $k_1(=1), k_2, \ldots, k_s$ form an S_n-system, and which yield precisely $m_n - 2^{k_2-2}$ sums. Let

$$\begin{array}{llll}
\alpha_1 & = \omega^{2s-2} & & \\
\alpha_2 & = \omega^{2s-3} + \omega^{2s-4} & & + \omega 100 \\
\alpha_3 & = \omega^{2s-3}2 + \omega^{2s-4} & & + \omega(2^{k_s} + 99) \\
\alpha_4 & = \omega^{2s-3}4 + \omega^{2s-4}2 & & \\
\alpha_5 & = \omega^{2s-3}8 + \omega^{2s-4}3 & & \\
\vdots & & & \\
\alpha_{k_1+k_2} & = \omega^{2s-3}2^{k_2-1} + \omega^{2s-4}(k_2 - 1) & & \\
\alpha_{k_1+k_2+1} & = & \omega^{2s-5} + \omega^{2s-6} & \\
\alpha_{k_1+k_2+2} & = & \omega^{2s-5}2 + \omega^{2s-6}2 & \\
\alpha_{k_1+k_2+3} & = & \omega^{2s-5}4 + \omega^{2s-6}3 & \\
\vdots & & & \\
\alpha_{k_1+k_2+k_3} & = & \omega^{2s-5}2^{k_3-1} + \omega^{2s-6}k_3 & \\
\vdots & & & \\
\alpha_{k_1+k_2+\ldots+k_{s-2}+1} & = & \omega^3 + \omega^2 & \\
\alpha_{k_1+k_2+\ldots+k_{s-2}+2} & = & \omega^3 2 + \omega^2 2 & \\
\alpha_{k_1+k_2+\ldots+k_{s-2}+3} & = & \omega^3 4 + \omega^2 3 &
\end{array}$$

$$
\begin{array}{lcll}
\vdots & & & \\
\alpha_{k_1+k_2+\ldots+k_{s-1}} & = & \omega^3 2^{k_{s-1}-1} + \omega^2 k_{s-1} & \\
\alpha_{k_1+k_2+\ldots+k_{s-1}+1} & = & & \omega + 1 \\
\alpha_{k_1+k_2+\ldots k_{s-1}+2} & = & & \omega 2 + 2 \\
\alpha_{k_1+k_2+\ldots+k_{s-1}+3} & = & & \omega 4 + 3 \\
\vdots & & & \\
\alpha_n & = & & \omega 2^{k_s-1} + k_s .
\end{array}
$$

Let $\beta_1, \beta_2, \ldots, \beta_n$ be as above. There are $k_j 2^{k_j-1}$ distinct possibilities for each such β_j. The only index set that we need consider with respect to repetition of sums is $\{1, 2, s\}$; for any other index set, $\{1, j_2, \ldots, j_t\}$, the corresponding sum (5) has exactly

$$\prod_{\mu=2}^{t} k_{j_\mu} 2^{k_{j_\mu}-1}$$

possibilities. Let $\gamma = 2$; then $\gamma' = 1$ and $\gamma^+ = s$. Let $u = 2$, $v = 3$,

$$z = 1 + \sum_{\mu=1}^{s-1} k_\mu , \quad J = \left\{2 + \sum_{\mu=1}^{s-1} k_\mu , \ 3 + \sum_{\mu=1}^{s-1} k_\mu , \ \ldots , n\right\},$$

$K = \phi$. Then (11) and (12) hold for these values of u, v, z, J, K, and for no others. There are 2^{k_2-2} subsets of $\{2, 3, \ldots, 1 + k_2\} - \{u, v\}$ as well as exactly one value of r satisfying (13); thus exactly 2^{k_2-2} of the sums (8) are eliminated. It follows that $s_n = m_n - 2^{k_2-2}$ for $n \geq 9$.

For an S_n-system, we reindex if necessary so that $k_2 = \min\{k_2, k_3, \ldots, k_s\}$. By [3], p. 260, we have $k_2 = 4$ for $n = 9$, 10, 14, 15, and 20; $k_2 = 5$ for $n = 11$, 12, 16, 17, 18, and for $n \geq 21$; $k_2 = 6$ for $n = 13$ and 19.

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