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RECURSIVE AND RECURSIVELY ENUMERABLE MANIFOLDS. I

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Foreword In [1] I have presented a sketch of the *Local Recursive Theory*—a generalization of the Recursive Theory, which is quite different from other generalizations: instead of being a study in definability (as, for example, [6] of Platek), or a concrete interpretation (as the Metarecursive Theory of Kreisel-Sacks in [7]), or an abstract axiomatization (as the Theory of Uniformly Reflexive Structures of Wagner in [8]), Local Recursive Theory is the study of sets which admit a *local recursive structure*; this structure is induced via appropriate enumerations of *local neighborhoods* and an effective patching of such neighborhoods.

Local Recursive Theory, or the *Theory of Recursive and Recursively Enumerable Manifolds*, is a further development of the *Theory of Enumerations*, of an integral part of the Recursive Theory, which was systematically studied by Malcev and his students, especially by Yu. Ershov; in [1] I presented a first draft for such a development, considering only a very special case (of injective local enumerations). Here, I develop the Local Recursive Theory in its full generality and in many directions which were not even mentioned in [1].

With the exception of a few pages, the material of this monograph has not been published previously. The monograph was drafted for a course in Generalized Recursive Theory, at the Graduate School of Mathematics at the University of Notre Dame in the first semester of 1974/1975 year.

CHAPTER I—BASIC NOTIONS

Every map $u: N \to U$ of the set N of non-negative integers onto an at most denumerable, non-empty set U, is called an *enumeration* of U; if it is bijective it will be called an *indexing* of U. Using enumerations we can extend recursive notions to any enumerated set U. For example, a map $f: U \to U$ of U into U will be called *u-recursive* iff (if and only if) there is an r. (recursive) function $f^*: N \to N$, such that, for all $n \in N$,

$$f(u(n)) = u(f^*(n)), \tag{1.1}$$

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i.e., such that the diagram in Figure 1.1 commutes.

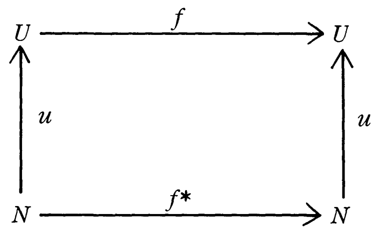

Figure 1.1

In case u is an indexing, this situation may be expressed by

(1.2) $$u^{-1} \circ f \circ u \text{ is an r. function.}$$

(Obviously, u^{-1} is the inverse of u, and $\circ$ denotes composition of functions.)

The Theory of Enumerations is exposed in the fundamental paper [2] of Malcev, and in the monograph [5] of Ershov. For the results I shall refer to both of those expositions. The fundamental idea of the Local Recursive Theory is the following one: suppose, for each $p \in P$, $\alpha_p: N \to A_p$ is an enumeration of the set A_p; thus, for each $p \in P$, one can pursue some recursive theory on A_p, using the enumeration α_p; now, if $A = \bigcup_{p \in P} A_p$, can one use the same enumerations to introduce some recursive theory on A? My answer is "yes", if one supposes the *local neighborhoods* A_p to be patched in an *effective* way—whenever their intersections are not empty. (By $\emptyset$ I shall denote the empty set.)

Definition 1.1 A non-empty set A is called a *Recursive Manifold* (an **RM**) iff:

(i) There is a family $\mathfrak{A}$ of enumerations $\alpha_p: N \to A_p$, $p \in P$, where each A_p is a non-empty subset of A and $A = \bigcup_{p \in P} A_p$.

(ii) For every pair $\langle p, p_1 \rangle \in P^2$ such that $A_p \cap A_{p_1} \neq \emptyset$, both $\alpha_p^{-1}(A_{p_1})$ and $\alpha_{p_1}^{-1}(A_p)$ are recursive sets, and there are numerical p.r. (partial recursive) functions

$$f_p: \alpha_p^{-1}(A_{p_1}) \to \alpha_{p_1}^{-1}(A_p) \text{ and } f_{p_1}: \alpha_{p_1}^{-1}(A_p) \to \alpha_p^{-1}(A_{p_1})$$

such that

(1.3) $$\alpha_p(n) = \alpha_{p_1}(f_p(n)), \text{ for all } n \in \alpha_p^{-1}(A_{p_1}),$$

and

(1.4) $$\alpha_{p_1}(n) = \alpha_p(f_{p_1}(n)), \text{ for all } n \in \alpha_{p_1}^{-1}(A_p).$$

In Figure 1.2 (see p. 267) the relations (1.3) and (1.4) are represented graphically.

The sets A_p are called *Local Neighborhoods*, the enumerations α_p are called *Local Enumerations* and the family $\mathfrak{A} = \{\alpha_p \mid p \in P\}$ is called the *Atlas*

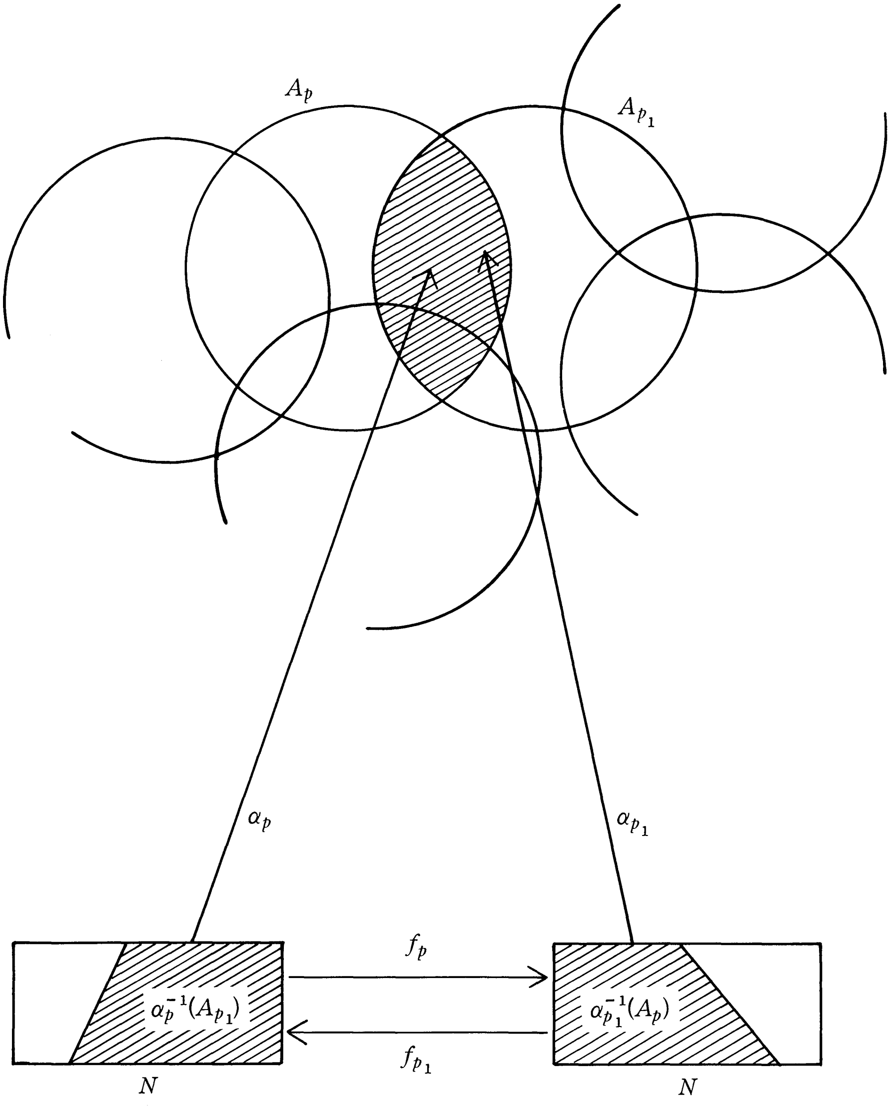

Figure 1.2

of the **RM** $\langle A, \mathfrak{A}\rangle$. In general, $\langle A, \mathfrak{A}\rangle$, $\langle B, \mathfrak{B}\rangle$, $\langle C, \mathfrak{C}\rangle$, . . ., will denote **RM**'s with atlases $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, . . ., respectively. We shall call sets A, B, C, . . ., the *Carriers* of the corresponding **RM**'s. In case all α_p are indexings (i.e., injective enumerations), we call $\langle A, \mathfrak{A}\rangle$ an *Injective* **RM** (an **IRM**). For such manifolds, (1.3) and (1.4) can be shortened to

(1.5) $\quad \alpha_{p_1}^{-1} \circ \alpha_p$ and $\alpha_p^{-1} \circ \alpha_{p_1}$ are p.r. functions with r. domains.

Every enumerated set $\langle U, \{u\}\rangle$ is an **RM**; in case u is bijective, $\langle U, \{u\}\rangle$ is an **IRM**. By **n** I shall denote the **IRM** $\langle N, \{\mathsf{I}\}\rangle$, where I is the identity on N. (In general, I_A will denote the identity on the set A.)

Example 1.1 Let A be a non-empty (infinite) set, and let $\alpha: N \to U$ be an enumeration (an indexing) of a subset U of A (which is infinite). If $A = U$, $\langle A, \{\alpha\}\rangle$ is an **RM** (an **IRM**). If $A \neq U$, let $P = A - U$ and to every $p \in P$ correspond the local neighborhood $A_p = U \cup \{p\}$ and the enumeration (the indexing) $\alpha_p: N \to A_p$, defined by

$$\alpha_p(n) = \begin{cases} p \text{ for } n = 0 \\ \alpha(n - 1) \text{ for } n \geqslant 1. \end{cases}$$

Let $\mathfrak{A} = \{\alpha_p | p \in P\}$. Then $\langle A, \mathfrak{A}\rangle$ is an **RM** (an **IRM**). Further, for all $p \neq p_1$, $A_p \cap A_{p_1} = U$, and $\alpha_p^{-1}(A_{p_1}) = \alpha_{p_1}^{-1}(A_p) = N^+ = N - \{0\}$, and $\alpha_p(n) = \alpha_{p_1}(n)$ for all $n \in N^+$. Figure 1.3 represents this last case.

