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VARIATIONS IN DEFINITION OF ULTRAPRODUCTS OF A FAMILY OF FIRST ORDER RELATIONAL STRUCTURES

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Two variations are made in the standard definitions, *cf.* [1], of an ultraproduct of a family of first order relational structures with respect to a chosen ultrafilter X of the index set I. The first variation, following a method used by W. A. J. Luxemburg, *cf.* [2] in the construction of higher order ultraproducts, relaxes the requirement of similarity on the members of the family. The second variation uses subfilters of X to define the individuals and relations of the ultraproduct.

In section 1 the construction of the ultraproduct with these variations is set out and some consequences developed, particularly those relating to the identity relation. In section 2 a family of similar structures is taken and a necessary and sufficient condition is established under which the first variation produces more relations, from an extensional view-point, than the standard definition.

1 Let $\{\mathbf{M}_i : i \in \mathsf{I}\}$ be a collection of first order relational structures. For each i, let $\mathbf{M}_i = \{R_i^0; R_i^1, R_i^2, \ldots\}$, where R_i^0 is the class (non empty) of individuals in the i^{th} structure and, for each positive integer k, R_i^k is the class of k-placed relations of the structure. Each R_i^k contains at least the empty relation and each R_i^2 contains the identity relation denoted by e_i. It is further assumed that the distinct members of each R_i^k are distinct from a set-theoretic and extensional point of view. Finally, if $a_1, \ldots, a_k \in R_i^0$ and $s^k \in R_i^k$ then "$s^k(a_1, \ldots, a_k)$" denotes the fact that $a_1, \ldots, a_k$ are related by s^k.

Let X be an ultrafilter defined on I. For each $k \geq 0$, X^k is a subfilter of X; that is X^k is a subclass of X and is a filter. For each $k \geq 0$, let R_{I}^k be the class $\{f^k : f^k : \mathsf{I} \to \bigcup \{R_i^k : i \in \mathsf{I}\}$ and for all $i \in \mathsf{I}, f^k(i) \in R_i^k\}$. Let $\sim_k$ denote the relation defined on R_{I}^k by: for all $f^k, g^k \in R_{\mathsf{I}}^k$, $f^k \sim_k g^k$ if, and only if, $\{i : f^k(i) = g^k(i)\} \in X^k$.

Lemma 1. *For each integer $k \geq 0$, $\sim_k$ is an equivalence relation.*

Proof: Immediate.

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For each integer $k \geq 0$, let $R_X^k k$ denote the quotient class of R_I^k with respect to $\sim_k$ and if $f^k \epsilon R_I^k$ let $\bar{f}^k$ denote its equivalence class. The next lemma prepares the way for the definition of the individuals and relations of the ultraproduct.

Lemma 2. *For each integer* $k \geq 0$, $\{i : f^k(i)(f_1^0(i), \ldots, f_k^0(i))\} \epsilon X$ *if, and only if,* $\{i : g^k(i)(g_1^0(i), \ldots, g_k^0(i))\} \epsilon X$, *where* $f_j^0 \sim_0 g_j^0$, *for each* j *from* 1 *to* k, *and and* $f^k \sim_k g^k$.

Proof: Let $F_j^0 = \{i : f_j^0(i) = g_j^0(i)\}$, for each j from 1 to k, and $F^k = \{i : f^k(i) = g^k(i)\}$. Now $F^k \cap F_1^0 \cap .. \cap F_k^0 \cap F_1 \subseteq F_2$ and $F^k \cap F_1^0 \cap .. \cap F_k^0 \cap F_2 \subseteq F_1$, where $F_1 = \{i : f^k(i)(f_1^0(i), \ldots f_k^0(i))\}$ and $F_2 = \{i : g^k(i)(g_1^0(i), \ldots g_k^0(i))\}$. But X^0 and X^k are subfilters of X. Hence $F_1 \epsilon X$ if, and only if, $F_2 \epsilon X$.

The ultraproduct, denoted by $\pi \mathbf{M}_i/(X; X^0, \ldots)$ can now be defined. The class of individuals is $R_{X^0}^0$. For each integer $k > 0$, and for each $\bar{f}^k \epsilon R_{X^k}^k$, a k-placed relation of the ultraproduct, denoted by the same symbol $\bar{f}^k$, is defined by: $\bar{f}^k(\bar{f}_1^0, \ldots, \bar{f}_k^0)$ if, and only if, $\{i : f^k(i)(f_1^0(i), \ldots, f_k^0(i))\} \epsilon X$, for all $\bar{f}_1^0, \ldots, \bar{f}_k^0 \epsilon R_{X^0}^0$. The symbol $R_{X^k}^k$ is also used to denote the class of k-placed relations of the ultraproduct.

