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A MODAL LOGIC WITH TEMPORAL VARIABLES

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There appears thus far to have been four general methods suggested in the literature for formalizing indeterministic tense-modal logics.[1] The first consists in the introduction of a third truth-value, but this has the absurd consequence that the law of non-contradiction must be denied since the conjunction of any two propositions, having the indeterminate truth-value, must be indeterminate rather than false. (The absurdity is apparent from the fact that in this context "having an indeterminate truth-value" means the same as, possibly being true; in other contexts the denial that '$p \wedge \sim p$' is false may, of course be correct, e.g., where 'p' has no definite meaning.) A second alternative called by Prior a "Peircean tense logic" has the very awkward result that true, contingent propositions about the future cannot be stated at all. The third method which Prior calls "Ockhamist"[2] is the one considered in the first section below. A consideration of the fourth is best left to the end of this paper.

1 The Ockhamist system is formalized by Prior in the following way. He takes over the third sort of metric tense logic discussed in [4] Chapter VI containing standard axioms such as *FP*3: $FnPnp \supset p$. This is then modified in allowing for two sorts of propositional variables, a, b . . . which always signify necessary truths, and p, q . . . which may or may not go proxy for statements containing L. As will become clear this convention cannot be used here; so I mention this simply to avoid confusion: Prior's a and b

1. Three of these and the variations within them are discussed in [4]. I have left out of account the very interesting systems due to C. A. Meredith and R. Suszko ([4], Chapter V) since these would require a very long separate discussion. (As accounts of the behaviour of propositions about future contingencies, though not in general, they seem to me to open objections similar to those mentioned above.)

2. It is based on that reading of Aristotle's *De Interpretatione* IX which takes him to be holding that all propositions are truė or if not true then false (i.e., both bivalence and excluded third are maintained) but that some propositions about the future are *also* only contingent.

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statements will be symbolized as Lp, LFq, LPr etc. (where F and P mean as usual, "is future that" and "is part that" respectively). Prior's non-standard postulates for his (Ockhamist) system **L** are $LA{:}\, a \supset La$ which we will modify slightly to $Pp \supset LPp$; and L_3: $\sim Lp \supset L \sim Lp$. Prior's system contains no operator M defined in terms of L. As L means "now unpreventable", I do not believe such a definition is possible; initially M will be informally defined as "causally indeterminate". This formula shows the connection between M and L: $Mp \supset \sim Lp$. (The highly counter-intuitive appearance of the converse of this formula disappears if it is remembered that M and L have the causal senses intended.) There appears to be a simple but important objection to Prior's L_3 as an expression of indeterminism. L_3 asserts that if something is not causally determinate then it is causally determinate that it is not so. There are two different states of affairs which can make $\sim Lp$ true, (1) p is about the past and false; (2) p is about the future and true (or false) but now causally indeterminate (only two truth-values are required in Prior's Ockhamist system; the modal operators are not truth-functional). Suppose that the antecedant of L_3 is true for the second reason given above, then the consequent says, in effect, that it is *now causally determinate that* p is causally indeterminate. But this sounds really like an affirmation of *determinism* not indeterminism since, although it allows for events the occurrences of which are only probable, it asserts that *this* fact is determinate. Hence, in the only interesting sense, the putatively indeterminist system is not indeterminist at all; in fact it is not even neutral on this score (as ordinary logic presumably is). But in one respect this criticism is certainly unfair. If at instant t_0 p is indeterminate then admittedly there is nothing that can happen *at that instant* to make it determinate; so again L_3 appears to be true. And this would be consistent with the claim that it is never *p itself* which can be "changed" from indeterminate to determinate but rather some other proposition "q" which describes an event which causes p. But in that case we of course just have the same problem again with q. Thus if indeterminism is to be true and formally representable there must be some way of stating that during some temporal interval an event is indeterminately so. Using **IND** (for "indeterminist logical system") I think we would want to have the postulate already mentioned, viz.,

IND 1 $Mp \supset \sim Lp$
2 $Mp \supset M \sim p$
3 $Mp \supset M \sim Lp$
4 $Fp \supset MFp$

(**IND** 4 could be replaced by $(\exists p)(Fp \supset MFp)$ if some future events are now causally determinate.) If these postulates are combined with Prior's L_3 we obtain absurd results, e.g., the result that if any proposition about the future is true then any proposition whatever is true. The solution, however, is not to drop L_3 in favour of **IND** 3 above since L_3 is true as Prior states it. Instead of L_3 we might try **IND** 3′: $(\exists p)\ Mp \supset M \sim Lp$ *and* **IND** 3″ $(\exists p) \sim Lp \supset L \sim Lp$. The present indeterminacy but future truth of a proposition ("grammatical form" might be a better expression in this context) could then be expressed by formulas like $MFnq \wedge Fn + mPLq$ (i.e., it is indeterminate that it will be the case n time units from now that q and it will be the case $n + m$ units from now that it was the case that q was determinate) and time's causal asymmetry might be represented by formulas like $Pp \supset LPp$, $Lp \supset LLp$, and $Mp \supset MMp$. But I will not try to develop a system of this sort for the following reason: 3′ and 3″ above do not express precisely what is needed. It is not that there are *some propositions* which are indeterminately indeterminate and others which are necessarily so, *but rather that one and the same proposition* may assert what is, relative to one time, necessarily indeterminate while being *at that and every other time*, if true, necessary relative to another time. In the next section I attempt the construction of a logic designed to express this point of view.

2 *The System* **IND** The symbolism of **IND** includes the usual singulary and binary truth-functional constants and the non-truth-functional operators M and L. The propositional variables are pab, qnb, qba . . ., i.e., the usual propositional variables followed by two of four other "variables"[3] a, b, m, and n (in any order); α and β are also propositional variables (abbreviations for pab, qba etc.). *Well-formed* formulas include all those which are well-formed according to truth-functional logic except that pab, pba etc. or α or β must occur where p, q etc. ordinarily occur. Truth-functionally wffs preceded by M or by L or by both are well-formed subject to the following qualifications. M and L must be followed immediately by only one of a, b, n, or m plus *a* wff except where the wff is α or β (or a truth-functional compound expression containing only α and β as atomic propositions). Ma, Ln etc. can be followed by a $\sim$ or a finite number of $\sim$s or by one occurrence of La or Mn etc. The last M- or L- in a formula must be followed by pab or pba etc. or a truth-functional compound of these.

Axioms:

1 $Mapnb$
2 $\sim Mb\alpha$
3 $\sim Mapba$
4 $\sim Lapab$
5 $Lnpmn \equiv pmn$
6 $Ln\alpha \supset \alpha$
7 $Ln\alpha \supset \sim M_m \sim \alpha$
8 $Mapnb \equiv LnMapnb$
9 $Ln\alpha \equiv LnLn\alpha$

Rules:

1. Any (set of) rule(s) for the propositional calculus.

3. a, b are best construed as "arbitrary constants" in Kleene's sense (v. [2], p. 150); m and n are genuine variables taking a and b as values.

2. Substitution:

(i) a or b can be substituted for m and n; the same variable can be substituted for m and n and must be substituted for different occurrences of n or m in the same formula.
(ii) Formulas which are truth-functionally equivalent or equivalent by Axiom 5 can be substituted for one another.
(iii) Where a formula contains M or L without a subscript and α or β then if equivalent formulas not containing α or β are substituted for α and β then M and L must be replaced by Ma and La or by Mb and Lb etc. Formulas containing M and L cannot be substituted for α and β (nor for p and q given the information rules) unless such substitution is justified through **IND** 5 where α and β are governed by modal operators. Ma can *not* be substituted for Mb etc.
(iv) Substitution of, e.g., pab for pba is precluded since, where δ is one or a series of non-truth-functional propositional operators, the truth-value of δpmn depends on which variables are substituted in what order for m and n.
(v) Atomic propositions within complex propositions not containing modal operators can be replaced by propositions containing M, L, a, b, and $\sim$ providing the substitution preserves the truth-value of the original.
(vi) The negations of propositions containing modal operators cannot be substituted for their non-negative equivalents in propositions not containing binary truth-functional connectives.