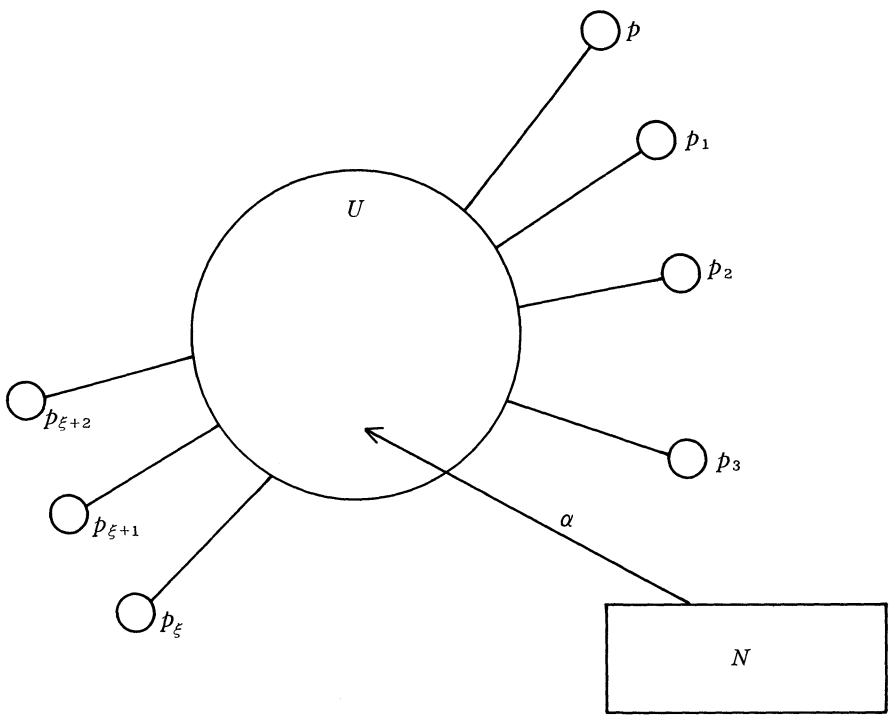

Figure 1.3

Example 1.2 Let $\langle A_i\rangle_{i\in N}$ be a sequence of non-empty recursive subsets of N. Let $A = \bigcup_{i=0}^{\infty} A_i$, let $\alpha_i: N \to A_i$ be recursive, with A_i as range, and let $\mathfrak{A} = \{\alpha_i | i \in N\}$. Then $\langle A, \mathfrak{A}\rangle$ is an **RM**. Namely, if $i \neq j$ and $A_i \cap A_j \neq \emptyset$, then $A_i \cap A_j$ is recursive, and both $\alpha_i^{-1}(A_j)$ and $\alpha_j^{-1}(A_i)$ are recursive. Define then

$$f_i(n) = \mu_y(\alpha_i(n) = \alpha_j(y)) \text{ for all } n \in \alpha_i^{-1}(A_j)$$

and

$$f_j(n) = \mu_y(\alpha_j(n) = \alpha_i(y)) \text{ for all } n \in \alpha_j^{-1}(A_i).$$

Then $\alpha_i(n) = \alpha_j(f_i(n))$ for all $n \in \alpha_i^{-1}(A_j)$, and $\alpha_j(n) = \alpha_i(f_j(n))$ for all $n \in \alpha_j^{-1}(A_i)$. In the case in which $i \neq j$ implies $A_i \cap A_j = \emptyset$ for all $i, j \in N$, we may suppose that all A_i are only r.e. sets. In the case in which all A_i are infinite and recursive we may suppose that all α_i are recursive and increasing; in this case $\langle A, \mathfrak{A}\rangle$ becomes an **IRM**.

Example 1.3 Let Ω be the class of all ordinals, and let Ω_0 be its subclass consisting of zero and of all limit-ordinals. To every $\xi \in \Omega_0$ there corresponds an enumeration

$$\alpha_\xi \colon N \to \{\xi + n \mid n \in N\}.$$

Let $H = \{\alpha_\xi \mid \xi \in \Omega_0\}$. Then $\langle \Omega, H\rangle$ is an **RM**, very trivial indeed, since $\xi \neq \eta$, $\xi, \eta \in \Omega_0$, imply $U_\xi \cap U_\eta = \emptyset$, where U_ξ and U_η are ranges of α_ξ and α_η respectively. In case each α_ξ is injective (for example, if one defines $\alpha_\xi(n) = \xi + n$), $\langle \Omega, H\rangle$ is an **IRM**. Initial segments of $\langle \Omega, H\rangle$ are **RM**'s too. If Ω_σ is the set of all ordinals $<\sigma$, and $\Omega_{\sigma;0}$ the subset of Ω_0 consisting of zero and of all limit-ordinals which are $<\sigma$, with $H_\sigma = \{\alpha_\xi \mid \xi \in \Omega_{\sigma;0}\}$, $\langle \Omega_\sigma, H_\sigma\rangle$ is an **RM** (an **IRM** in case all α_ξ are indexings).

Example 1.4 Let H be a (non-immune and infinite) subset of N. Let $h_0 \colon N \to H$ be a recursive (increasing) function with a recursive (infinite) range H_0. Let $h \colon N \to H - H_0$ be an enumeration of $H - H_0$ (an indexing of $H - H_0$, in which case we suppose it infinite). In a trivial way, $\langle H, \{h_0, h\}\rangle$ is an **RM** (an **IRM**).

A more interesting manifold is constructed as follows (supposing that h is injective): define, for $n \geq 0$, the enumeration (indexing) h_{n+1} by

$$h_{n+1}(i) = h(i) \text{ for } 0 \leq i \leq n,$$
$$h_{n+1}(i) = h_0(i - n - 1) \text{ for } n < i.$$

Let H_n be the range of h_n, and $\mathfrak{H} = \{h_n \mid n \in N\}$. Then, $\langle H, \mathfrak{H}\rangle$ is an **RM** (an **IRM**). This is easily checked: $n < m$ implies $H_n \cap H_m = H_n$ and

$$H_m - H_n = \{h(n), h(n+1), \ldots, h(m-1)\}.$$

Thus, defining, for $n < m$,

$$f_m(k) = \begin{cases} k \text{ for } 0 \leq k \leq n-1, \\ k + n - m \text{ for } m \leq k, \\ \text{undefined for } n \leq k \leq m - 1, \end{cases}$$

and

$$f_n(k) = \begin{cases} k \text{ for } 0 \leq k \leq n-1, \\ \\ k - n + m \text{ for } n \leq k, \end{cases}$$

we have

$$h_m(k) = h_n(f_m(k)) \text{ for all } k \in D_{f_m},$$

and

$$h_n(k) = h_m(f_n(k)) \text{ for all } k \in D_{f_n},$$

where D_f denotes the domain of the function f. (We shall write R_f for the range of f.) This r. manifold $\langle H, \mathfrak{H}\rangle$ has the property that $H_n \subset H_{n+1}$ for all $n \in N$.

Example 1.5 An **RM** (**IRM**) $\langle A, \mathfrak{A}\rangle$ will be called an *amalgam* iff $A_p \cap A_{p_1} \neq \emptyset$ implies that $\alpha_p(n) = \alpha_{p_1}(n)$ for both $n \in \alpha_p^{-1}(A_{p_1})$ and $n \in \alpha_{p_1}^{-1}(A_p)$. On such a manifold, in case it is injective, we can define $\overline{\overline{P}}$ additive operations $\oplus_p$ and $\overline{\overline{P}}$ multiplicative operations $\odot_p$, $p \in P$, ($\overline{\overline{P}}$ = the cardinal of the set P), by

$$\alpha_p(n) \oplus_p \alpha_p(m) = \alpha_p(n + m),$$

and

$$\alpha_p(n) \odot_p \alpha_p(m) = \alpha_p(n \cdot m).$$

It is interesting to note: if $\alpha_p(n)$, $\alpha_p(m)$, and $\alpha_p(n + m)$ are in $A_p \cap A_{p_1}$ then $\alpha_p(n) \oplus_p \alpha_p(m) = \alpha_{p_1}(n) \oplus_{p_1} \alpha_{p_1}(m)$, and similarily for $\odot_p$. Thus, one can consider A_p's as "sheets" of the amalgam $\langle A, \mathfrak{A}\rangle$; on each sheet A_p one can develop an arithmetic which will be compatible with the arithmetic on another sheet A_{p_1}, in case $A_p \cap A_{p_1} \neq \emptyset$.

Let me introduce now some first effective notions on manifolds. In the following (if not indicated otherwise) $\langle A, \mathfrak{A}\rangle$, $\langle B, \mathfrak{B}\rangle$, $\langle C, \mathfrak{C}\rangle$, . . ., will denote **RM**'s (or **IRM**'s); then

$$\mathfrak{A} = \{\alpha_p \mid p \in P\},\ \mathfrak{B} = \{\beta_q \mid q \in Q\},\ \mathfrak{C} = \{\gamma_r \mid r \in R\},\ \ldots,$$

A_p, B_q, C_r, . . ., will denote respective ranges of enumerations α_p, β_q, γ_r, ...

Definition 1.2 (i) The set $X \subset A$ is $\mathfrak{A}$-*recursively enumerable* ($\mathfrak{A}$-r.e.), respective $\mathfrak{A}$-*recursive* ($\mathfrak{A}$-r.) iff, for every $p \in P$, $\alpha_p^{-1}(X)$, the inverse image of X under α_p, is an r.e., respective an r. subset of N.
(ii) The map f: $X \to B$, $X \subset A$, is $\mathfrak{A}$-$\mathfrak{B}$-*partial recursive* ($\mathfrak{A}$-$\mathfrak{B}$-p.r.) iff X is an $\mathfrak{A}$-r.e. set and, for every pair $\langle p, q\rangle \in P \times Q$, there is a p.r. arithmetical function $f_{p,q}$, with domain $D_{f_{p,q}} = \alpha_p^{-1}(X \cap f^{-1}(B_q))$, such that

(1.6) $$f(\alpha_p(n)) = \beta_q(f_{p,q}(n)),\ \text{for all } n \in D_{f_{p,q}}.$$

(iii) If f is both $\mathfrak{A}$-$\mathfrak{B}$-p.r. and total it is called $\mathfrak{A}$-$\mathfrak{B}$-*recursive* ($\mathfrak{A}$-$\mathfrak{B}$-r.).

In Definition 1.2, in case $X \cap f^{-1}(B_q) = \emptyset$, $f_{p,q}$ is meant to be the nowhere defined p.r. function Λ.

In considering *functionals*, i.e., maps $f: X \to N$, $X \subset A$, and *anti-functionals*, i.e., maps f: $D \to A$, $D \subset N$, we shall consider N always as the **IRM n** $= \langle N, \{\mathsf{I}\}\rangle$, where I is the identity on N. In this way, every α_p: $N \to A_p$, as an anti-functional, is $\{\mathsf{I}\}$-$\mathfrak{A}$-recursive. In case it is injective, its inverse α_p^{-1}: $A_p \to N$, as a map from A into N, is an $\mathfrak{A}$-$\{\mathsf{I}\}$-partial recursive functional, with $\mathfrak{A}$-recursive domain A_p. Also, I_A, the identity on A, is an $\mathfrak{A}$-$\mathfrak{A}$-recursive map: if f_p and f_{p_1} are as in (1.3) and (1.4) then, in case $A_p \cap A_{p_1} \neq \emptyset$,

$$\mathsf{I}_A(\alpha_p(n)) = \alpha_{p_1}(f_p(n)),\ \text{for all } n \in \alpha_p^{-1}(A_{p_1}),$$

and

$$1_A(\alpha_{p_1}(n)) = \alpha_p(f_{p_1}(n)), \text{ for all } n \in \alpha_{p_1}^{-1}(A_p).$$

Similarly, every constant map $f: A \to \{a\}$, where a is a fixed element of A, is $\mathfrak{A}$-$\mathfrak{A}$-recursive.

For subsets of A^m and maps from A^m we enlarge Definition 1.2.

Definition 1.2′ (i) The set $X \subset A^m$, $m \geq 1$, is $\mathfrak{A}$-r.e. (respectively $\mathfrak{A}$-r.) iff, for every m-tuple $\langle p_1, \ldots, p_m\rangle \in P^m$, the set

$$(1.7) \qquad X_{p_1,\ldots,p_m}^{-1} = \{\langle n_1, \ldots, n_m\rangle \in N^m | \langle \alpha_{p_1}(n_1), \ldots, \alpha_{p_m}(n_m)\rangle \in X\}$$

is an r.e. (respectively r.) subset of N^m.
(ii) Let $X \subset A^m$. The map $f: X \to B$ is $\mathfrak{A}$-$\mathfrak{B}$-p.r. iff X is $\mathfrak{A}$-r.e. and, for every $(m+1)$-tuple $\langle p_1, \ldots, p_m; q\rangle \in P^m \times Q$, there is a p.r. function

$$f_{p_1,\ldots,p_m;q} \text{ with domain } \alpha_{p_1}^{-1}(f^{-1}(B_q)) \times \ldots \times \alpha_{p_m}^{-1}(f^{-1}(B_q))$$

such that

$$(1.8) \qquad f(\alpha_{p_1}(n_1), \ldots, \alpha_{p_m}(n_m)) = \beta_q(f_{p_1,\ldots,p_m;q}(n_1, \ldots, n_m)),$$

for all $\langle n_1, \ldots, n_m\rangle \in D_{f_{p_1,\ldots,p_m;q}}$.
(iii) Let X be a subset of A^m. A map $g: X \to B^n$, $g = \langle g_1, \ldots, g_n\rangle$, is $\mathfrak{A}$-$\mathfrak{B}$-p.r. iff X is an $\mathfrak{A}$-r.e. set and each $g_i: X \to B$ an $\mathfrak{A}$-$\mathfrak{B}$-p.r. map.