Lemma 2, which justifies the definitions as given, has not required the 'ultra' property of X. If this requirement is dropped the definition provides a variation to the standard construction of reduced products. Further, it is noted that Łoś's theorem as stated for an ultraproduct in relation to a suitable first order language still holds for an ultraproduct defined as above.

The first result below establishes that from a set theoretic and extensional viewpoint the use of subfilters X^k, for $k > 0$, adds no extra relations to those gained by taking $X^k = X$.

Theorem 1. *For each* $k > 0$, *if* $f^k, g^k \epsilon R_I^k$ *such that* $\bar{f}^k \neq \bar{g}^k$ *but* $\{i : f^k(i) = g^k(i)\} \epsilon X$ *then for all* $\bar{f}_1^0, \ldots, \bar{f}_k^0 \epsilon R_{X^0}^0, \bar{f}^k(\bar{f}_1^0, \ldots, \bar{f}_k^0)$ *if, and only if,* $\bar{g}^k(\bar{f}_1^0, \ldots, \bar{f}_k^0)$.

Proof: $\bar{f}^k(\bar{f}_1^0, \ldots, \bar{f}_k^0)$ if, and only if, $\{i : f^k(i)(f_1^0(i), \ldots, f_k^0(i))\} \epsilon X$; that is if, and only if, $\{i : g^k(i)(f_1^0(i), \ldots, f_k^0(i))\} \epsilon X$, as $\{i : f^k(i) = g^k(i)\} \epsilon X$; that is if, and only if, $\bar{g}^k(\bar{f}_1^0, \ldots, \bar{f}_k^0)$.

From now on for all $k > 0$, X^k will be X itself. The next theorem establishes that for all $k > 0$, the distinct members of R_X^k provide distinct k-placed relations on an extensional basis.

Theorem 2. *For each* $k > 0$ *and* $f^k, g^k \epsilon R_I^k$, $\bar{f}^k \neq \bar{g}^k$ *if, and only if, there exist* $\bar{f}_1^0, \ldots, \bar{f}_k^0 \epsilon R_{X^0}^0$ *satisfying one, and only one, of the relations* $\bar{f}^k, \bar{g}^k$.

Proof: Assume $\bar{f}^k \neq \bar{g}^k$ and let $G = \{i : f^k(i) \neq g^k(i)\}$. Hence $G \epsilon X$. For each $i \epsilon G$, there exists $a_1^i, \ldots, a_k^i \epsilon R_i^0$ which satisfy one, and only one, of the relations $f^k(i), g^k(i)$, as $f^k(i) \neq g^k(i)$. Let $G_0 = \{i : i \epsilon G$ and $f^k(i)(a_1^i, \ldots, a_k^i)\}$ and $G_1 = \{i : i \epsilon G$ and $g^k(i)(a_1^i, \ldots, a_k^i)\}$. Now $G = G_0 \cup G_1$ and so either $G_0 \epsilon X$, $G_1 \notin X$ or $G_1 \epsilon X$, $G_0 \notin X$. Define, for each j from 1 to k, f_j^0 as follows: for all $i \epsilon G$, put $f_j^0(i) = a_j^i$; for all $i \notin G$ choose $f_j^0(i)$ some arbitrary member of R_i^0.

Hence $\bar{f}_j^0$ is uniquely defined as $G \in X$. Further, if $G_0 \in X$, $G_1 \notin X$ then $\bar{f}_j^0, j$ from 1 to k, satisfy the relation $\bar{f}^k$ but not $\bar{g}^k$, but if $G_1 \in X$, $G_0 \notin X$ then they satisfy $\bar{g}^k$ but not $\bar{f}^k$. Conversely, if $\bar{f}^k = \bar{g}^k$ then for all $\bar{f}_j^0 \in R_{X^0}^0$, j from 1 to k, $\bar{f}^k(\bar{f}_1^0, \ldots, \bar{f}_k^0)$ if, and only if, $\bar{g}^k(\bar{f}_1^0, \ldots, \bar{f}_k^0)$.