It follows from these rules that axioms and theorems derived by the rules of substitution alone can replace tautologies within wffs. Similarly the negations of axioms and the negations of wffs derivable by substitution can replace contradictions.

3.1 *Semantical Interpretation of* **IND** An **IND**-model is an ordered triple $\langle T, <, \mathsf{V}\rangle$ where T is a non-empty ordered set of entities (interpreted here as times). For any $t \in T$ there is a t_i and a t_j such that $t_i\langle t\rangle t_j$. (This characterization is not complete and is enlarged on below. All we need note here is that the truth of $Mapab$ requires that there be no last temporal "object" to which a can refer.) $<$ is a dyadic, irreflexive, asymmetrical relation. Even though we only have the two temporal variables a and b, $<$ must also be qualified as transitive so as to avoid the possibility that time is circular. V is a value assignment, i.e., a function of two arguments the first of which ranges over formulas and the second over times. It is a formal and philosophical advantage of **IND** that the second argument is often irrelevant to the value of V.[4]

(1) $[\mathsf{V}\sim]$, $[\mathsf{V}\equiv]$, $[\mathsf{V}\supset]$ have their usual truth-functional definitions. There is no distinction in **IND** analogous to (for instance) the distinction Prior sometimes makes between $F \sim p$ and $\sim Fp$.

4. Some of these points of interpretation are discussed further in Section **4**. Once the use of a, b, m and n have been explained by means of the t_n notation the latter is dropped.

(2) $\ulcorner pt_j t_k \urcorner$ means that at t_j it is the case that p at t_k. Thus $\ulcorner pt_i t_k \urcorner$ where $t_i = t_k$ is what corresponds in **IND** to the present-tense. However it is not equivalent in one important respect since $\ulcorner pt_i t_k \urcorner$ contains no reference to the *present time* but only to two actually related times which are not characterized by means of tenses or by temporal indexical expressions. The point of the phrase "actually related times" is that **IND** does not require *prima facie* at least "possible times" or branching futures but just an ordinary linear time-series. This is discussed below. Similarly $\ulcorner pt_i t_k \urcorner$ where $t_i < t_k$ corresponds to a simple future tense and $\ulcorner pt_i t_k \urcorner$, $t_i > t_k$ to a past tense. For *any* $t_i \in T$, $\vee t_i [pt_i t_k] = 1$ or the $\vee t_i [pt_i t_k] = 0$. For purposes of interpretation (philosophical and semantical) it is worth noting that since this holds for any t_i the subscript on $\vee$ is irrelevant. Similarly for *any* $t_i \in T$, $\vee[pt_i t_k] = \vee[pt_k t_k] = 1$ or 0, i.e., the value of $pt_i pt_k$ does not depend on the relation between t_i and t_k nor when $pt_i t_k$ is asserted. Furthermore there is no third truth-value, nor a set of propositions which are neither true nor false. Both bivalence and the excluded third hold in this system. In **IND** only the modalities of propositions, not their truth-values, vary over times.

(3) $[\vee L]$ For any wff. α and t_i, t_j, $t_k \in T$, $\vee[Lt_i \alpha t_j t_k] = 1$ iff $t_i \geqslant t_k$ and $\vee[\alpha t_j, t_k] = 1$; otherwise $\vee[Lt_i \alpha t_j t_k] = 0$.

(3.1) $[\vee LL]$ The value of formulas containing iterated L's is defined inductively in the following way. $\vee[Lt_i Lt_j \alpha] = 1$ iff $t_i \geqslant t_j$ and $\vee[Lt_j \alpha] = 1$ (understanding here that α ranges over propositions containing the appropriate temporal terms); otherwise the $\vee[Lt_i, Lt_j \alpha] = 0$. $\vee[Lt_i Lt_j Lt_k \alpha] = 1$ iff $t_i \geqslant t_j$ and $\vee[Lt_j Lt_k \alpha] = 1$; otherwise $\vee[Lt_i Lt_j Lt_k \alpha] = 0$, and so on.

(4) $[\vee M]$ For any wff. α and t_i, t_j, $t_k \in T$, $\vee[Mt_i \alpha t_j t_k] = 1$ iff $t_i < t_k$; otherwise $\vee[Mt_i \alpha t_j t_k] = 0$. (Thus the truth-value of $\alpha t_j t_k$ is irrelevant to that of $Mt_i \alpha t_j t_k$.)

(4.1) $[\vee MM]$ For any wff α, $\vee[M - M - \alpha] = \vee[M - M - M - \alpha] = 0$ and so on for any number of M's.

(5) In **IND** a and b always take as values t_i and t_k respectively such that $t_i < t_k$. This is a special condition on **IND**. The above semantics would also work for systems slightly different from **IND** in that they contain more temporal variables.

(6) Negation is the only truth-functional constant allowed within the scope of a modal operator.

3.2 *Decidability in* **IND** Given the above rules for assigning truth-values and the convention that $a < b$ in any formula then we can construct truth-tables for formulas containing M and L, which include non-truth-functional assignments as well as the usual ones. Here are some examples. In some cases where the final truth-values assigned to a formula depend on the relations between M, L, a, and b and not on the truth-value assigned to *pnm* the truth-values assigned to *pnm* are put in brackets.

(1) $$\begin{array}{cc} Ma & pab \\ \begin{matrix}1\\1\end{matrix} & \begin{bmatrix}1\\0\end{bmatrix}\end{array}$$

(2) $$\begin{array}{cc} La & pab \\ \begin{matrix}0\\0\end{matrix} & \begin{bmatrix}1\\0\end{bmatrix}\end{array}$$

(3) $$\begin{array}{cc} Mb & pab \\ \begin{matrix}0\\0\end{matrix} & \begin{bmatrix}1\\0\end{bmatrix}\end{array}$$

(4) $$\begin{array}{cc} Lb & pab \\ 1 & 1 \\ 0 & 0\end{array}$$

(5) $$\begin{array}{ccccc} Ma & pab & \equiv & Ma & \sim\ pab \\ \begin{matrix}1\\1\end{matrix} & \begin{bmatrix}1\\0\end{bmatrix} & \begin{matrix}1\\1\end{matrix} & \begin{matrix}1\\1\end{matrix} & \begin{bmatrix}0&1\\1&0\end{bmatrix}\end{array}$$

(6) $$\begin{array}{ccccc} Ma & pba & \equiv & Ma & \sim\ La\ pba \\ \begin{matrix}0\\0\end{matrix} & \begin{bmatrix}1\\0\end{bmatrix} & \begin{matrix}1\\1\end{matrix} & \begin{matrix}0\\0\end{matrix} & \begin{bmatrix}0&1&1\\1&0&0\end{bmatrix}\end{array}$$

(7) $$\begin{array}{cccccccc} Ma & pab & \equiv & Ma & Lb\ pab & \& & Ma & \sim\ Lb\ pab \\ \begin{matrix}1\\1\end{matrix} & \begin{bmatrix}1\\0\end{bmatrix} & \begin{matrix}1\\1\end{matrix} & \begin{matrix}1\\1\end{matrix} & \begin{bmatrix}1&1\\0&0\end{bmatrix} & \begin{matrix}1\\1\end{matrix} & \begin{matrix}1\\1\end{matrix} & \begin{bmatrix}0&1&1\\1&0&0\end{bmatrix}\end{array}$$