For example, the projection $f: A^2 \to A$, defined by $f(x, y) = x$, is $\mathfrak{A}$-$\mathfrak{A}$-recursive. Defining $f_{p,p_1;p}(n, m) = n$, we have (in case $A_p \cap A_{p_1} \neq \emptyset$)

$$f(\alpha_p(n), \alpha_{p_1}(m)) = \alpha_p(f_{p,p_1;p}(n, m)) = \alpha_p(n), \text{ for all } n \in N.$$

Similar is the situation with $g(x, y) = y$. Remark that in case $\langle B, \mathfrak{B}\rangle$ is an **IRM**, one can define the p.r. function in (1.8) by

$$(1.9) \qquad f_{p_1,\ldots,p_m;q}(n_1, \ldots, n_m) \simeq \beta_q^{-1}(f(\alpha_{p_1}(n_1), \ldots, \alpha_{p_m}(n_m)).$$

Definition 1.3 Let $X \subset A$. χ_X, the *characteristic functional* of X, is defined by

$$(1.10) \qquad \chi_X(x) = \begin{cases} 0 \text{ if } x \in X, \\ 1 \text{ if } x \in \complement X = A - X. \end{cases}$$

Theorem 1.1 (i) *The set $X \subset A$ is $\mathfrak{A}$-recursive iff both X and $\complement X$ are $\mathfrak{A}$-r.e.*
(ii) *The set $X \subset A$ is $\mathfrak{A}$-recursive iff its characteristic functional χ_X is $\mathfrak{A}$-$\{1\}$-recursive.*

Proof: (i) If X is $\mathfrak{A}$-recursive then each $\alpha_p^{-1}(X)$ is recursive; therefore, each $\alpha_p^{-1}(\complement X) = \alpha_p^{-1}(A_p - X) = N - \alpha_p^{-1}(X)$ is recursive too. Conversely, if both $\alpha_p^{-1}(X)$ and $\alpha_p^{-1}(\complement X)$ are r.e. they are recursive, i.e., X is $\mathfrak{A}$-recursive.
(ii) Remark that a functional $f: A \to N$ is $\mathfrak{A}$-$\{1\}$-recursive iff to each $p \in P$ there corresponds a recursive arithmetical function f_p such that $f(\alpha_p(n)) = f_p(n)$, for all $n \in N$. Now, if X is recursive, $\chi_X \circ \alpha_p$ is just the characteristic function of the recursive set $\alpha_p^{-1}(X)$.

It is evident that $X \subset A$ is $\mathfrak{A}$-r.e. iff it is the domain of an $\mathfrak{A}$-$\mathfrak{A}$-p.r. map from A into A. Also, every such set is the range of such a map. However, *it is not necessarily true that the range of every* $\mathfrak{A}$-$\mathfrak{A}$-p.r. map $f: X \to A$ *is an* $\mathfrak{A}$-r.e. set. Namely,

$$f(X) = \bigcup_{p \in P} f(X \cap A_p). \tag{1.11}$$

Consider now $D_p = \alpha_p^{-1}(X \cap A_p)$, in case it is not empty. It is a r.e. subset of N and for every p_1, such that $X \cap f^{-1}(A_{p_1}) \neq \emptyset$, there is a p.r. function f_{p,p_1} with domain $\alpha_p^{-1}(X \cap f^{-1}(A_{p_1}))$ such that

$$f(\alpha_p(n)) = \alpha_{p_1}(f_{p,p_1}(n)), \text{ for all } n \in D_{f_{p,p_1}}.$$

This gives:

$$\alpha_{p_1}^{-1}(f(X \cap A_p)) = \alpha_p^{-1}(X \cap f^{-1}(A_{p_1})),$$

and this proves that each $f(X \cap A_p)$ is an $\mathfrak{A}$-r.e. set, since the set $\alpha_p^{-1}(X \cap f^{-1}(A_{p_1}))$, as the domain of a p.r. arithmetical function, is a r.e. set. However, for each $p_1 \in P$,

$$\alpha_{p_1}^{-1}(f(X)) = \bigcup_{p \in P} \alpha_{p_1}^{-1}(f(X \cap A_p)), \tag{1.12}$$

and, although each member in the union in (1.12) is a r.e. subset of N, the union itself is not necessarily a r.e. subset of N.

Example 1.6 Consider the IRM of Example 1.3, with $\alpha_\xi(n) = \xi + n$. Let ω be the smallest denumerable ordinal, let $\xi_0 = 0$ and $\xi_n = \omega \cdot n$ for $n \geq 1$. Let $d: N \to N$ be any increasing function whose range $\mathbf{D}$ is not a r.e. set, and $X = \{\xi_n | n \in N\}$. Then X is an H-r.e. subset of Ω. (Each $\alpha_\xi^{-1}(X)$ is either empty or a singleton.) Define $f: X \to \Omega$ by $f(\xi_n) = d(n)$. Then $f(\alpha_\xi(i))$ is defined only if $\xi = \xi_n$ for some $n \in N$, and $i = 0$; in such a case

$$f(\alpha_{\xi_n}(i)) = \begin{cases} \alpha_0(d(n)) \text{ for } i = 0, \\ \text{undefined otherwise,} \end{cases}$$

i.e., f is an H-H-p.r. map. Yet, $\alpha_0^{-1}(f(X)) = \mathbf{D}$ is not a r.e. set, i.e., $f(X)$ is not an H-r.e. set.

In view of the previous example, one may ask for the validity of the Graph-Theorem for maps of one RM into itself. Such a theorem is valid without additional suppositions for IRM's only; in the general case of RM's I need one condition more.

Definition 1.4 (i) We say that the atlas $\mathfrak{A}$ is *positive* iff, for every $p \in P$, the numerical predicate $\mathfrak{A}_p$ of two variables, defined by

$$\mathfrak{A}_p(n, m) \leftrightarrow \alpha_p(n) = \alpha_p(m), \tag{1.13}$$

is recursively enumerable, it is *negative* iff, for every $p \in P$, $\sim\mathfrak{A}_p$, the negation of $\mathfrak{A}_p$, is r.e.; it is *solvable* iff it is both negative and positive.
(ii) We say that the RM $\langle A, \mathfrak{A}\rangle$ is *positive, negative,* and *solvable* iff its atlas $\mathfrak{A}$ is positive, negative, and solvable respectively.

Definition 1.4 leaves a huge number of RM's outside of its scope; I shall call such RM's *neutral.* It is evident that all IRM's are solvable. (For these, $\mathfrak{A}_p(n, m) \leftrightarrow n = m$.)

Theorem 1.2 (Graph Theorem) *Let* $\langle A, \mathfrak{A}\rangle$ *be a positive* RM, *and let* X *be an* $\mathfrak{A}$-r.e. *subset of* A. *Then, a partial map* $f\colon X \to A$ *is* $\mathfrak{A}$-$\mathfrak{A}$-*partial recursive iff its graph* G_f *is an* $\mathfrak{A}$-*recursively enumerable subset of* A^2.

Proof: First, let f be $\mathfrak{A}$-p.r. For $\langle p, p_1\rangle \in P^2$ consider the set (subset of N^2)

$$(G_f)^{-1}_{p,p_1} = \{\langle n, m\rangle \mid \langle \alpha_p(n), \alpha_{p_1}(m)\rangle \in G_f\} = \{\langle n, m\rangle \mid \alpha_{p_1}(m) = f(\alpha_p(n))\}.$$

Let f_{p,p_1} be p.r. with domain $\alpha_p^{-1}(X \cap f^{-1}(A_{p_1}))$, and such that

$$f(\alpha_p(n)) = \alpha_{p_1}(f_{p,p_1}(n)), \text{ for all } n \in D_{f_{p,p_1}}.$$

Then

$$\begin{aligned}(G_f)^{-1}_{p,p_1} &= \{\langle n, m\rangle \mid \alpha_{p_1}(m) = \alpha_{p_1}(f_{p,p_1}(n))\}\\ &= \{\langle n, m\rangle \mid \mathfrak{A}_{p_1}(m, f_{p,p_1}(n)) \wedge n \in D_{f_{p,p_1}}\},\end{aligned}$$

which proves (since $\langle A, \mathfrak{A}\rangle$ is positive) that each $(G_f)^{-1}_{p,p_1}$ is r.e., i.e., that G_f is $\mathfrak{A}$-r.e. (The sign $\wedge$ above denotes conjunction.)

Conversely, suppose that G_f is $\mathfrak{A}$-r.e., i.e., that each $(G_f)^{-1}_{p,p_1}$ is a r.e. subset of N^2. By definition of this set we have

$$f(\alpha_p(n)) = \alpha_{p_1}(m) \leftrightarrow \langle n, m\rangle \in (G_f)^{-1}_{p,p_1}.$$

Define f_{p,p_1} by

$$f_{p,p_1}(n) \simeq \text{some } m \text{ such that } \langle n, m\rangle \in (G_f)^{-1}_{p,p_1}.$$

f_{p,p_1} is p.r. and $f(\alpha_p(n)) = \alpha_{p_1}(f_{p,p_1}(n))$ for all $n \in D_{f_{p,p_1}}$, which proves that f is $\mathfrak{A}$-$\mathfrak{A}$-partial recursive. (The symbol $\simeq$ denotes conditional equality.)

Corollary 1.2.1 *For every* IRM $\langle A, \mathfrak{A}\rangle$ *and any* $\mathfrak{A}$-r.e. *set* $X \subset A$, *the partial map* $f\colon X \to A$ *is* $\mathfrak{A}$-$\mathfrak{A}$-p.r. *iff its graph is* $\mathfrak{A}$-r.e.

Remark that the proof of Theorem 1.2 establishes a sharper one-sided result: *in any* RM $\langle A, \mathfrak{A}\rangle$, *if* D_f *is* $\mathfrak{A}$-r.e. *and* G_f $\mathfrak{A}$-r.e. *then* f *is* $\mathfrak{A}$-p.r.

Similar to the case of direct images, we cannot say anything definite about inverse images of $\mathfrak{A}$-r.e. sets under $\mathfrak{A}$-$\mathfrak{A}$-p.r. maps. The following theorem is the only exception I know.

Theorem 1.3 *The inverse image of a* r.e. *subset of* N *under an* $\mathfrak{A}$-$\{I\}$-p.r. *functional* $f\colon X \to N$, $X \subset A$, *is an* $\mathfrak{A}$-r.e. *set.*

Proof: To each $p \in P$ there corresponds a recursive function f_p such that $f(\alpha_p(n)) = f_p(n)$, for all $n \in N$. Let $E \subset N$ be r.e. and such that $f(X) \cap E \neq \emptyset$. Then $\alpha_p^{-1}(f^{-1}(E)) = f_p^{-1}(E)$, which is a r.e. subset of N.

Theorem 1.4 *If* $X \subset A$ *is* $\mathfrak{A}$-r.e. *then there is a map* $\varphi\colon P \to N$ *such that*

$$X = \bigcup_{p \in P} \alpha_p(\omega_{\varphi(p)}), \tag{1.14}$$

where $\omega_i = \{n \mid V_y T_1(i, n, y)\}$ *is the i'th* r.e. *subset of N in the standard enumeration of all such subsets.*

Proof: Set $\omega_{\varphi(p)} = \alpha_p^{-1}(X)$, and remark that $\alpha_p(\alpha_p^{-1}(X)) = A_p \cap X$.