The next results are concerned with the way the identity relations in the component structures transfer to the ultraproduct. For technical reasons a short lemma is set out.

Lemma 3. *Let* $G = \{i : |R_i^0| = 1\}$. *If* X^0 *is a subfilter of ultrafilter* X *then* $R_{X^0}^0 \neq R_X^0$ *if, and only if, there is an* $F \in X$ *such that* $F \supseteq G$ *and* $F \notin X^0$.

Proof: Assume that $R_{X^0}^0 \neq R_X^0$ and so there exist $f^0, g^0 \in R_{\mathsf{I}}^0$ such that $f^0 \sim g^0$ but $f^0 \not\sim_0 g^0$. Let $F = \{i : f^0(i) = g^0(i)\}$ and so $F \supseteq G$, $F \in X$ but $F \notin X^0$. Conversely, assume there exists an $F \in X$ such that $F \supseteq G$ but $F \in X^0$. Define $f^0, g^0 \in R_{\mathsf{I}}^0$ by: for all $i \in F$ put $f(i) = g(i)$; for all $i \notin F$ take $f(i) \neq g(i)$. Thus $f^0 \sim g^0$ but $f^0 \not\sim_0 g^0$ and so $R_{X^0}^0 \neq R_X^0$.

A subfilter X^0 of an ultrafilter X will be called *distinct* if $R_{X^0}^0 \neq R_X^0$, otherwise it will be called *indistinct*.

Theorem 3. *If* $f^2 \in R_{\mathsf{I}}^2$ *is defined by*: $f^2(i) = \mathsf{e}_i$, *for all* $i \in \mathsf{I}$, *then* $\bar{f}^2$ *is the identity relation of* $\pi\mathbf{M}_i/(X; X^0)$ *if, and only if,* X^0 *is an indistinct subfilter of* X.

Proof: Assume X^0 is an indistinct subfilter of X and so $R_{X^0}^0 = R_X^0$. For all $\bar{f}^0, \bar{g}^0 \in R_{X^0}^0$, $\bar{f}^2(\bar{f}^0, \bar{g}^0)$ if, and only if, $\{i : \mathsf{e}_i(f^0(i), g^0(i))\} \in X$; that is if, and only if, $f^0 \sim g^0$; that is if, and only if, $\bar{f}^0 = \bar{g}^0$, as $R_{X^0}^0 = R_X^0$. Hence $\bar{f}^2$ is the identity relation. Conversely, assume X^0 is a distinct subfilter of X. Hence, as in Lemma 3, there exist $\bar{f}^0, \bar{g}^0 \in R_{X^0}^0$ such that $\bar{f}^0 \neq \bar{g}^0$ but $f^0 \sim g^0$. Thus $\{i : f^0(i) = g^0(i)\} \in X$ and so $\bar{f}^2(\bar{f}^0, \bar{g}^0)$. Hence $\bar{f}^2$ is not the identity relation.

It should be noted that $\bar{f}^2$ as defined in the above theorem is always an equivalence relation and moreover one with the general substitution property. Thus the theorem has given that a distinct subfilter gives rise to a non-normal structure. The next theorem sets out the expected relationship between such a non-normal structure and the normal ultraproduct got by putting X^0 equal to X.

Theorem 4. $\pi\mathbf{M}_i/(X; X)$ *is isomorphic to a quotient structure of* $\pi\mathbf{M}_i/(X; X^0)$.

Proof: Define a map $\beta : R_{X^0}^0 \to R_X^0$ by: for each $\bar{f}^0 \in R_{X^0}^0$, put $\beta(\bar{f}^0) = [f^0]$, where $[f^0] \in R_X^0$. β is well defined and surjective. Further, for all $\bar{f}^k \in R_X^k$, and for all $\bar{f}_1^0, \ldots, \bar{f}_k^0 \in R_{X^0}^0$, $\bar{f}^k(\bar{f}_1^0, \ldots, \bar{f}_k^0)$ if, and only if, $\bar{f}^k(\beta(\bar{f}_1^0), \ldots, \beta(\bar{f}_k^0))$. Let $\sim_\beta$ be the binary relation defined on $R_{X^0}^0$ by: $\bar{f}^0 \sim_\beta \bar{g}^0$ if, and only if, $\beta(\bar{f}^0) = \beta(\bar{g}^0)$. Now $\sim_\beta$ as defined is a congruence of $\pi\mathbf{M}_i/(X; X^0)$ and it can be immediately checked that the quotient structure with respect to this congruence is isomorphic to $\pi\mathbf{M}_i/(X; X)$.