(8) $$\begin{array}{cccccc} Ma & pab & \equiv & \sim & La & pab \\ \begin{matrix}1\\1\end{matrix} & \begin{bmatrix}1\\0\end{bmatrix} & \begin{matrix}1\\1\end{matrix} & \begin{matrix}1\\1\end{matrix} & \begin{matrix}0\\0\end{matrix} & \begin{bmatrix}1\\0\end{bmatrix}\end{array}$$

(9) (oddly)

$$\begin{array}{cccccc} La & pba & \supset & \sim & Ma & pba \\ \begin{matrix}1\\0\end{matrix} & \begin{matrix}1\\0\end{matrix} & \begin{matrix}1\\1\end{matrix} & \begin{matrix}1\\1\end{matrix} & \begin{matrix}0\\0\end{matrix} & \begin{bmatrix}1\\0\end{bmatrix}\end{array}$$

(10) we do *not* have,

$$\begin{array}{ccccc} Ma & pab & \supset & Lb & pab \\ \begin{matrix}1\\1\end{matrix} & \begin{bmatrix}1\\0\end{bmatrix} & \begin{matrix}1\\0\end{matrix} & \begin{matrix}1\\0\end{matrix} & \begin{matrix}1\\0\end{matrix}\end{array}$$

(11) so (perhaps oddly but see (8) through (11)) we do not have

$$\begin{array}{cccccc} Ma & pab & \supset & Ma & Ma & pab \\ \begin{matrix}1\\1\end{matrix} & \begin{bmatrix}1\\0\end{bmatrix} & \begin{matrix}0\\0\end{matrix} & \begin{matrix}0\\0\end{matrix} & \begin{matrix}1\\0\end{matrix} & \begin{bmatrix}1\\0\end{bmatrix}\end{array}$$

A preliminary justification for (11) can be given in the following way. As would be expected, it does not follow from the fact that something is possible that it is necessary (10) but in **IND** we also have that what is possible is (relative to a certain a or b) not necessary and what is necessary is (relative to a certain a or b) not possible ((9) and (10)). But we also have that what is possible, *may* be necessary (or not necessary). Therefore in the sense of "possible" relevant to **IND**, what is possible is not possibly possible for the latter would have the consequence, excluded in **IND**, that something can, relative to the same a or b be both necessary and possible. (A further explanation of this is given in the philosophical interpretation.) Hence,

$$(12)\quad \begin{array}{ccccc} Ma\ pab & \supset & La & Ma & pab \\ & \begin{array}{c}1\\1\end{array} & \begin{array}{c}1\\1\end{array} & \begin{array}{c}1\\1\end{array} & \begin{bmatrix}1\\0\end{bmatrix} \end{array}$$

Formulas like $\ulcorner L\alpha \supset \alpha \urcorner$ in effect summarize formulas containing the a-b notation and so their truth-tables can only be provided through providing a list of the tables they summarize, e.g.,

$$(13)\ (\mathrm{i})\quad \begin{array}{cccc} La & paa & \supset & paa \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{array}$$

$$(\mathrm{ii})\quad \begin{array}{cccc} La & pab & \supset & pab \\ \begin{array}{c}0\\0\end{array} & \begin{bmatrix}1\\0\end{bmatrix} & \begin{array}{c}1\\1\end{array} & \end{array} \qquad \text{etc.}$$

$$(14)\quad L\alpha \supset {\sim} M \sim \alpha$$

$$(\mathrm{i})\quad \begin{array}{ccccccc} La & paa & \supset & \sim & Ma & \sim & paa \\ \begin{array}{c}1\\0\end{array} & \begin{array}{c}1\\0\end{array} & \begin{array}{c}1\\1\end{array} & \begin{array}{c}1\\1\end{array} & \begin{array}{c}0\\0\end{array} & \multicolumn{2}{c}{\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}} \end{array}$$

$$(\mathrm{ii})\quad \begin{array}{ccccccc} La & pbb & \supset & \sim & Ma & \sim & pbb \\ 0 & 1 & 1 & & & & \\ 0 & 0 & 1 & & & & \end{array}$$

$$(\mathrm{iii})\quad \begin{array}{ccccccc} La & pab & \supset & \sim & Ma & \sim & pab \\ 0 & 1 & 1 & & & & \\ 0 & 0 & 1 & & & & \end{array}$$

$$(\mathrm{iv})\quad \begin{array}{ccccccc} La & pba & \supset & \sim & Ma & \sim & pba \\ \begin{array}{c}1\\0\end{array} & \begin{array}{c}1\\0\end{array} & \begin{array}{c}1\\1\end{array} & \begin{array}{c}1\\1\end{array} & \begin{array}{c}0\\0\end{array} & \multicolumn{2}{c}{\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}} \end{array}$$

The other cases are,

(v) $Lb\ paa \supset \sim Mb \sim paa$
(vi) $Lb\ pbb \supset \sim Mb \sim pbb$
(vii) $Lb\ pab \supset \sim Mb \sim pab$
(viii) $Lb\ pba \supset \sim Mb \sim pba$

In general, the first case and in (iv)-(viii) the consequents are always true; in (ii) and (iii) the antecedents are false. Similarly, sets of truth-tables can be constructed for Axioms 1, 5, 8, and 9 and formulas like them.

These truth-tables provide an effective decision procedure for determining the (in)validity of wffs. The formation and substitution rules do not allow for wffs with M or L qualifying propositions containing binary connectives (i.e., we do not have compound propositions of the form $L(\alpha \supset \beta)$ or $M(\alpha \vee \sim\alpha)$). The rules (1)-(5) for assigning truth-values enable us to assign a truth-value to any non-compound wff containing some combination of M, L, a, and b. (Where a wff contains m or n we can just substitute a's and b's or both according to the rules). Any complex wff made up of these can be assigned truth-values according to the ordinary truth-functional rules. Hence it can be decided in a finite number of steps whether any **IND** wff is valid or invalid.

3.3 *Negation—consistency of* **IND**[5] By "valid" will be meant **IND**-tautologous (true in every case according to the **IND** truth-tables). "Derivable" will mean deducible from the axioms by means of the rules of substitution and inference.

(1) Consider first the **IND** 1-4 axioms. These are all **IND**-tautologous. Next consider the set of wffs derivable by the **IND**-rules of substitution alone (i.e., derivable without the aid of the other axioms or of any purely truth-functional rules of inference). These will all obviously be valid. (Ignoring the irrelevant fact that $\ulcorner p \urcorner$ can be substituted for $\ulcorner q \urcorner$ *ad infinitum* there are a small number of cases which can all be checked.) (a) Hence if A is a theorem (of this type) then A is valid; so if it is not the case that A is valid then A is not a theorem. Given the rules governing the formation of **IND**-truth-tables, the negations of **IND** 1-4 and the negations of wffs derivable from them by substitution alone will be invalid. Therefore if A is a thesis (of the type under consideration) then $\sim A$ is not valid and so (by (a) above) it is not a thesis. (Of course both A and $\sim A$ may be invalid and we do not yet know whether every valid wff is a theorem.)