The following example exhibits the perils of replacing maps with functionals for **RM**'s $\langle A, \mathfrak{A}\rangle$, where $A \subset N$; it illustrates also bad sides of atlases whose cardinal is larger than the cardinal of the carrier A.

Example 1.7 To every arithmetical function $\alpha: N \to N$ there corresponds its bar-function $\overline{\alpha}$ by

$$\overline{\alpha}(n) = \prod_{i<n} p_i^{1+\alpha(i)},$$

where $p_0 = 2$ and p_i = the i'th odd prime. Let U_α be the range of $\overline{\alpha}$ and $\mathfrak{A} = \{\overline{\alpha} \mid \text{for all } \alpha: N \to N\}$. Let $A = \bigcup_{\alpha \in \mathfrak{A}} U_\alpha$. Obviously, A is the set of all sequence numbers.

Let me prove that $\langle A, \mathfrak{A}\rangle$ is an **IRM**. Since $\overline{\alpha}(0) = 1$ for all α, then for all functions α, β the intersection $U_\alpha \cap U_\beta$ is not empty; as is easily checked, either $U_\alpha = U_\beta$ or $U_\alpha \cap U_\beta$ is a finite set. In the former case, $(\overline{\beta})^{-1} \circ \overline{\alpha}$ and $(\overline{\alpha})^{-1} \circ \overline{\beta}$ are identities on N, and in the second case they are identities on their domains. Thus, $\langle A, \mathfrak{A}\rangle$ is even an amalgam (see Example 1.5).

Consider now a functional $f: A \to N$. It is recursive iff every $f \circ \overline{\alpha}$ is a recursive function (see Figure 1.4). Let now f be defined by $f(x) = x$ for all $x \in A$. Then $f \circ \overline{\alpha}(n) = \overline{\alpha}(n)$, i.e., $f \circ \overline{\alpha}$ is the bar-function $\overline{\alpha}$ itself. Since most of $\overline{\alpha}$'s are not recursive, f is not an $\mathfrak{A}$-$\{\mathsf{I}\}$-recursive functional.

However, as identity I_A on A, f is an $\mathfrak{A}$-$\mathfrak{A}$-recursive map, since

$$(\overline{\beta})^{-1} \circ f \circ \overline{\alpha} = (\overline{\beta})^{-1} \circ \overline{\alpha},$$

which is a p.r. function for all α, β.

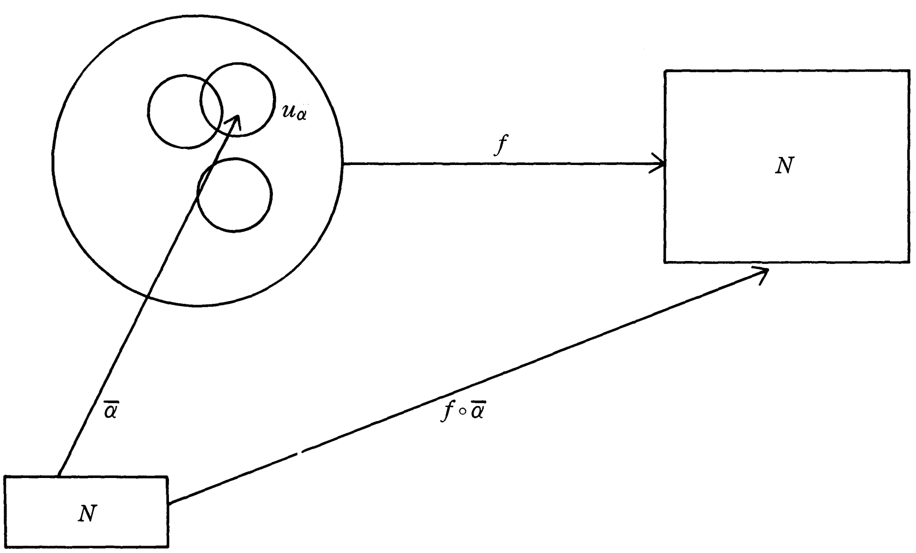

Figure 1.4

In the theory of enumerations, a set $X \subset U$, where $\langle U, \{u\}\rangle$ is an enumerated set (i.e., an **RM** with singleton-atlas), is called weakly u-r.e. iff there is a r.e. set $\omega_i \subset N$ such that $X = u(\omega_i)$. We can introduce a similar notion.

Definition 1.5 The set $X \subset A$ is *weakly* $\mathfrak{A}$-r.e. iff there is a φ: $P \to N$ such that $X = \omega_\varphi$, where

$$\omega_\varphi = \bigcup_{p \in P} \alpha_p(\omega_{\varphi(p)}). \tag{1.15}$$

By Theorem 1.4, every $\mathfrak{A}$-r.e. set is also weakly $\mathfrak{A}$-r.e.; however, the converse statement does not hold even in the case of enumerated sets (see, for example, [5], page 312).

Until now I have imposed the demand that for every **RM** $\langle A, \mathfrak{A}\rangle$, each A_p be essentially a "recursive" set (and so $A_p \cap A_{p_1}$ is also "recursive"). Now I shall reduce this demand to recursive enumerability only.

Definition 1.6 A set A is called a *Recursively Enumerable Manifold* (an **REM**) iff:

(i) There is a family $\mathfrak{A}$ of enumerations α_p: $N \to A_p$, $p \in P$, where each A_p is a subset of A and $A = \bigcup_{p \in P} A_p$.

(ii) For every pair $\langle p, p_1\rangle \in P^2$ such that $A_p \cap A_{p_1} \neq \emptyset$, both $\alpha_p^{-1}(A_{p_1})$ and $\alpha_{p_1}^{-1}(A_p)$ are recursively enumerable subsets of N, and there are numerical partial recursive functions

$$f_p\colon \alpha_p^{-1}(A_{p_1}) \to \alpha_{p_1}^{-1}(A_p) \text{ and } f_{p_1}\colon \alpha_{p_1}^{-1}(A_p) \to \alpha_p^{-1}(A_{p_1}),$$

such that (1.3) and (1.4) hold.

(iii) If all α_p are injective we say that A is an *Injective* **REM** (an **IREM**).

Example 1.8 Every sequence $\langle A_i\rangle_{i \in N}$ of non-empty r.e. subsets of N is an **REM**: let α_i: $N \to A_i$ be recursive, with A_i as range, let $A = \bigcup_{i=0}^{\infty} A_i$ and $\mathfrak{A} = \{\alpha_i \mid i \in N\}$. Then $\langle A, \mathfrak{A}\rangle$ is an **REM** which, in general, is not an **RM**. In case all A_i are infinite and all α_i injective (and recursive) $\langle A, \mathfrak{A}\rangle$ is an **IREM**.

All definitions of this chapter are applicable to **REM**'s, and I shall use them without further notice. Also all theorems of this chapter hold for **REM**'s without change; in using them I shall refer to the number of the theorem for **RM**'s.

Note that a disjoint **REM** $\langle A, \mathfrak{A}\rangle$, i.e., for which $p \neq p_1$ implies $A_p \cap A_{p_1} = \emptyset$, is always an **RM**.

CHAPTER II—GENERATION OF REM's AND RM's

In this chapter I shall exhibit several ways of obtaining new **REM**'s and **RM**'s from given ones.

Theorem 2.1 (Duplication) *Let* $\langle A, \mathfrak{A}\rangle$ *be an* **REM** (*an* **RM**), *let* B *be any set* *of the same cardinality as* A, *and let* $f: A \to B$ *be a bijective map of* A *onto* B. *Define the family*

$$\mathfrak{B} = \{\beta_p \mid p \in P\} \text{ of maps by } \beta_p = f \circ \alpha_p.$$

Then, $\langle B, \mathfrak{B}\rangle$ *is an* **REM** (*an* **RM**), f *is an* $\mathfrak{A}$-$\mathfrak{B}$-*recursive map and* f^{-1} *is a* $\mathfrak{B}$-$\mathfrak{A}$-*recursive map.*

Proof: Let A_p and B_p denote the respective ranges of α_p and β_p. It is obvious that $B = \bigcup_{p \in P} B_p$. Suppose that $D_{p,p_1} = B_p \cap B_{p_1}$ is not empty. Then:

$$\begin{aligned}\beta_p^{-1}(D_{p,p_1}) &= (f \circ \alpha_p)^{-1}(D_{p,p_1}) \\ &= \{n \mid \bigvee_m f(\alpha_p(n)) = f(\alpha_{p_1}(m))\} \\ &= \{n \mid \bigvee_m \alpha_p(n) = \alpha_{p_1}(m)\} \\ &= \alpha_p^{-1}(A_p \cap A_{p_1}) = \alpha_p^{-1}(A_{p_1}).\end{aligned}$$

If $\alpha_p^{-1}(A_{p_1})$ is r.e. (r.) then $\beta_p^{-1}(D_{p,p_1})$ is r.e. (r.). ($\bigvee_m$ denotes the existential quantifier.) Let now f_{p,p_1} be partial recursive, with $\alpha_p^{-1}(A_{p_1})$ as domain (and $\alpha_{p_1}^{-1}(A_p)$ as range), and such that

$$\alpha_p(n) = \alpha_{p_1}(f_{p,p_1}(n)) \text{ for all } n \in \alpha_p^{-1}(A_{p_1}).$$

Then, for all $n \in \beta_p^{-1}(D_{p,p_1})$

$$\beta_p(n) = f(\alpha_p(n)) = f(\alpha_{p_1}(f_{p,p_1}(n))) = \beta_{p_1}(f_{p,p_1}(n)),$$

(and similarily for $\beta_{p_1}(n)$). This proves that $\mathfrak{B}$ is an atlas on B. At last, for all $n \in N$, and $p_1 \in P$

$$f(\alpha_{p_1}(n)) = \beta_{p_1}(\mathsf{I}(n)),$$

where I is the identity on N; similarly, if $D_{p,p_1} \neq \emptyset$ then

$$f(\alpha_{p_1}(n)) = \beta_p(f_{p_1}(n)), \text{ for all } n \in \alpha_{p_1}^{-1}(A_p),$$

where f_{p_1} satisfies $\beta_{p_1}(n) = \beta_p(f_{p_1}(n))$ for all $n \in \beta_{p_1}^{-1}(D_{p,p_1})$. The statement about f^{-1} is proved in a similar way.

Remark: If $\langle A, \mathfrak{A}\rangle$ is an **IREM** (an **IRM**) then $\langle B, \mathfrak{B}\rangle$ is an **IREM** (an **IRM**).

Construction in Theorem 2.1 is suitable for situations in which we need replicas of an **REM** which are disjoint from it. Another simple construction is given by the next theorem.

Theorem 2.2 *Let* $\langle A, \mathfrak{A}\rangle$ *be a positive* **REM** (*a solvable* **RM**). *For every* $p \in P$ *define* β_p *by*

$$\beta_p(n) = \langle n, \alpha_p(n)\rangle \text{ for all } n \in N.$$

Let B_p *be the range of* β_p, *let* $B = \bigcup_{p \in P} B_p$ *and* $\mathfrak{B} = \{\beta_p \mid p \in P\}$. *Then,* $\langle B, \mathfrak{B}\rangle$ *is an* **IREM** (*an* **IRM**).