2. Let $\{\mathbf{M}_i : i \in \mathsf{I}\}$ now be a family of similar structures. For each $k > 0$, let the symbols $\{r_j^k : j < \alpha_k\}$ denote the k-placed relations of each $\mathbf{M}_i$, where α_k is the common (i.e. for all $i \in \mathrm{I}$) cardinality of each R_X^k, and each symbol $r_j^k, j < \alpha_k$: denotes the corresponding relation in each structure under the similarity correspondence. In Robinson *cf.* [3] the individuals of the ultraproduct are defined as in section 1 but with $X^0 = \{\mathrm{I}\}$. But in Bell-Slomson *cf.* [1], the individuals are defined by taking $X^0 = X$. It is this which is called the standard definition. Further, the k-placed relations of the ultraproduct in this standard definition, denoted by the same symbols, $\{r_j^k : j < \alpha_k\}$ as used for the component structures are defined by: $r_j^k(\overline{f}_1^0, \ldots, \overline{f}_k^0)$ if, and only if, $\{i : r_j^k(f_1^0(i), \ldots, f_k^0(i))\} \in X$. Now in terms of section 1 this definition has selected from R_I^k the subclass $S_\mathrm{I}^k = \{h_j^k : h_j^k : \mathrm{I} \to \bigcup\{R_i^k : i \in \mathrm{I}\}$ and for all $i \in \mathrm{I}$, $h_j^k(i) = r_j^k, j < \alpha_k\}$ and associated with each member of this subclass a k-placed relation of the ultraproduct. The following theorem establishes a necessary and sufficient condition under which the construction of section 1 applied to this family of similar structures reproduces only the standard relations. Of course at least the standard relations will always be produced for if $h_m^k \neq h_n^k$ then $\overline{h}_m^k \neq \overline{h}_n^k$.

Theorem 5. *For all $k > 0$, there exists an $f^k \in R_\mathrm{I}^k$ such that $\overline{f}^k \neq \overline{h}^k$ for any $h^k \in S_\mathrm{I}^k$ if, and only if, X is α_k-incomplete.*

Proof: Assume that X is α_k-incomplete and so let β_k be the first cardinal, $\beta_k \leq \alpha_k$, such that X is β_k-incomplete. Thus there exists, (with a permutation of the index set of R_i^k if necessary), for each $j < \beta_k$, an $F_j \in X$ such that $\bigcap\{F_j : j < \beta_k\} = \phi$. Construct f^k inductively as follows: for all $i \notin F_0$, put $f^k(i) = r_0^k$; for all $i \in F_0 - F_1$, put $f^k(i) = r_1^k$; assume that $f^k(i)$ has been defined for all $i \in \bigcup\{\complement F_t : t < \delta\}$ for some ordinal $\delta < \beta_k$, where $\complement F_t$ is the complement of F_t, and define $f^k(i) = r_\delta^k$ for all $i \in \bigcap\{F_t : t < \delta\} - F_\delta$. By induction f^k is well defined and domain $f^k = \mathrm{I}$, as $\bigcap\{F_j : j < \beta_k\} = \phi$. Now $\{i : f^k(i) = r_0^k\} = \complement F_0$ and so $\overline{f}^k \neq \overline{h}_0^k$ as $\complement F_0 \notin X$. For $0 < j < \beta_k$, $\{i : f^k(i) = r_j^k\} = \bigcap\{F_t : t < j\} - F_j$ and so $\overline{f}^k \neq \overline{h}_j^k$ as $\complement F_j \notin X$. Finally if $\beta_k \leq j < \alpha_k$, $\{i : f^k(i) = r_j^k\} = \phi$ and so $\overline{f}^k \neq \overline{h}_j^k$ as $\phi \notin X$. Conversely, assume there is an $f^k \in R_\mathrm{I}^k$ such that for all $h_j^k \in S_\mathrm{I}^k$, $\overline{f}^k \neq \overline{h}_j^k$. For each $j < \alpha_k$, define $G_j = \{i : f^k(i) = r_j^k\}$. Now $\bigcup\{G_j : j < \alpha_k\} = \mathrm{I}$ and so $\bigcap\{\complement G_j : j < \alpha_k\} = \phi$. But for all $j < \alpha_k$, $\complement G_j \in X$ and so X is α_k-incomplete.