(2) Next consider any wff derivable from **IND** 1-4 using truth-functional methods (with or without the **IND** substitution rules). The formation rules do not allow for the construction of wffs containing a modal operator qualifying an expression containing a binary truth-functional connective. The **PC**-transform of a wff in **IND** of this sort (i.e., a formula obtained by deleting every occurrence of a modal operator) will not necessarily be a wff of the propositional calculus but we can obtain something exactly analogous. This will be called a **PC**-*semantical picture* of an **IND** wff containing binary truth-functional connectives. This picture is obtained by replacing the modal singular propositions in such a formula with $\gamma_1\ \gamma_2 \ldots$ and writing underneath these symbols their possible truth-values as determined by the truth-tables for modal singular propositions. For example consider a wff like (A) $\ulcorner \sim Mapnb \supset \delta \urcorner$ where δ may or may not contain modal operators and/or truth-functional connectives but such that it does not have a unique truth-value determined by its internal logical structure. Given the rules for the formation of **IND** truth-tables the only possible value for $\ulcorner \sim Map\ nb \urcorner$ is 0 since the two possible truth-tables are:

$\sim$	Ma	pnb		$\sim$	Ma	pbb
0	1	1	and	0	1	1
0	1	0		0	1	0

5. Clearly **IND** is not *regular* in Sobociński's sense (v. [5]); so we cannot proceed by simply showing that if a wff α is derivable, $\sim\alpha$ is not derivable. If we did not have restricted rules of substitution inconsistency would be provable using Post's criterion.

Hence the **PC** semantical picture of $\ulcorner\sim Mapab \supset \delta\urcorner$ is:

γ	$\supset$	δ
0	1	1
0	1	0

which is to say that A is a truth-functional tautology in the sense that given the possible truth-values of its modal components the whole formula is truth-functionally valid. But since in **IND** modal operators never qualify propositions containing binary truth-functional connectives we can always construct a **PC** semantical picture of any **IND** wff containing truth-functional operators in the above way. In general consider any wff $\Delta_1\Delta_2 \ldots \gamma_1\gamma_2 \ldots$ where $\Delta_1\Delta_2 \ldots$ are the truth-functional operators and $\gamma_1\gamma_2 \ldots$ are the atomic propositions which may or may not contain modal operators. Using the **IND** truth-tables we can obtain the possible values of those γ's containing modal operators. Then by assigning values to the other γ's so as to cover all the possibilities we obtain the **PC**-picture of the formula which will uniquely determine whether the original wff is valid or invalid on purely truth-functional grounds.

We can conclude (a) the **PC**-picture of any wff of **IND** will be an ordinary **PC** truth-table (except that there may be columns containing only 1's or 0's where we would ordinarily expect both); (b) every wff of **IND** (of the relevant type) will have one and only one **PC**-picture (though, of course, two or more wffs of **IND** may have the same **PC**-picture); and (c) if the **PC**-picture of an **IND** wff A is

$$\begin{array}{l} \Delta_1\Delta_2 \ldots \alpha,\ \beta \\ | \\ | \quad \text{(i.e., all 1's in the last column)} \end{array}$$

then the **PC**-picture of $\sim A$ will have all 0's in the last column.

We now have to show that the **PC**-picture of every thesis of **IND** depicts the truth-table of a valid wff of **PC** (with the proviso that there may be all 1's or all 0's in some columns). Every thesis of **IND** containing a binary truth-functional connective is obtained from one or more axioms by a set of Rules of Inference for **PC** plus the Rules of Substitution for **IND** (which includes the **PC** rule of uniform substitution). We have already seen that the rules of substitution preserve the truth or falsity of singular modal propositions and that these are the only sort of proposition to which these rules are directly relevant (i.e., these rules have application only to the component propositions of propositions containing binary operators); so we need only consider the rules of inference and uniform substitution. But these preserve validity in the **PC**. Therefore the PC-picture of every thesis of **IND** (containing binary connectives) depicts the truth-table of a valid wff of **PC** (with perhaps some of the apparently possible truth-values missing because of the restrictions on the assignment of truth-values to some of the modal components). It is easily seen that these considerations also apply to **IND** 5-9 and to wffs derivable from them; hence we can now conclude the following. There are two relevant types of wffs of **IND**, A_1: those containing no binary connectives; A_2: those containing binary connectives. In the case of an A_1, if it is a thesis it is **IND**-valid in which case (as we have already seen) $\sim A_1$ will be invalid. Therefore A_1 and $\sim A_1$ cannot both be theses. In the case of an A_2, if A_2 is a thesis then its **PC**-picture A_2''[6] is a (partial) truth-table of a **PC** valid formula. It follows that for every wff (of type A_2), A_2 of **IND**, A_2 and $\sim A_2$ are not both theses; for if they were then **PC**-pictures A_2'' and $\sim A_2''$ would both represent (partial) truth-tables of valid **PC** formulae which we already know to be impossible. Hence, in general, for every wff A of **IND**, A and $\sim A$ are not both theses. Hence **IND** is consistent with respect to negation. (**IND** is not strongly complete since other wffs having valid **PC**-pictures could be added to **IND**.)

3.4 *Completeness* To prove completeness we need a list of all the possible types of well-formed non-theorems (hereafter referred to as **As**).

(I) Wffs containing no modal operators.

(II) Wffs containing only some combination of M, L, a, b, p, and $\sim$, but no binary truth-functional connectives.

(1) Then (since A is a wff non-theorem) A cannot be equivalent by the substitution rules to any of **IND** 1-4. But then A must be either (i) the negation of one of **IND** 1-4, or (ii) a wff of the same form as **IND** 1-4 but containing the a's and b's in a different order; (iii) a wff satisfying both (i) and (ii) but in such a way as not to be derivable, i.e., A cannot be $Mapbb$ nor the negations of **IND** 3 or 4 with the a's and b's reversed. (iv) Let A be a wff of type (1) but with the non-modal term negated. Then in order to be non-derivable it must simply satisfy the conditions mentioned in (i)-(iii).
(2) Let A be a wff containing more than one Ln but no negation signs or Ms. Since wffs of the sort being considered here do not contain binary truth-functional connectives A will just be of the form $Ln\alpha$ or $LnLn\alpha$. Then, providing α satisfies the conditions mentioned under (1) above, A will not be derivable.
(3) Wffs the last operator on the left-hand side of which is an Mn but which are not equivalent to wffs of types (1) and, of course not axioms nor derivable by substitution from axioms, i.e., a wff of this type will contain two modal operators the first of which is an Mn.
(4) Wffs the last modal operator on the left-hand side of which is Ln and which are not identical with the wffs mentioned under II (1) and (2) above. This requires that there must be one Mn after the Ln in the A's of this sort. So we have these possibilities (where α does not contain a modal operator):

6. Notation consisting of capital letters with a single prime is introduced later.

(i) $Ln\ Mm\ \alpha$
(ii) $\sim Ln\ Mn\ \alpha$
(iii) $Ln\ Mm \sim \alpha$
(iv) $\sim Ln\ Mm \sim \alpha$

This completes the specification of non-theorems which either do not contain modal operators or contain modal but no truth-functional connectives. Clearly **As** of sort (I) are not valid since the **PC** is complete. (Since the system is consistent singular propositional variables not containing modal operators are not valid.) The consistency proof shows that wffs of sort (II) (1) (i) are not valid. Similarly it is easily confirmed by inspecting the rules of assignments of truth-values that **As** of sorts II (1) (ii) (iii) and (iv) are not valid. As of type II (2) are the formulas of the form $Ln\alpha$ or $LnLn\alpha$ where α is a wff of the sort II (i)-(iv). By the above reasoning α will contain at least one 0 in its last column. But then by the rules for the assignment of truth-values $Ln\alpha$ or $LnLn\alpha$ will also be invalid. As of type II (3) will be either of the form (a) $MnMm\alpha$ or (b) $MnLm\alpha$. Wffs of type (a) are not valid. In order to be a non-theorem of type (b) it must be that $n \geqslant m$, it is easily confirmed that any wff of that sort is not **IND**-tautologous. As of type II (4). To see that these are invalid, try assuming that they are valid: it will be easy to show that they are then theorems, contrary to the hypothesis. For example, in order for a wff of type II (4) (i) to be valid Mn cannot be Mb (for it is easily confirmed that $LnMm\alpha$ will then be 0 no matter what the values of n and α); so m must be a and α must be pnb (since $Mapna$ is 0) but then the result will be a theorem by **IND** 1 and 8. Hence if a wff of this sort is to be an A (a non-theorem) it must be invalid.