Proof: Each β_p is obviously injective. Suppose $D_{p,p_1} = B_p \cap B_{p_1}$ is not empty. Then

$$\beta_p^{-1}(D_{p,p_1}) = \beta_p^{-1}(\{\langle n, \alpha_p(n)\rangle \mid \alpha_{p_1}(n) = \alpha_p(n)\}) = \{n \mid \alpha_{p_1}(n) = \alpha_p(n)\}.$$

Let f_{p_1}: $\alpha_{p_1}^{-1}(A_p) \rightarrow \alpha_p^{-1}(A_{p_1})$ be partial recursive with r.e. domain (with r.) and range, and such that $\alpha_{p_1}(n) = \alpha_p(f_{p_1}(n))$ for all $N \in \alpha_{p_1}^{-1}(A_p)$. Then

$$\begin{aligned}\beta_p^{-1}(D_{p,p_1}) &= \{n \mid \alpha_p(f_{p_1}(n)) = \alpha_p(n)\} \\ &= \{n \in \alpha_{p_1}^{-1}(A_p) \mid \mathfrak{A}_p(f_{p_1}(n), n)\},\end{aligned}$$

where $\mathfrak{A}_p$ is the predicate from Definition 1.4. In case $\langle A, \mathfrak{A}\rangle$ is a positive **REM** this proves that $\beta_p^{-1}(D_{p,p_1})$ is a r.e. subset of N (and similarly for $\beta_{p_1}^{-1}(D_{p,p_1})$). In case $\langle A, \mathfrak{A}\rangle$ is a solvable **RM** we must proceed further. Define g_{p_1} by

$$g_{p_1}(n) = \begin{cases} f_{p_1}(n) & \text{for } n \in \alpha_{p_1}^{-1}(A_p), \\ b & \text{for } n \notin \alpha_{p_1}^{-1}(A_p), \end{cases}$$

where b is any fixed element of $\complement\alpha_{p_1}^{-1}(A_p)$. g_{p_1} is recursive, since $\alpha_{p_1}^{-1}(A_p)$ is now recursive. Then

$$\beta_p^{-1}(D_{p,p_1}) = \{n \mid n \in \alpha_{p_1}^{-1}(A_p) \wedge \mathfrak{A}_p(g_{p_1}(n), n)\}.$$

Since $\mathfrak{A}_p$ and g_{p_1} are recursive, $\beta_p^{-1}(D_{p,p_1})$ is now recursive. (Similarly for $\beta_{p_1}^{-1}(D_{p,p_1})$.) At last

$$\beta_p^{-1}(\beta_{p_1}(n)) = \beta_p^{-1}(\langle n, \alpha_{p_1}(n)\rangle) = n \text{ if } n \in \beta_{p_1}^{-1}(D_{p,p_1})$$

(undefined otherwise), which shows that each $\beta_p^{-1} \circ \beta_{p_1}$ is a p.r. function.

I shall call the **IREM** (the **IRM**) $\langle B, \mathfrak{B}\rangle$ from Theorem 2.2 and *The Graph of the* **REM** (*of the* **RM**) $\langle A, \mathfrak{A}\rangle$.

Example 2.1 I call $\langle B, \mathfrak{B}\rangle$ from Theorem 2.2 a graph, because, for the manifold $\langle A, \{\alpha\}\rangle$, where α is an enumeration of A, the corresponding B is just the graph of α.

Definition 2.1 Let $\langle A, \mathfrak{A}\rangle$ and $\langle B, \mathfrak{B}\rangle$ be **REM**'s, let $\mathfrak{A} = \{\alpha_p|_{p \in P}\}$, $\mathfrak{B} = \{\beta_q|_{q \in Q}\}$ and let A_p and B_q be the respective ranges of α_p and β_q. Set $\mathbf{C} = A \times B$ and, for each pair $\langle p, q\rangle \in P \times Q$, define the enumeration $\gamma_{p,q}$: $N \rightarrow A_p \times B_q$ by

$$\gamma_{p,q}(\sigma^2(n, m)) = \langle \alpha_p(n), \beta_q(m)\rangle, \tag{2.1}$$

where σ^2: $N^2 \rightarrow N$ is the well-known bijective, recursive map of N^2 onto N. (I shall induce its inverses σ_1^2 and σ_2^2 by $\sigma_1^2(\sigma^2(n, m)) = n$ and $\sigma_2^2(\sigma^2(n, m)) = m$; they are recursive and of large oscillation: they take each natural number as value infinitely many times.) Set

$$\mathfrak{C} = \{\gamma_{p,q} \mid \langle p, q\rangle \in P \times Q\}$$

and denote the range of $\gamma_{p,q}$ by $\mathbf{C}_{p,q}$. Then the pair $\langle \mathbf{C}, \mathfrak{C}\rangle$ is called the *Direct Product* of $\langle A, \mathfrak{A}\rangle$ and $\langle B, \mathfrak{B}\rangle$.

Theorem 2.3 *The direct product of two* **REM**'s (*respectively* **RM**'s, **IREM**'s, *and* **IRM**'s) *is an* **REM** (*respectively* **RM**, **IREM**, *and* **IRM**).

Proof: Using notations of Definition 2.1, remark first that $\gamma_{p,q}$'s are injective in case both α_p's and β_q's are injective. Suppose now that $D = \mathbf{C}_{p,q} \cap \mathbf{C}_{p_1,q_1} \neq \emptyset$. Then:

$$\gamma_{p,q}^{-1}(D) = \sigma^2(\alpha_p^{-1}(A_p \cap A_{p_1}),\ \beta_q^{-1}(B_q \cap B_{q_1}));$$

if $\alpha_p^{-1}(A_p \cap A_{p_1})$ and $\beta_q^{-1}(B_q \cap B_{q_1})$ are r.e. (r.) so is $\gamma_{p,q}^{-1}(D)$. Moreover, if f_{p_1}: $\alpha_{p_1}^{-1}(A_p) \to \alpha_p^{-1}(A_{p_1})$ and f_{q_1}: $\beta_{q_1}^{-1}(B_q) \to \beta_q^{-1}(B_{q_1})$ satisfy

$$\alpha_{p_1}(n) = \alpha_p(f_{p_1}(n)) \text{ for all } n \in \alpha_{p_1}^{-1}(A_p)$$

and

$$\beta_{q_1}(m) = \beta_q(f_{q_1}(m)) \text{ for all } m \in \beta_{q_1}^{-1}(B_q),$$

then

$$\begin{aligned}\gamma_{p_1,q_1}(\sigma^2(n, m)) &= \langle \alpha_{p_1}(n),\ \beta_{q_1}(m)\rangle \\ &= \langle \alpha_p(f_{p_1}(n)),\ \beta_q(f_{q_1}(m))\rangle \\ &= \gamma_{p,q}(\sigma^2(f_{p_1}(n)),\ f_{q_1}(m)),\end{aligned}$$

for all $\sigma^2(n, m) \in \gamma_{p_1,q_1}^{-1}(D)$; thus, with

$$f_{p_1,q_1}(u) = \sigma^2(f_{p_1}(\sigma_1^2(u)),\ f_{q_1}(\sigma_2^2(u)))$$

we obtain

$$\gamma_{p_1,q_1}(n) = \gamma_{p,q}(f_{p_1,q_1}(n)),$$

for all $n \in \gamma_{p_1,q_1}^{-1}(D)$.

Example 2.2 Let $\langle A, \mathfrak{A}\rangle$, $\langle B, \mathfrak{B}\rangle$ be **REM**'s and let $\langle \mathbf{C}, \mathfrak{C}\rangle$ be their direct product. Define the projections p_0: $\mathbf{C} \to A$ and p_1: $\mathbf{C} \to B$ by $p_0(x, y) = x$ and $p_1(x, y) = y$. Since

$$p_0(\gamma_{p,q}(\sigma^2(n, m))) = p_0(\alpha_p(n),\ \beta_q(m)) = \alpha_p(n),$$

p_0 is $\mathfrak{C}$-$\mathfrak{A}$-recursive. Similarily, p_1 is $\mathfrak{C}$-$\mathfrak{B}$-recursive.

Let now $\langle \mathbf{D}, \mathfrak{D}\rangle$, $\mathfrak{D} = \{\delta_s \mid s \in S\}$, $\mathbf{D}_s$ = range of δ_s, be another **REM**, such that there are two maps, g_0: $\mathbf{D} \to A$ which is $\mathfrak{D}$-$\mathfrak{A}$-recursive, and g_1: $\mathbf{D} \to B$ which is $\mathfrak{D}$-$\mathfrak{B}$-recursive. These two maps determine in a unique way the map f: $\mathbf{D} \to C$, defined by $f(x) = \langle g_0(x),\ g_1(x)\rangle$, which satisfies both $g_0 = p_0 \circ f$ and $g_1 = p_1 \circ f$ (see Figure 2.1).

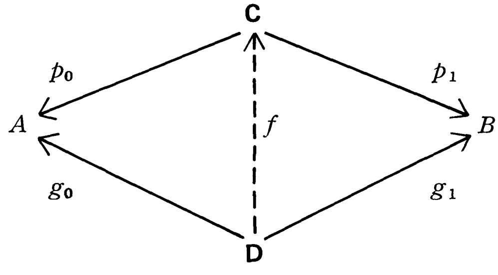

Figure 2.1

Now,

$$f(\delta_s(n)) = \langle g_0(\delta_s(n)),\ g_1(\delta_s(n))\rangle = \gamma_{p,q}(\sigma^2(f_s(n),\ h_s(n))),$$

where f_s and h_s satisfy

$$g_0(\delta_s(n)) = \alpha_p(f_s(n)),$$

and

$$g_1(\delta_s(n)) = \beta_q(h_s(n))$$

on corresponding domains. This proves that f is $\mathfrak{D}$-$\mathfrak{C}$-recursive.

The dual notion to the direct product is the direct sum.

Definition 2.2 Let $\langle A, \mathfrak{A}\rangle$ and $\langle B, \mathfrak{B}\rangle$ be **REM**'s. A **REM** $\langle \mathbf{C}, \mathfrak{C}\rangle$ is called the *Direct Sum* of $\langle A, \mathfrak{A}\rangle$ and $\langle B, \mathfrak{B}\rangle$ iff there are two maps, f_0: $A \to \mathbf{C}$, which is $\mathfrak{A}$-$\mathfrak{C}$-recursive, and f_1: $B \to \mathbf{C}$, which is $\mathfrak{B}$-$\mathfrak{C}$-recursive, with the following property: for any **REM** $\langle \mathbf{D}, \mathfrak{D}\rangle$ and any two maps, g_0: $A \to \mathbf{D}$, which is $\mathfrak{A}$-$\mathfrak{D}$-recursive, and g_1: $B \to \mathbf{D}$, which is $\mathfrak{B}$-$\mathfrak{D}$-recursive, there is a uniquely determined $\mathfrak{C}$-$\mathfrak{D}$-recursive map f: $\mathbf{C} \to \mathbf{D}$, such that $g_0 = f \circ f_0$ and $g_1 = f \circ f_1$ (see Figure 2.2).

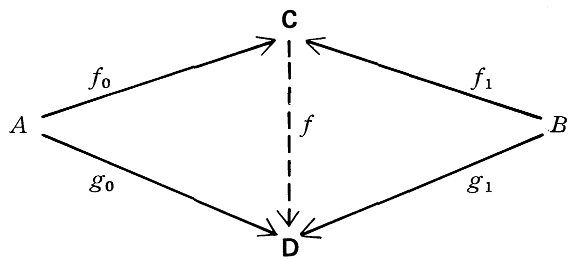

Figure 2.2

Theorem 2.4 *The direct sum of any two* **REM**'*s exists; it is of the same kind* (**REM**, **IREM**, **RM**, **IRM**) *as those two.*

Proof: I shall use notations of Definition 2.2. Consider first the case of disjoint A and B. In this case, set $\mathbf{C} = A \cup B$ and $\mathfrak{C} = \mathfrak{A} \cup \mathfrak{B}$. Trivially, $\langle \mathbf{C}, \mathfrak{C}\rangle$ is an **REM** of the same kind as both $\langle A, \mathfrak{A}\rangle$ and $\langle B, \mathfrak{B}\rangle$. (In case those two are not of the same kind, then $\langle \mathbf{C}, \mathfrak{C}\rangle$ is of the kind of the "worse" one of those two.)