While the above theorem establishes the distinctness of $\overline{f}^k$ in terms of an equivalence class of maps Theorem 2 ensures that the distinctness is carried over to the relations of the ultraproduct on an extensional basis.

Corollary 1. *For each $k > 0$, if α_k is finite then for each $f^k \in R_\mathrm{I}^k$, there exists some $h^k \in S_\mathrm{I}^k$ such that $\overline{f}^k = \overline{h}^k$.*

Proof: If α_k is finite then X is α_k-complete.

Corollary 2. *If X is a principal ultrafilter then for all integers $k > 0$, and for all $f^k \in R_\mathrm{I}^k$, there exists some $h^k \in S_\mathrm{I}^k$ such that $\overline{f}^k = \overline{h}^k$.*

Proof: A principal ultrafilter is α_k-complete for all α_k.

The final theorem concerns the relationship between two ultraproducts, each formed by the standard definition from the same family of similar structures with respect to the same ultrafilter, but where in the case of the second ultraproduct the similarity correspondence may, for each $k > 0$, link different k-placed relations from each structure from those linked in the first case.

Let $\pi\mathbf{M}_i/X$ be the standard ultraproduct formed as noted at the beginning of section 2. Let $\pi'\mathbf{M}_i/X$ be a second ultraproduct formed by the standard definition but following possible rearrangements of the relations connected under the similarity correspondence; that is, for each $i \in \mathrm{I}$, and for each $k > 0$, if β_i^k is a permutation of the set $\{j : j < \alpha_k\}$ then the k-placed relations of $\pi'\mathbf{M}_i/X$ are given by r'^k_j, $j < \alpha_k$, where $r'^k_j(\overline{f}^0_1, \ldots, \overline{f}^0_k)$ if, and only if, $\{i : r^k_{\beta_i^k(j)}(f^0_1(i), \ldots, f^0_k(i))\} \in X$.

Theorem 6. *There exists such a $\pi'\mathbf{M}_i/X$ as above non-isomorphic to $\pi\mathbf{M}_i/X$ if, and only if, there exists some $k > 0$ such that X is α_k-incomplete.*

Proof: Assume that for each $k > 0$, X is α_k-complete. Associate each standard relation r^k_j, $j < \alpha_k$, in $\pi\mathbf{M}_i/X$ with $\overline{h}^k_j$, where for all $i \in \mathrm{I}$, $h^k_j(i) = r^k_j$. Associate r'^k_j, $j < \alpha_k$, in $\pi'\mathbf{M}_i/X$ with $\overline{h}'^k_j$, where for all $i \in \mathrm{I}$, $h'^k_j(i) = r^k_{\beta_i^k(j)}$. From Theorem 5 it follows that $\{\overline{h}^k_j : j < \alpha_k\} = \{\overline{h}'^k_j : j < \alpha_k\}$. Hence $\pi\mathbf{M}_i/X$ is the same structure as $\pi'\mathbf{M}_i/X$. Conversely, assume that for some $k > 0, X$ is α_k-incomplete. Hence from Theorem 5 there exists $\overline{f}^k \in R^k_{\underline{X}}$ such that $\overline{f}^k$ is distinct from each of the standard k-placed relations of $\pi\mathbf{M}_i/X$. For each $j < \alpha_k$, let $G_j = \{i : f^k(i) = r^k_j\}$. Thus $\{G_j : j < \alpha_k\}$ partitions X and for each $j < \alpha_k$, $G_j \notin X$. For each $i \in \mathrm{I}$, and each $m \neq k$, $m > 0$, take β_i^m as the identity permutation of α_m. For each $i \in \mathrm{I}$, take β_i^k as one of the permutations of α_k such that $\beta_i^k(0) = j$, where $i \in G_j$. Hence the relation r'^k_0 of $\pi'\mathbf{M}_i/X$ is associated with $\overline{f}^k$ and so is distinct from all the k-placed relations of $\pi\mathbf{M}_i/X$. Thus $\pi'\mathbf{M}_i/X$ is not isomorphic to $\pi\mathbf{M}_i/X$.

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