(III) We have yet to consider wffs containing binary truth-functional connectives and modal operators. We have already seen that any wff of this type will have a corresponding **PC**-semantical-picture (**PC-SP**). For convenience let us call the **PC-SP** of a wff B (whether a theorem or a non-theorem) B'' and the formula occurring in the picture which results from replacing the modal singular propositions with designations of them, B'. We also have to replace certain complex propositions the components of which are modal; these cases are of the following sort. Where Δ is a truth-functional operator and δ a modal operator and $\gamma_1\gamma_2$ . . . are propositions not containing truth-functional operators, there may be components of B, $\Delta\delta\alpha_1\alpha_2$ or $\Delta\delta_1\beta_1\delta_2\beta_2$ which have only the value 1 or the value 0 because of the *connection* (as given in the rules for the assignment of truth-values) between $\delta\alpha_1$ and α_2 in the first case, or $\delta_1\beta_1$ and $\delta_2\beta_2$ in the second case. Here we have to replace the whole complex wff within B in order to get B'.

The set of **PC-SP**s of *valid* wffs of **IND** consists of the set of **PC** truth-tables depicting valid **PC**-wffs plus a set of truth-tables which correspond to invalid **PC** truth-tables in the following way. First consider a *valid* formula B containing modal propositions of both sorts. Its **IND** truth-table can be written:

$\Delta_1\Delta_2 \ldots$	$\Delta\delta\,\alpha_1$	$\alpha_2 \ldots$	$\Delta\ \delta\alpha_1$	$\delta\alpha_2 \ldots \delta$		α_3	$\gamma_1\gamma_2 \ldots$
1	1	0	1	0	1	0	1
1	1 or 0		1 or 0		1 or 0 or 0		
1	1	0	1	0	1	0	1
1	⋮	⋮	⋮	⋮	⋮	⋮	0
⋮	⋮	⋮	⋮	⋮	⋮	⋮	1

Now replace all the (complex and singular) modal propositions by ordinary propositional variables γ and assign values in the usual way. This is formula B. Its truth-table will be that of an *invalid* wff, say like this:

$\Delta_1\Delta_2 \ldots \gamma_1 \ldots \gamma_2 \ldots \gamma_3 \ldots \gamma_4\gamma_5$
1
1
0
0

But we can get the truth-table of B from that of B' by simply eliminating those rows which have a 0 in the last column. So these truth-tables correspond to the subset of invalid **PC** truth-tables obtained in this way. But this subset can be transformed into a subset of *valid* formulas of **PC**: instead of replacing every modal proposition in B by γ to obtain B' proceed in this way. Assign truth-values to the modal components of B. Where the component has all 1's replace it with a **PC** tautology, where it has all 0's replace it with a **PC**-contradiction, where it has 1's and 0's replace it simply with γ. Let us call this result B^*. Its truth-table will be the same as that of β' with the rows having a 0 in the last column eliminated; so it will be the same as the **PC-SP** of B. For example, consider the wff $[B_1]$ $\ulcorner(\sim Mapnb \supset q) \vee \sim q\urcorner$. Let $\supset = \Delta_1$, $\vee = \Delta_2$, $\sim = \Delta_3$, $\wedge = \Delta_4$. B' will be the invalid formula $\Delta_2\Delta_1\gamma_1\gamma_2\Delta_3\gamma_2$ using a Polish-style notation without brackets and letting γ_1 be $\ulcorner\sim Mapnb\urcorner$ and γ_2 be $\ulcorner q\urcorner$. B^* will be the formula resulting from replacing the **IND**-contradiction $\ulcorner\sim Mapnb\urcorner$ with a **PC** contradiction, e.g., it could be,

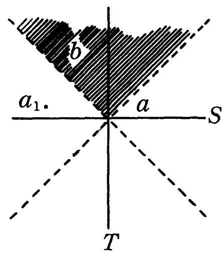
$$\Delta_2\Delta_1\Delta_4\gamma\Delta_3\gamma\gamma\gamma \text{ (i.e., a designation of } \ulcorner((q \wedge \sim q) \supset q) \vee \sim q\urcorner)$$
which is a **PC**-tautology

In general every **IND** wff B will have a corresponding B^* formula, the **PC** truth-table of which will depict a valid or invalid formula of **PC**. Now suppose that B has a valid **IND** truth-table, then (given the above) the corresponding **PC** formula B^* will have a valid **PC** truth-table and since the **PC** is complete, B^* will be derivable. *But this shows that every wff* **IND**-*valid formula B is derivable in* **IND** *as well:* all we need do is derive B^* and then replace tautologies and contradictions in it by the appropriate modal formulas according to the rules of substitution. (Contingent modal wffs can be substituted for non-modal singular propositional variables.)

Every wff of **IND** falls into one of the classes (I), (II), or (III). In the case of the first two we have seen that every wff non-theorem is invalid, in the case of wffs of type III that every valid wff of this sort is a theorem. This completes the proof that **IND** is complete.

4 *Further Interpretation of* **IND**. *The Meanings of M, L, n, b, and the relation* $<$. Von Wright (in [8]) has pointed out the following difficulty for any logic which has the law of non-contradiction given that time is continuous. Consider any (tenseless) proposition pt_1 which is not true at every time. Then for a time $t_2 \sim pt_2$. Then it appears that there must be an instant, the "last" instant of the interval t_1, at which p is true and such that at the "next" instant $\sim p$ is true. But between any two instants (in this case the last instant of t_1 and the first of t_2) there is another instant, at which (it appears) $p \wedge \sim p$ will be true. Alternatively we might try to say that the last instant of t_1 is the first instant of t_2 which will again give us the result that for that instant $p \wedge \sim p$. An exactly analogous problem arises for tense logics and for **IND**; in the former the temporal point, for example, at which Fp if true, at some time ceases to be true and p becomes true; in the latter where $M\_p\_$ ceases to be true and $L\_p\_$ becomes true or false. C. L. Hamblin (in [1]) has in effect solved this problem in his formalization of the topological properties of temporal intervals. His account provides us with some of the characteristics of the sequence T over which a and b of **IND** range. The time scale need not, of course, be discrete providing it is made up of intervals each of which are infinitely divisible ([1], p. 94): **IND** does not require that time be discrete or continuous, or if the latter it does not specify to which sort of continuum time must correspond. (Since this is a non-logical question this seems to me to be as it should be.)