Let f_0 be the identity on A and f_1 the identity on B (both maps satisfy conditions of Definition 2.2). For given g_0 and g_1 as in Definition 2.2, define f: $\mathbf{C} \to \mathbf{D}$ by

$$f(x) = \begin{cases} g_0(x) \text{ for } x \in A, \\ g_1(x) \text{ for } x \in B. \end{cases}$$

f is $\mathfrak{C}$-$\mathfrak{D}$-recursive and $g_0 = f \circ f_0$, $g_1 = f \circ f_1$. Suppose now that $A \cap B \neq \emptyset$. Let A' be any set disjoint from $A \cup B$, of the same cardinality as A; take any bijective φ: $A \to A'$ and construct, as in Theorem 3.1, the replica $\langle A', \mathfrak{A}'\rangle$ of $\langle A, \mathfrak{A}\rangle$. (Thus $\mathfrak{A}' = \{\alpha'_p | p \in P\}$ and $\alpha'_p = \varphi \circ \alpha_p$.) Let then $\langle \mathbf{C}, \mathfrak{C}\rangle$ be the direct sum of $\langle A', \mathfrak{A}'\rangle$ and $\langle B, \mathfrak{B}\rangle$, i.e., $\mathbf{C} = A' \cup B$ and $\mathfrak{C} = \mathfrak{A}' \cup \mathfrak{B}$. Define f_0: $A \to \mathbf{C}$ by $f_0 = \varphi$, and let f_1 be the identity on B. (By Theorem 3.1

f_0 is $\mathfrak{A}$-$\mathfrak{C}$-recursive.) For $g_0\colon A \to \mathbf{D}$ and $g_1\colon B \to \mathbf{D}$ as in Definition 2.2, construct f as in the first part of this proof. Anew, f is $\mathfrak{C}$-$\mathfrak{D}$-recursive and $g_0 = f \circ f_0$, $g_1 = f \circ f_1$.

Definition 2.3 Let $\langle A, \mathfrak{A}\rangle$ and $\langle B, \mathfrak{B}\rangle$ be **REM**'s. We say that $\langle B, \mathfrak{B}\rangle$ is a *submanifold* of $\langle A, \mathfrak{A}\rangle$ iff $B \subset A$ and to every $q \in Q$ there corresponds a $p \in P$ such that $B_q \subset A_p$.

(Obviously, I use in Definition 2.3 the notations $\mathfrak{A} = \{\alpha_p | p \in P\}$, $\mathfrak{B} = \{\beta_q | q \in Q\}$, A_p = the range of α_p and B_q = the range of β_q.)

Lemma 2.1 *An* **REM** $\langle B, \mathfrak{B}\rangle$ *is a submanifold of the* **REM** $\langle A, \mathfrak{A}\rangle$ *iff to every* $q \in Q$ *there corresponds some* $p \in P$ *and a function* $f_p\colon N \to N$ *such that* $\beta_q = \alpha_p \circ f_p$.

Proof: Define $f_p(n)$ = any m such that $\beta_q(n) = \alpha_p(m)$.

In view of Lemma 2.1 I shall say that $\langle B, \mathfrak{B}\rangle$ is *effectively a submanifold of* $\langle A, \mathfrak{A}\rangle$ iff each f_p in Lemma 2.1 is recursive (or can be chosen recursive). In the case in which $\langle A, \mathfrak{A}\rangle$ and $\langle B, \mathfrak{B}\rangle$ are injective, all f_p must be injective.

Example 2.3 One may conjecture that the fact of $\langle B, \mathfrak{B}\rangle$ being an **REM** would imply the recursiveness of each f_p in Lemma 2.1. Let me give an example that it is not so.

Let B be any subset of N which is not r.e. and let b be the *principal function* of B, i.e., b is an increasing function with B as its range. Then the **IRM** $\langle B, \{b\}\rangle$ is a submanifold of the **IRM** $\langle N, \{\mathsf{I}\}\rangle$ (I is the identity on N), and $b = \mathsf{I} \circ f$, where f can never be chosen recursive ($f = b$).

Example 2.4 Let $\langle A, \mathfrak{A}\rangle$ be an **REM**. Let $P_0 \subset P$ be non-empty. To each $p \in P_0$ there corresponds an injective recursive function g_p; define $\beta_p = \alpha_p \circ g_p$, $\mathfrak{B} = \{\beta_p | p \in P_0\}$, B_p = range of β_p, and $B = \bigcup_{p \in P_0} B_p$. Then $\langle B, \mathfrak{B}\rangle$ is an **REM** which is effectively a submanifold of $\langle A, \mathfrak{A}\rangle$.

Example 2.5 Let $\langle M, \mathfrak{M}\rangle$, $\mathfrak{M} = \{\mu_t | t \in T\}$, M_t = range of μ_t, be a positive **REM**. Define $\mu_t^{(0)}$ and $\mu_t^{(1)}$ by $\mu_t^{(0)}(n) = \mu_t(2n)$ and $\mu_t^{(1)}(n) = \mu_t(2n+1)$. Let $M_t^{(0)}$ and $M_t^{(1)}$ be the respective ranges of $\mu_t^{(0)}$ and $\mu_t^{(1)}$, set $M^{(0)} = \bigcup_{t \in T} M_t^{(0)}$, $M^{(1)} = \bigcup_{t \in T} M_t^{(1)}$, $\mathfrak{M}^{(0)} = \{\mu_t^{(0)} | t \in T\}$ and $\mathfrak{M}^{(1)} = \{\mu_t^{(1)} | t \in T\}$. Then $\langle M^{(0)}, \mathfrak{M}^{(0)}\rangle$ and $\langle M^{(1)}, \mathfrak{M}^{(1)}\rangle$ are **REM**'s which are effectively submanifolds of $\langle M, \mathfrak{M}\rangle$.

It is enough to prove that $\langle M^{(0)}, \mathfrak{M}^{(0)}\rangle$ is an **REM**. Suppose that $D = M_t^{(0)} \cap M_{t_1}^{(0)} \neq \emptyset$. Then

$$(\mu_t^{(0)})^{-1}(D) = \{n | \bigvee_u \mu_t(2n) = \mu_{t_1}(2u)\}.$$

Let $f_{t_1}\colon \mu_{t_1}^{-1}(M_t) \to \mu_t^{-1}(M_{t_1})$ be partial recursive and such that

$$\mu_{t_1}(n) = \mu_t(f_{t_1}(n)) \text{ for all } n \in D_{f_{t_1}}.$$

Then

$$(\mu_t^{(0)})^{-1}(D) = \{n | \bigvee_u \mu_t(2n) = \mu_{t_1}(f_{t_1}(2u))\} = \{n | \bigvee_u \mathfrak{M}_t(2n, f_{t_1}(2u))\},$$

where $\mathfrak{M}_t(u, u) \leftrightarrow \mu_t(u) = \mu_t(u)$ is a r.e. predicate. Thus, $(\mu_t^{(0)})^{-1}(D)$ is r.e. The remaining part of the proof is left to the reader.

Let me introduce a less strict notion of submanifold.

Definition 2.4 Let $\langle A, \mathfrak{A}\rangle$ and $\langle B, \mathfrak{B}\rangle$ be **REM**'s. We say that $\langle B, \mathfrak{B}\rangle$ is a *quasi-submanifold* of $\langle A, \mathfrak{A}\rangle$ iff $B \subset A$ and to every $q \in Q$ there corresponds a finite set $P_q \subset P$ such that

$$B_q = B \cap \bigcup_{p \in P_q} A_p. \tag{2.2}$$

Lemma 2.2 *An* **REM** $\langle B, \mathfrak{B}\rangle$ *is a quasi-submanifold of the* **REM** $\langle A, \mathfrak{A}\rangle$ *iff* $B \subset A$ *and to every* $q \in Q$ *there corresponds a finite family, say* $\{f_{p_1}^{(q)}, \ldots, f_{p_m}^{(q)}\}$ *of partial functions, such that*

$$\beta_q(n) = \alpha_{p_i}(f_{p_i}^{(q)}(n)) \text{ for } n \in \beta_q^{-1}(A_{p_i}),$$

and for $i = 1, 2, \ldots, m$, *and such that* (2.2) *holds with* $P_q = \{p_1, \ldots, p_m\}$.

Anew, if all $f_{p_i}^{(q)}$ in Lemma 2.2 are recursive (or can be chosen recursive), we shall say that $\langle B, \mathfrak{B}\rangle$ is *effectively* a quasi-submanifold of $\langle A, \mathfrak{A}\rangle$.

CHAPTER III--ATLASES AND THEIR DEGREES

In this chapter I shall consider relations between different atlases on one and the same set, and two fundamental relations between such atlases: *compatibility* and *reducibility*. Compatibility is concerned with the recursive structure imposed by a given atlas, and reducibility helps introduce a classification of atlases on one and the same set. Both notions can be introduced with various degrees of strength.

I consider a fixed non-empty set A, and atlases

$$\mathfrak{A} = \{\alpha_p \mid p \in P\},\ \mathfrak{B} = \{\beta_q \mid q \in Q\},\ \mathfrak{C} = \{\gamma_r \mid r \in R\},\ \ldots,$$

which are all atlases on A. I shall say that an atlas $\mathfrak{A}$ is an **RE**-atlas (respectively **IRE**-atlas, **R**-atlas and **IR**-atlas) iff $\langle A, \mathfrak{A}\rangle$ is an **REM** (respectively **IREM**, **RM**, and **IRM**). If I do not mention the special structure of the atlas, I always consider it to be an **RE**-atlas; I shall mainly be interested in such, most general atlases.

Definition 3.1 Two atlases $\mathfrak{A}$ and $\mathfrak{B}$ (on A) are *compatible* iff they induce the same "r.e.", "p.r.", and "r."-notions for sets and anti-functionals in both $\langle A, \mathfrak{A}\rangle$ and $\langle A, \mathfrak{B}\rangle$.

Thus, compatible atlases induce the same "effective" structures on a given set A, at least for its subsets and for maps of N into A.

Theorem 3.1 $\mathfrak{A}$ *and* $\mathfrak{B}$ *are compatible iff their union* $\mathfrak{A} \cup \mathfrak{B}$ *is an atlas on* A, *which is compatible with both* $\mathfrak{A}$ *and* $\mathfrak{B}$.

Proof: Suppose first that $\mathfrak{A}$ and $\mathfrak{B}$ are compatible. Consider α_p as an anti-functional $f: N \to A$, with A_p as range. It is, trivially, an $\{\mathrm{I}\}$-$\mathfrak{A}$-recursive anti-functional. But then it must be also $\{\mathrm{I}\}$-$\mathfrak{B}$-recursive; thus, whenever $A_p \cap B_q \neq \emptyset$ there is a p.r. function f_q, with domain $\alpha_p^{-1}(B_q)$, such that $f(n) = \beta_q(f_q(n))$, i.e., such that $\alpha_p(n) = \beta_q(f_q(n))$ for all $n \in \alpha_p^{-1}(B_q)$. With a similar consideration for β_q's, we conclude that $\mathfrak{A} \cup \mathfrak{B}$ is an atlas on A. It is obviously compatible with both $\mathfrak{A}$ and $\mathfrak{B}$. Converse evident.