Given this, the first point with regard to the reference of $\ulcorner a \urcorner$ and $\ulcorner b \urcorner$ is that these can refer either to temporal intervals or to instants depending on the types of states of affairs described by the propositions to which the variables attach. Similarly a and b can be separated by either intervals or instants but for any proposition $\ulcorner pab \urcorner$ there will be temporal instants or intervals t_1 and t_2 which *could* be designated by $\ulcorner a \urcorner$ and $\ulcorner b \urcorner$ such that t_1 is next to t_2 (as defined in [1]). $\ulcorner M\alpha \urcorner$ is taken to mean that α is 'causally open', i.e., α can be brought about or prevented; $\ulcorner L\alpha \urcorner$ means that it is causally closed *and true that* $\ulcorner \alpha \urcorner$. Correspondingly $\ulcorner a < b \urcorner$ does not simply mean that a is temporally prior to b, but, rather, that b is absolutely future relative to a. Where the times which are the range of the variables $\ulcorner a \urcorner$ and $\ulcorner b \urcorner$ are defined in terms of events then the events which define the set of times which define the possible designata of b are those events which are in the absolute future relative to the point-instant referred to by $\ulcorner a \urcorner$ (or the first instant of the closed temporal interval referred to by $\ulcorner a \urcorner$). Thus an a-event might in one sense be *after* a b-event but it would then have to be (in the terminology used by philosophical commentators or Relativity Theory) *topologically simultaneous* with the event defining the time referred to by $\ulcorner a \urcorner$. This could be summed up in the following diagram (for convenience taking space to be one-dimensional):



The dotted lines represent particles moving with the velocity of light: $\ulcorner a < b \urcorner$ means that $b \in H$ where H is the set of times defined by the set of events "occurring within" the shaded area. Thus a_1 on the diagram, though it occurs later than a, does not define a time which is a possible designatum of $\ulcorner b \urcorner$; $\ulcorner a \urcorner$ could be used to refer to a_1, but then, of course, we would have a different set of b-events. Thus an a-event and any event in a's absolute past is a *possible cause* of a b-event. Thus **IND** is *indeterministic* in a sense consistent with Special Relativity (although the latter as usually set up is in fact, deterministic)[7]: (i) causes must come before their effects[8]; (ii) past events are causally closed (nothing can 'now' effect or affect them); future events are causally open.

The components of a semantical model needed to achieve a full assignment of truth-values to wffs of **IND** are then, [1] a set of events (the temporal "duration" of the existence of an object counts as an event). Each of these events in conjunction with the others defines a linear ordered time (an ordered set of times). Thus relative to any instant (point-event) a there is a set T of times ordered by the relation $<$ (as defined above). [2] A valuation V which assigns truth-values $V[A]$ to wffs A of **IND** for *any* $t \in T$. The set T plus $<$ is a model structure $\mathfrak{M}$. Strictly speaking, in order to preserve the truth of the first axiom we would also need to specify that for all $a \in T$ there is a $b \in T$ such that $a < b$, i.e., there is no last time. (T is not the set of all times but a subset forming a local linear time.) A valuation V of **IND** on $\mathfrak{M}$ is a function which for each t, $t^1 \in T$ designated by a and b (according to the rules governing $<$) and sentence variable p, M-p--, L-p-- etc., of **IND** assigns a unique value 1 or 0 in a and b. The rules for these assignments have already been given. It is worth noting that **IND** has a linear model structure as opposed to a *branching* one. (An attempt to justify this philosophically is given below.) Axiom 1 of **IND** amounts to the presupposition that the whole of the "future" is causally open. It might be claimed that this presupposition rests on a confusion between the ontological and the epistemological: although any future event will have a probability slightly less than one relative to our knowledge, it

7. I have tried to show that these can be made consistent in "Special Relativity and Indeterminism," *Ratio*, vol. XV (1973), pp. 107-110.

8. This is just a presupposition of **IND** (I am not claiming that it has been established that this is a necessary truth.)

may "really" be necessary that it is going to occur. But here the ontological-epistemological distinction seems to me quite pointless for the epistemological includes what it is *possible* to know: hence if the ontological is understood to fall outside this realm it is not something of which we can intelligently speak at all.

One of the major difficulties for tense modal logics, which are not neutral with regard to determinism, is supposed to be in discovering truth-conditions for the future tense. (See, for instance [6]). **IND** presents no difficulties in this regard. The notation $\ulcorner pba \urcorner$ corresponds to the (simple) past tense and $\ulcorner pab \urcorner$ to the future tense. $\vee a[pab] = 1$ ($\vee$ gives the value true to $\ulcorner pab \urcorner$ at a) if $\vee b[pbb] = 1$ for some b such that $a < b$. This is also the condition for giving truth to $\vee b[pab]$ and, in fact, for *any* time t, $\vee t[pnb] = 1$ iff $\vee b[pnb] = 1$; so (as we have seen) the subscripts on $\vee$ can be dropped. What this comes to is that although something equivalent to tenses and indeterminism are expressed in **IND**, there are no special conditions on the assignment of truth-values to non-modal "tensed" propositions. This accords best, I think, with both the requirements of intuition (or ordinary language), of science and the theoretical requirements of "simplicity": ordinarily we would allow that if it is now the case that p, then it was the case that it would be the case that p and that it was the case that it would be the case that it would be the case that p etc. (See [5] and [6]). The fact that it is scientifically impossible to have known that p just shows that it is the (epistemically) modal proposition which has a truth-value other than true. Similarly it seems to me exceedingly artificial and non-intuitive to draw a distinction in a formal system between "it is not the case that it will be that p" and "it will be the case that it is not the case that p (see [5]) if the fact this distinction is supposed to reflect can be expressed formally in some other way. Lastly to say that it is possible (or probable to a certain degree) that p will be true is not to say that p *is* true in an "alternative future" (or in a "probable-to-the-nth degree future") but rather to say that p may be in the *actual* future. **IND** purports to provide a semantical model of this independent of appeal to alternative futures. Generally in **IND** wffs true at one time are true at every other time, i.e., we do *not* have this sort of case: it is correct to say at time t_2 that p was going to be true at t_1 but it cannot be asserted at t_1 that p is going to be true. The same holds with respect to modalities: any modality a proposition has at time t it has *at time* t "from the point of view" of any other time t' but propositions may have different modalities at the different times they are asserted.

I will now briefly discuss some theorems and non-theorems of **IND** to show that these accord with what we should want in an indeterministic system. $\ulcorner MaMapab \urcorner$ comes out *false*, i.e., if a state of affairs, in this case pab, is causally open at time a; then the fact *that* it is causally open cannot itself be causally open since this would permit the possibility that *at time a pab* is causally closed, which is not possible. We *do* have (which is exactly analogous to L_3 in Prior's Ockhamist system) $\ulcorner Mapnb \supset LnMapnb \urcorner$ (**IND** 8):

if anything is causally open at a time t then it is causally open relative to any other time which belongs to the same local time. Of course relative to some event at a time *outside* T, pnb may be topologically simultaneous and hence not causally open, but since it cannot be absolutely past this could not be the basis of an objection to **IND**. Given that something is indeterminate (causally open) then it is also indeterminate that it is causally closed at a later time, i.e., we have T_1: $\ulcorner Mapnb \equiv MaLbpnb \urcorner$ (this can be proved using **IND** 1 and 5), and T_2: $\ulcorner Mapnb \equiv \sim Lapnb \urcorner$ (trivially since the antecedant is a substitution in **IND** 1 and the consequent is **IND** 4); so we have T_3: $\ulcorner \sim Lapab \equiv \sim LaLbpab \urcorner$. T_1 is important to the extent that it illustrates how **IND** constitutes a change in and an extension of Prior's Ockhamist system; its (inexact) analogue, $\ulcorner Mp \equiv MLp \urcorner$ cannot (as we have already seen) be consistently adjoined to Prior's system. We also have T_4: $\ulcorner Mapab \equiv Ma \sim pab \urcorner$.

It might seem that the truth of some of these theorems simply arises from the fact that in **IND** we just have the two temporal variables a and b. Presuming if we added a denumerably infinite set of such variables we might, instead of rejecting a wff like $\ulcorner MaMapab \urcorner$, have formulas like: $\ulcorner Mt_1pt_3t_4 \supset Mt_1Mt_2pt_3t_4 \wedge Mt_1Lt_2pt_3t_4 \urcorner$, the first conjunct of the consequent being true since $t_1 < t_2 < t_3 < t_4$. But given our supposition that (the whole of) the future is causally open and that what is causally open at a given time t is causally open *at that time* relative to any other time this formula is simply false: we just have such formulas as $\ulcorner Mt_1pt_3t_4 \supset L_nMt_1pt_3t_4 \urcorner$ which add nothing significant to what can be said in the restricted notation of **IND**. We also want T_5: $\ulcorner (Mapab \wedge pab) \equiv Lbpab \urcorner$ (from **IND** 1, 5 using conditional proof), and T_6: $\ulcorner \sim pab \supset Mapab \urcorner$.

In this last section I want to say a bit more about what I think are the advantages of **IND** and indicate some difficulties that seem to arise with other formalizations. In this connection we should consider an ingenious and elegant system of tense-indeterministic logic suggested by Thomason ([8]). I will try to show that there are sufficient difficulties with this system to at least make **IND** worth considering as an alternative. Thomason's system rests on a distinction which certaintly has application in other contexts (e.g., V. van Fraassen [9]), viz., the distinction between implication, $\Vdash A \supset B$ and *semantic consequence*. $A \Vdash B$ "It always is the case that if [the first] holds so does [the second] but depending on the nature of the Language $\mathcal{L}$ and the bivalent valuations of $\mathcal{L}$, the converse may fail" ([8], p. 273). A future-tense proposition is true if it is true in *any* alternative future.[9] L is taken to mean "unavoidability or *inevitability*; a thing is inevitable if it is the case with respect to all alternative

9. Truth-conditions for the future tense are given in Thomason's system in this way. $V\alpha(Fp) = 1$ iff for all $h \in H\alpha$ there is a $\beta \in h$ such that $\alpha < \beta$ and $V\beta L(p) = 1$ ([8], p. 274). $H\alpha$ is a set of histories containing the time α, i.e., a linear pathway through a model structure: the part of h beyond α corresponds to a *possible future* for α (for details *cf.* [8], p. 267). The idea of possible futures is discussed further below.

futures"[10] ([8], p. 275) Thomason gives us the following formulas defining the properties of L (I have numbered them differently):

(1) $A \Vdash LA$
(2) $Fp \Vdash LFp$
(3) $\Vdash p \supset Lp$
(4) $\nVdash Fp \supset LFp$
(5) $\nVdash PFp \supset PLFp$ (where F and P are the future- and past-tense operators indicates a rejected formula).

Now this distinction between semantical consequence as in (1) and implication as in (3) and (4) depends, as Thomason explains on pages 272, 273, on a rejection of bivalence (although excluded middle is preserved): "Any formula $A \vee \sim A$ will be valid . . . whereas in many cases it will be possible for neither A nor $\sim A$ to be true" ([8], p. 272). It is important to realize that this requires that where T = 'it is true that' Tarski's principle holds only as a consequence, $A \Vdash TA$ but not as an implication; so that for some $A \nVdash A \supset TA$ ([8], p. 273).[11] To avoid the implication of the negation of (4) above from (3) Thomason also must introduce special restrictions on the substitution of propositional variables. Nonetheless there seem to be pretty obvious objections to (1)-(5) above, e.g., by using (1) and conditional proof we would seem to be able to infer the negation of (4). Furthermore L and T come to the same thing in Thomason's system: "We have $TA \Vdash LA$ and $LA \Vdash TA$. In this sense truth and inevitability are coincident" ([8], p. 278). Then given that $\ulcorner p \supset Tp \urcorner$ Thomason's system would appear to have the absurd consequence that it is not the case that Fp implies Fp. These criticisms as stated are incorrect; the way in which Thomason makes his system consistent can be fairly summarized I think in three closely related points which are explained in terms of each other: (1) a distinction between supposition and assertion; (2) the rejection of bivalence; (3) a kind of difference between T and L based on the fact that the model structures in Thomason's semantics contain alternative futures. "[1] To suppose that p will be is to posit that we will be in a situation in which p is true, that we will follow a history h in which p is sooner or later satisfied. But this is quite different from positing that such histories are the only alternatives now open; this would amount to positing that p is inevitable. [2] In our semantic theory this difference between supposing that p will be and supposing that it is now true that p will be is represented by the difference between making Fp an antecedent of an implication as in (4) (p. 562) above making it a premiss of the consequence relation as in (2) (p. 562 above) ([8], p. 276). [3] Our theory . . . allows (indeed, forces) us to say that *having been true* is different from *having been inevitable*, at least as far as future-tense statements go . . ., [i.e.], $PTFp \nVdash PLFp$" ([8], p. 279).

10. The truth-conditions for L: $V\alpha h[Lp] = 1$ iff $V\alpha g(p) = 1$ for all $g \in H\alpha$.

11. *Cf.* also Thomason's footnote 13.

Concerning [1] and [2] above, there is of course a distinction between supposing a proposition is true and asserting that it is true, but the only way we can make sense of holding that $A \Vdash LA$ but that in some cases $\Vdash A \supset LA$ (e.g., where A is Fp) is, as Thomason points out, to hold that in some cases $\Vdash A \supset TA$. *But clearly the only way a denial of $A \supset TA$ can make sense is to allow a third truth-value.* Thomason in effect says this when he admits his theory requires a rejection of bivalence.[12] Let us call this third truth-value "$\frac{1}{2}$"; then where A has the value $\frac{1}{2}$, TA will have the value 0 and $A \supset TA$ will come out false as Thomason's system requires. But this leaves Thomason's account open to the standard objection to three-valued systems as explanations of the logic of future contingents viz., where α and β each have the value $\frac{1}{2}$, so must $\alpha \wedge \beta$ from which the absurdity follows that a contradiction is not necessarily false in the system. I do not, of course, mean that a third truth-value explicitly occurs in Thomason's system—it does not—but that this is what the rejection of the for all A, $A \supset TA$ may come to. A slightly different way of looking at it would be to say that where Fp occurs as the antecedent of a conditional it really means MFp or (which in Thomason's system amounts to the same thing), Fp is "true" only relative to a time on one arm of an alternative future. This is what permits Thomason to assert $PTFp \nVdash PLFp$: there are no alternative pasts hence if Fp *was* true it must have been true in what was the actual future; but the actual future is only one out of a number of alternatives, so this does not entail that it was the case that LFp. But Fp's *present* truth in some alternative future does not permit us to say bluntly that Fp is true for this would permit the assertion of $Fp \wedge \sim Fp$. Thus this "true in an alternative future or true only *qua* antecedent of a conditional" can only mean possibly true and might as well be symbolized as such. What Thomason explicitly says that he is doing is not introducing a third truth-value but providing "a strategy for introducing truth-value gaps since we are now determined that [future-tense] formulas . . . should be neither true no false under certain conditions." ([8], p. 272) The method (which we have already roughly described) is the one used by van Fraassen in dealing with existence presuppositions and the paradoxes of self-reference. It would thus be better perhaps to use '__' rather than '$\frac{1}{2}$' for the tacit third truth-value of Thomason's system. I mention this merely to point out that any criticism of Thomason's method of dealing with future contingents does not apply to van Fraassen's general method of employing truth-value gaps for to say (for example) that '$F(a) \wedge \sim F(a)$' where 'a' lacks reference is neither true nor false makes perfect sense as does "'$p \wedge \sim p$' is neither true nor false" where p is the sentence 'this sentence is false'. But the

12. Thomason makes a rather desperate attempt (p. 276) to distinguish between saying "if p is true then . . . " and "supposing p is true then . . . ". This appears to me unintelligible unless glossed in the above way. I realize that Thomason's system is *not* explicitly three-valued: what I am arguing is that, despite appearances, no consistent interpretation can be given it without the tacit introduction of a third truth-value.

truth-value "gap" in Thomason's system (unlike that in van Fraassen's) amounts to "is possibly true" and it seems to me to be unintelligible to suggest that a statement like "It will be that p throughout the interval t_5 and it will not be that p throughout the interval t_5'' is possibly true.