Let me point out that the condition on anti-functionals cannot be omitted from Definition 3.1. To see this, let A be a denumerable set and $\alpha: N \to A$, $\beta: N \to A$ two indexings of A. By Theorem 3.1, $\{\alpha\}$ and $\{\beta\}$ are compatible iff there is a recursive permutation $p: N \to N$ such that $\beta = \alpha \circ p$. By a theorem of Kent ([9], p. 233) there exists a non-recursive permutation $f: N \to N$ such that, for every r.e. set $E \subset N$, both $f(E)$ and $f^{-1}(E)$ are r.e. Thus, if $\beta: N \to A$ is defined by $\beta = \alpha \circ f$, β and α induce the same notions "r.e." and "r." for subsets of A. However, for anti-functionals this is not true. Define $\varphi: N \to A$ by $\varphi = B \circ f^{-1}$. Then φ is not $\{\mathsf{I}\}$-$\{\beta\}$-recursive; namely, if there is a recursive, injective $\varphi^*: N \to N$ such that $\varphi(n) = \beta(\varphi^*(n))$ for all $n \in N$, this would imply that $f^{-1} = \beta^{-1} \circ \varphi = \varphi^*$ is recursive, and so that f is recursive. However, since $\varphi = \alpha \circ \mathsf{I}$, where I is the identity on N, we obtain that φ is $\{\mathsf{I}\}$-$\{\alpha\}$-recursive.

Corollary 3.1.1 *If $\mathfrak{A}$ and $\mathfrak{B}$ are compatible (on A) then I_A, the identity on A, is both $\mathfrak{A}$-$\mathfrak{B}$-recursive and $\mathfrak{B}$-$\mathfrak{A}$-recursive.*

Proof: $\mathsf{I}_A: A \to A$ is $\mathfrak{A}$-$\mathfrak{B}$-recursive iff for every $p \in P$ and $q \in Q$, such that $A_p \cap B_q \neq \emptyset$, there is a p.r. function $f_{p,q}$ with domain $D_{p,q} = \alpha_p^{-1}(B_q)$ such that $\mathsf{I}_A(\alpha_p(n)) = \beta_q(f_{p,q}(n))$ for all $n \in D_{p,q}$, i.e., such that $\alpha_p(n) = \beta_q(f_{p,q}(n))$ for all $n \in \alpha_p^{-1}(B_q)$. Since $\mathfrak{A}$ and $\mathfrak{B}$ are compatible such $f_{p,q}$'s always exist.

Definition 3.2 Two atlases $\mathfrak{A}$ and $\mathfrak{B}$ (on A) are *strongly compatible* iff they are compatible and, for every **REM** $\langle M, \mathfrak{M}\rangle$, "$f$ is $\mathfrak{A}$-$\mathfrak{M}$-p.r. map" $\leftrightarrow$ "f is $\mathfrak{B}$-$\mathfrak{M}$-p.r. map" and "f is $\mathfrak{M}$-$\mathfrak{A}$-p.r. map" $\leftrightarrow$ "f is $\mathfrak{M}$-$\mathfrak{B}$-p.r. map".

It is difficult to find necessary and sufficient conditions for strong compatibility; they may depend on the structure of atlases in question. I am able to provide a fairly general sufficient condition in Corollary 3.2.1.

Theorem 3.2 *Let $\mathfrak{A}$ and $\mathfrak{B}$ be compatible (on A), and suppose that each B_q meets only finite many A_p's. Then, for any* **REM** $\langle M, \mathfrak{M}\rangle$, *every* $\mathfrak{A}$-$\mathfrak{M}$-p.r. *map is also a* $\mathfrak{B}$-$\mathfrak{M}$-p.r. *map, and every* $\mathfrak{M}$-$\mathfrak{A}$-p.r. *map is also an* $\mathfrak{M}$-$\mathfrak{B}$-p.r. *map.*

Proof: Let $f: X \to M$, $X \subset A$, be an $\mathfrak{A}$-$\mathfrak{M}$-p.r. map. Thus, for every pair $\langle p, t\rangle \in P \times T$ (we suppose $\mathfrak{M} = \{\mu_t | t \in T\}$) there is a p.r. function $f_{p,t}$, with domain $D_{p,t} = \alpha_p^{-1}(X \cap f^{-1}(M_t))$, where M_t = range of μ_t, and such that

$$f(\alpha_p(n)) = \mu_t(f_{p,t}(n)) \text{ for all } n \in D_{p,t}.$$

Let $q \in Q$ be such that $A_p \cap B_q \neq \emptyset$. By supposition, there is a p.r. function g_q, with domain $\beta_q^{-1}(A_p)$, such that

$$\beta_q(m) = \alpha_p(g_q(m)) \text{ for all } m \in \beta_q^{-1}(A_p).$$

Now, if B_q is covered by $A_{p_1}, A_{p_2}, \ldots, A_{p_s}$, we have s p.r. functions g_{q_j}, $j = 1, 2, \ldots, s$, such that

(3.1) $$\beta_q(m) = \alpha_{p_j}(g_{q_j}(m)) \text{ for all } m \in \beta_q^{-1}(A_{p_j}).$$

Then,

$$f(\beta_q(m)) = \mu_t(f_{p_j,t}(g_{q_j}(m))) \text{ for all } m \in \beta_q^{-1}(A_{p_j} \cap f^{-1}(M_t)),$$

and $j = 1, \ldots, s$. By the uniformization theorem of the classical recursive theory there is a p.r. function $f_{q,t}$ defined on

$$\beta_q^{-1}(X \cap f^{-1}(M_t)) = \bigcup_{j=1}^{s} \beta_q^{-1}(X \cap A_{p_j} \cap f^{-1}(M_t))$$

such that, for every $n \in \beta_q^{-1}(X \cap f^{-1}(M_t))$, $f_{q,t}(n)$ is one of the values $f_{p_j,t}(g_{q_j}(n))$ which are defined at the point n. Then

$$f(\beta_q(n)) = \mu_t(f_{q,t}(n)) \text{ for all } n \in D_{f_{q,t}}.$$

Suppose now that $f: Y \to A$, $Y \subset M$, is an $\mathfrak{M}$-$\mathfrak{A}$-p.r. map. Thus, for every pair $\langle t, p\rangle \in T \times P$ there is a p.r. function $f_{t,p}$, with domain $D_{t,p} = \mu_t^{-1}(Y \cap f^{-1}(A_p))$, such that

$$f(\mu_t(n)) = \alpha_p(f_{t,p}(n)) \text{ for all } n \in D_{t,p}.$$

Suppose now anew that $A_{p_1}, \ldots, A_{p_s}$ cover B_q. Since $\mathfrak{A}$ and $\mathfrak{B}$ are compatible, there are p.r. functions h_i such that

$$\alpha_{p_i}(m) = \beta_q(h_i(m)) \text{ for } m \in \alpha_{p_i}^{-1}(B_q).$$

Then

$$f(\mu_t(n)) = \beta_q(h_i(f_{t,p_i}(n)))$$

for $n \in \mu_t^{-1}(Y \cap f^{-1}(B_q \cap A_{p_i}))$, $i = 1, \ldots, s$. As in the first part of the proof, there is a p.r. function $f_{t,q}$, with domain $\mu_t^{-1}(Y \cap f^{-1}(B_q))$ such that

$$f(\mu_t(n)) = \beta_q(f_{t,q}(n)) \text{ for all } n \in D_{f_{t,q}},$$

which proves that f is also an $\mathfrak{M}$-$\mathfrak{B}$-p.r. map.

Corollary 3.2.1 *Let $\mathfrak{A}$ and $\mathfrak{B}$ be compatible and such that each A_p meets at most finite many B_q's and each B_q meets at most finite many A_p's. Then $\mathfrak{A}$ and $\mathfrak{B}$ are strongly compatible.*

Most pleasant atlases are the finite ones. The following theorem demonstrates why it is so.

Theorem 3.3 *If $\mathfrak{A} = \{\alpha_i \mid 0 \leq i \leq n\}$ is a finite atlas on A, then there is an enumeration $\alpha: N \to A$ of A such that $\mathfrak{A}$ and $\{\alpha\}$ are strongly compatible.*

Proof: By induction. Let $n = 1$, i.e., $\mathfrak{A} = \{\alpha_0, \alpha_1\}$. Set $\alpha(2n) = \alpha_0(n)$ and $\alpha(2n+1) = \alpha_1(n)$. $\mathfrak{A}$ and $\{\alpha\}$ are trivially compatible; by Corollary 3.2.1 they are strongly compatible. Induction now completes the proof.

If we apply the construction in the proof of Theorem 3.3 to the case in which $\langle A, \mathfrak{A}\rangle$ is an **IREM**, the corresponding α will be not an indexing but an enumeration only. I can prove that in the case in which $\langle A, \mathfrak{A}\rangle$ is an **IRM** the corresponding α may be chosen so as to be an indexing.

Theorem 3.4 *If* $\langle A, \mathfrak{A}\rangle$ *is an* **IRM** *with finite atlas* $\mathfrak{A} = \{\alpha_i \mid 0 \leq i \leq n\}$, *then there is an indexing* α: $N \to A$ *such that* $\mathfrak{A}$ *and* $\{\alpha\}$ *are strongly compatible.*

Proof: By induction. Let $n = 1$, i.e., $\mathfrak{A} = \{\alpha_0, \alpha_1\}$. Consider $D = A_0 \cap A_1$ (A_i = range of α_i). $E = \alpha_1^{-1}(D)$ is either empty, finite or infinite and recursive; if it is empty apply the construction in the proof of Theorem 3.3, and if it is not empty consider $N - E$. This is a recursive set. If it is finite, say $N - E = \{e_0, \ldots, e_s\}$, define $\alpha(i) = \alpha_1(e_i)$ for $i = 0, \ldots, s$ and $\alpha(s+1+i) = \alpha_0(i)$ for $i \geq 0$. If $N - E$ is infinite, let f: $N \to N - E$ be recursive, increasing, with $N - E$ as range. Set $\alpha(2i) = \alpha_0(i)$ and $\alpha(2i + 1) = \alpha_1(f(i))$. It is easy to show that $\mathfrak{A}$ and $\{\alpha\}$ are compatible. Then, they are strongly compatible. Now, apply induction.

Theorems 3.3 and 3.4 show that, as far as "effective" structure is in question, finite atlases can always be replaced by enumerations, respectively by indexings. However, this situation should not suggest that denumerable sets A should be considered only as **REM**'s $\langle A, \{\alpha\}\rangle$, where α is an enumeration or an indexing. I shall give later important instances in which denumerable atlases on such a set A are essentially different from possible enumerations of A (i.e., from singleton-atlases on A).

In the Theory of Enumerations one of the fundamental problems is the so-called problem of *reducibility* for enumerations of one and the same set. If α: $N \to A$ and β: $N \to A$ are enumerations of the set A, and there is a recursive (and injective) function f, such that $\alpha = \beta \circ f$, then we say that α is *reducible* (*uni-reducible*) to β. In a natural way, this notion leads to a notion of *equivalence* (*and uni-equivalence*) and to the notion of *degrees* (*one-degrees*) of enumerations of A. (For example, the whole content of [5] consists in an elaboration of this notion of reducibility.)

In the Theory of **REM**'s we have several possible notions of reducibility of atlases, all of which fall back to the reducibility of enumerations in case of singleton-atlases. I shall expose now some of these possibilities.

We consider a non-empty set A and the class a_A of all atlases $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \ldots$, on A. (See the beginning of this chapter for notations.)

Definition 3.3 $\mathfrak{A}$ is *strongly reducible* (*strongly one-reducible*) to $\mathfrak{B}$, in symbol $\mathfrak{A} << \mathfrak{B}$ ($\mathfrak{A} <<_1 \mathfrak{B}$), iff $\mathfrak{A} = \{\alpha_p \mid p \in P\}$, $\mathfrak{B} = \{\beta_p \mid p \in P\}$ and there is a family $F = \{f_p \mid p \in P\}$ of recursive (and injective) arithmetical functions, such that

$$\alpha_p = \beta_p \circ f_p, \text{ for all } p \in P. \tag{3.2}$$

Strong reducibility is an immediate generalization of the reducibility of enumerations, and it is not difficult to pursue its study along the same lines as in the classical recursive theory. In the next chapter I shall show the naturalness of the demand that atlases be enumerated by same indices—at least for the sake of comparison of **REM**'s; however, I will not enter into any detailed discussion of the strong reducibility.