It has already been mentioned that Thomason's system becomes straightforwardly inconsistent if conditional proof is allowed. The only rationale for this rejection which is not purely *ad hoc* seems to be the tacit admission of a third truth-value; so the above mentioned objection appears to arise again.

I mention lastly a more-or-less intuitive disagreement with Thomason. He holds that present truth (in the ordinary sense) amounts to *inevitability*; so that if it is allowed that any future-tense proposition is true or false (with no third alternative "true-or-false") then determinism follows. This seems to amount to the unproved assertion that the modal operators are redundant, and that α entails $L\alpha$ (reflected in Thomason's system by $\alpha \Vdash L\alpha$). Here I can only appeal to counter-intuition: to say (correctly) that its true that q will be is surely not to say its inevitable, for it is not even to say (necessarily) that its probable; very improbable things do happen and are occasionally even correctly predicted.

In general the idea of alternative futures seems, in one of its applications, to be quite vacuous. Possible futures cannot be held consistently to literally exist (since we would then have $Fp \wedge \sim Fp$ for any wfed Fp); so "true in a possible future" can only mean, "possibly true but, perhaps, actually false". The introduction of alternative futures seems to be partly motivated by the desire to provide for modal logic something as semantically definite as truth-tables are for sentential logic; but this can be done by simply introducing the requisite truth-tables and suitably restricted rules of substitution as in **IND**. It might be objected that **IND** itself has in effect a branching time since to say that '*Mapab*' is true seems to amount to '*pab* is true in a possible future'. But '*Mapab*' can be just as well glossed as, '*pab* is possible at an actual future time'. (As Thomason in effect points out it is difficult to see how a possible time series could have any *temporal relation*, such as *being future*, to any *actual* time.) However, given that facts are what answer to true propositions it could be objected that **IND** is committed to the existence of *possible facts* which are surely as problematic as alternative futures. In a sense I think this is so, but the sense is quite innocuous. **IND** does not commit us to the *existence* of facts which have among their properties "possibility" but only to the possible existence of things, properties, events, fields or whatever one cares to allow in one's ontology i.e., where f ranges over facts and ϕ is a property of facts, we are *not* committed to statements like $(\exists f)[\phi(f) \wedge M(f)]$ but only, where f is an *ordinary* property, to statements like '$M(\exists x)f(x)$'. Of course given that times are defined in terms of events we could define a "possible time-series" in terms of a possible "series" of events but the interpretation of **IND** does not *require* such a procedure for terms referring to actual times can be meaningfully appended to statements such as $M(\exists x)fx$.

In general the advantages of **IND** seem to me to be that it preserves what we *need* to preserve of the peculiarity of the future tense, only through the employment of causal operators, but without (a) the idea of alternative futures, (b) the idea of propositions changing in truth-value, (c) the difficulties that appear to arise in multi-valued logics when these are used to express indeterminism. **IND** might be extended in various ways. We could add as an axiom something analogous to $L(\alpha \supset \beta) \supset (L\alpha \supset L\beta)$. This would complicate the system a good deal since the first L would have to be followed by temporal variables; so that rules for the application of modal operators to truth-functionally complex propositions and rules governing the relations between the temporal variables following these modal operators and those (perhaps different ones) following the operators within the complex assertions would have to be developed.[13] Another obvious way of extending **IND** would be to add an infinite number of temporal variables (which infinity being dependent on which continuum time was thought to correspond to). It seems, however, unclear whether anything substantial would be accomplished by this extension. There are, of course, many types of assertion involving temporal variables or constants and modal terms which cannot be formulated in **IND**. A simple example would be: '$Mt_1pt_2t_3 \supset (Mt_3qt_4t_6 \supset Mt_7rt_8t_q)$' but clearly the truth-conditions of such a proposition are not purely logical and temporal, i.e., do not just depend on the functions of M, $\supset$ and the temporal variables $t_1 \ldots t_q$ but depend on the empirical content of p, q, and r; so it is unclear what point the ability to formulate such propositions in a logical calculus would have. Of course the addition of temporal variables would allow for the addition of wffs containing more iterated modal operators, e.g., we could have '$Lt\alpha \supset Lt''Lt'Lt\alpha$' where $t < t' < t''$ but, again, I do not know what the point of this would be. It is worth noting that even with a modification in the first axiom to permit determinate propositions about the future we would *not* have theorems of the sort, $\ulcorner Mt_0pt_1t_2 \supset Mt_{-1}Mt_0pt_1t_2 \urcorner$ $(t_{-1} < t_0 < t_2)$. Intuitively this is false since, from the fact that α is causally open *at a specific time t*, it does not follow that this fact is changeable *at t* but only that the non-modal part might be necessary at a *later* time. Thus all we have is the (significantly different) type of formula, $\ulcorner Mt_0pt_1t_2 \supset M_npt_1t_2 \urcorner$ $(n < 0)$; hence this does not provide clearly sufficient grounds for introducing more temporal variables either.[14]

There are two types of complication that would arise, I think, with the introduction of quantification into **IND**. (I just mention these here and do not

13. If the introduction of $L(\alpha \supset \beta) \supset (L\alpha \supset L\beta)$ were to have any significance the first asiom would have to be dropped in favour of one allowing for the possibility that part of the future is causally closed. On the various possibilities here see [3], pp. 120, 121.

14. It might be objected to formulas of this sort and to **IND** in general that α_{t4} could be M at t_1 (say), L at t_2 and M at t_3 again if for instance a causal sequence "in progress" at t_2 were interfered with at t_3 but it would seem to me that since this possibility of interference must have existed at t_2, αt_4 was not really L at that time.

attempt to develop a quantified **IND**.) It can be argued that there is a logical asymmetry of time with respect to *reference* consisting in the fact that there are at least two sorts of proper names which would have to be reflected in a temporal logistic by two types of constant: (1) names which are merely shorthand for definite descriptions (in Russell's sense), (2) names which simply refer without any one definite description or set of definite descriptions being understood to constitute the sense of the name. There cannot be names in the second sense for objects which do not yet exist: this will generate difficulties with regard to instantiation. Secondly we would not have in **IND** an analogue of the Barcan formula since (for instance) $Ma(\exists x)Fxab$ clearly does not imply $(\exists x)MaFxab$; so special rules would have to be introduced governing the order of modal operators and quantifiers.[15] **IND** is, of course, formally a very weak system but it has, I think, an important philosophical basis. As I have tried to show stronger formal systems which purport to express indeterminism either fail to do so at all or can be seen to entail absurdities. In fact it *seems* that any formally rich system will have these difficulties; so that if indeterminism is to be expressed at all formally it must be in a system very much like **IND**.

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15. **IND** is not "trivial" (is not equivalent to a system where L and M are redundant) because of the first four axioms. On this see [4], Chapter V.