Definition 3.4 $\mathfrak{A}$ is *finitely reducible* (*finitely one-reducible*) to $\mathfrak{B}$, in symbol $\mathfrak{A} \underset{F}{\leqslant} \mathfrak{B}$ ($\mathfrak{A} \underset{F-1}{\leqslant} \mathfrak{B}$), iff each A_p can be covered by finite many B_q's, say by $B_{q_1}^{(p)}, \ldots, B_{q_s}^{(p)}$, and there are (injective) p.r. functions $f_1^{(p)}, \ldots, f_s^{(p)}$, such that for every $i = 1, 2, \ldots, s$

$$\alpha_p(n) = \beta_{q_i}(f_i^{(p)}(n)) \text{ for } n \in \alpha_p^{-1}(B_{q_i}). \tag{3.3}$$

One should remark that the Definition 3.4 does not demand the covering neighborhoods $B_{q_1}^{(p)}, \ldots, B_{q_s}^{(p)}$ to be disjoint in pairs. Thus, if $n \in \alpha_p^{-1}(B_{q_i}^{(p)} \cap B_{q_j}^{(p)})$, we will have

$$\alpha_p(n) = \beta_{q_i}(f_i^{(p)}(n)) = \beta_{q_j}(f_j^{(p)}(n)).$$

It is evident that $\underset{F}{\leqslant}$ and $\underset{F-1}{\leqslant}$ are both reflexive and transitive. Defining

$$\mathfrak{A} \underset{F}{\equiv} \mathfrak{B} \longleftrightarrow \mathfrak{A} \underset{F}{\leqslant} \mathfrak{B} \wedge \mathfrak{B} \underset{F}{\leqslant} \mathfrak{A}, \tag{3.4}$$

and

$$\mathfrak{A} \underset{F-1}{\equiv} \mathfrak{B} \longleftrightarrow \mathfrak{A} \underset{F-1}{\leqslant} \mathfrak{B} \wedge \mathfrak{B} \underset{F-1}{\leqslant} \mathfrak{A}, \tag{3.5}$$

we define *Finitary Atlas-Degrees* (on A), respectively *Finitary Atlas-One-Degrees* (on A), in short **FAD**'s, respective **FAOD**'s, as equivalence classes of a_A under $\underset{F}{\equiv}$, respectively under $\underset{F-1}{\equiv}$. The **FAD** of $\mathfrak{A}$ will be denoted by $\mathfrak{A}_F$, and its **FAOD** will be denoted by $\mathfrak{A}_{F-1}$.

By Corollary 3.2.1 if two compatible atlases are in the same **FAD**, then they may eventually be strongly compatible. In principle, one may expect that a **FAD** contains non-compatible atlases. For example, if in (3.3) one of the sets $\alpha_p^{-1}(B_{q_i})$, $i = 1, \ldots, s$, is not r.e., then $\mathfrak{A}$ and $\mathfrak{B}$ are not compatible.

Example 3.1 Let me consider **FAD**'s on N, the set of non-negative integers. In order to eliminate pathological atlases, I shall consider only at most denumerable atlases $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \ldots$, on N, which are *genuine* in the following sense: if just one of A_p's, or B_q's, or C_r's, ..., is removed, the remaining local neighborhoods of the respective atlas do not cover N.

A singleton-atlas $\{\alpha\}$ is an atlas on N iff α is an enumeration of N; thus

$$\{\alpha\} \underset{F}{\leqslant} \{\beta\} \longleftrightarrow \alpha = \beta \circ f,$$

where f is a recursive function.

Now, suppose that α is an indexing of N. If β is another indexing of N and $\{\alpha\} \underset{F}{\leqslant} \{\beta\}$, then $\alpha = \beta \circ p$, where p is a recursive permutation. This implies also that $\beta = \alpha \circ p^{-1}$, i.e., $\{\beta\} \underset{F}{\leqslant} \{\alpha\}$. Thus, we have the result:

(i) *If two singleton injective atlases on* $\{\alpha\}$ *and on* $\{\beta\}$ *are comparable* (*under* $\underset{F}{\leqslant}$) *then they are in the same* **FAD**.

In particular, all singleton-atlases each consisting of one recursive permutation are in the same **FAD**, say $\{I\}_F$; this **FAD** is incomparable (under the obvious sense of this word) with any **FAD** which contains a singleton-atlas consisting of one non-recursive permutation. This already proves:

(ii) *There is a continuum of mutually incomparable* **FAD**'*s on* N.

Now consider a genuine denumerable atlas $\mathfrak{A} = \{\alpha_i \mid i \in N\}$ on N, and a singleton-atlas $\{\alpha\}$ on N. We can never have $\{\alpha\} \underset{F}{\leq} \mathfrak{A}$, since no finite number of A_i's (A_i = range of α_i) can cover N (which has to be the case if $\{\alpha\} \underset{F}{\leq} \mathfrak{A}$). Thus, we obtain:

(iii) *The* **FAD**'*s of genuine denumerable atlases on* N *never contain finite atlases and, if comparable with* **FAD**'*s of finite atlases, the* **FAD**'*s of genuine denumerable atlases are smaller than the* **FAD**'*s of finite atlases.*

(I have taken for granted that the reader realizes that, by Theorem 3.3, finite atlases fall into **FAD**'s of singleton-atlases, i.e., they do not produce any new **FAD**'s on N.)

Let now $\{\alpha\}$ be an injective singleton-atlas on N. Let $\langle E_i \rangle_{i \in N}$ be a sequence of infinite recursive sets, such that $N = \bigcup_{i=0}^{\infty} E_i$, but such that for every $j \in N$, $N - E_j \neq N$. Let E_i be the range of the increasing recursive function f_i, let $\alpha_i = \alpha \circ f_i$ and $\mathfrak{A} = \{\alpha_i \mid i \in N\}$. Then $\mathfrak{A}$ is a genuine atlas on N, and $\mathfrak{A} \underset{F}{\leq} \{\alpha\}$. This gives:

(iv) *To every injective singleton-atlas on* N *one can correspond a genuine denumerable atlas of a lower* **FAD**.

Theorem 3.5 *The* **FAD**'*s on a fixed set* A *form an upper semi-lattice*, i.e., *to every two* **FAD**'*s* $\mathfrak{A}_F$ *and* $\mathfrak{B}_F$ *there corresponds their least upper bound* $\mathfrak{A}_F \vee \mathfrak{B}_F$.

Proof: If the atlases $\mathfrak{A}$ and $\mathfrak{B}$ are given, $\mathfrak{A} = \{\alpha_p \mid p \in P\}$, $\mathfrak{B} = \{\beta_q \mid q \in Q\}$, consider the cardinalities $\overline{\overline{P}}$ and $\overline{\overline{Q}}$. Suppose $\overline{\overline{P}} \leq \overline{\overline{Q}}$; then we can assume that $P \subset Q$. Define $\mathfrak{C} = \{\gamma_q \mid q \in Q\}$ as follows: if $q \in P$ then $\gamma_q(2n) = \alpha_q(n)$ and $\gamma_q(2n+1) = \beta_q(n)$; and if $q \in Q - P$ then $\gamma_q = \beta_q$. Trivially, $\mathfrak{A} \underset{F}{\leq} \mathfrak{C}$ and $\mathfrak{B} \underset{F}{\leq} \mathfrak{C}$. Suppose now that an atlas $\mathfrak{D} = \{\delta_s \mid s \in S\}$ is such that both $\mathfrak{A} \underset{F}{\leq} \mathfrak{D}$ and $\mathfrak{B} \underset{F}{\leq} \mathfrak{D}$. Then one obtains easily that $\mathfrak{C} \underset{F}{\leq} \mathfrak{D}$ i.e., $\mathfrak{C}_F = \mathfrak{A}_F \vee \mathfrak{B}_F$.

In an analogy with the notion of a cylinder I shall introduce a notion of cylindrification for atlases.

Definition 3.5 Let $\mathfrak{A} = \{\alpha_p \mid p \in P\}$ be an atlas on A. Then, $\mathrm{Cyl}_{\mathfrak{A}}$, the *cylindrification* of $\mathfrak{A}$, is the atlas $\mathrm{Cyl}_{\mathfrak{A}} = \{\overline{\alpha}_p \mid p \in P\}$, where

$$\overline{\alpha}_p(\sigma^2(n, m)) = \alpha_p(m) \text{ for all } n, m \in N. \tag{3.6}$$

(σ^2, σ_1^2, and σ_2^2 and σ_2^2 are as in Definition 2.1.)

Since $\alpha_p(m) = \overline{\alpha}_p(\sigma^2(0, m))$ and $\overline{\alpha}_p(n) = \alpha_n(\sigma_2^2(n))$ we have always $\mathfrak{A} \underset{F-1}{\leq} \mathrm{Cyl}_{\mathfrak{A}}$ and $\mathrm{Cyl}_{\mathfrak{A}} \underset{F}{\leq} \mathfrak{A}$.

Lemma 3.1 *Let* $\mathfrak{A}$ *and* $\mathfrak{B}$ *be atlases on* A. *Then:*

(i) $\mathfrak{A} \underset{F-1}{\leq} \mathrm{Cyl}_{\mathfrak{A}}$ *and* $\mathrm{Cyl}_{\mathfrak{A}} \underset{F}{\leq} \mathfrak{A}$.

(ii) $\mathfrak{B} \underset{\mathsf{F}}{\leqslant} \mathfrak{A}$ *implies* $\mathfrak{B} \underset{\mathsf{F}^{-1}}{\leqslant} \mathsf{Cyl}_{\mathfrak{A}}$.
(iii) $\mathfrak{B} \underset{\mathsf{F}}{\leqslant} \mathfrak{A} \leftrightarrow \mathsf{Cyl}_{\mathfrak{B}} \underset{\mathsf{F}^{-1}}{\leqslant} \mathsf{Cyl}_{\mathfrak{A}}$.

Proof: (i) was already proved. (ii) Let $q \in Q$ and let $A_{p_1}, \ldots, A_{p_s}$ cover B_q so that

$$\beta_q(n) = \alpha_{p_i}(f_i^{(q)}(n)), \text{ for } n \in \beta_q^{-1}(A_{p_i}), \tag{3.7}$$

where $f_1^{(q)}, \ldots, f_s^{(q)}$ are p.r. functions. Define $g_i^{(q)}(n) = \sigma^2(n, f_i^{(q)}(n))$, $i = 1, \ldots, s$. Then:

$$\beta_q(n) = \bar{\alpha}_r(g_i^{(q)}(n)), \text{ for } n \in \beta_q^{-1}(A_{p_i}),$$

which proves that $\mathfrak{B} \underset{\mathsf{F}^{-1}}{\leqslant} \mathsf{Cyl}_{\mathfrak{A}}$.
(iii) If $\mathfrak{B} \underset{\mathsf{F}}{\leqslant} \mathfrak{A}$ then $\mathsf{Cyl}_{\mathfrak{B}} \underset{\mathsf{F}}{\leqslant} \mathfrak{A}$, since $\mathsf{Cyl}_{\mathfrak{B}} \underset{\mathsf{F}}{\leqslant} \mathfrak{B}$. Thus, by (ii), $\mathsf{Cyl}_{\mathfrak{B}} \underset{\mathsf{F}^{-1}}{\leqslant} \mathsf{Cyl}_{\mathfrak{A}}$. Conversely, if $\mathsf{Cyl}_{\mathfrak{B}} \underset{\mathsf{F}^{-1}}{\leqslant} \mathsf{Cyl}_{\mathfrak{A}}$ we have

$$\mathfrak{B} \underset{\mathsf{F}^{-1}}{\leqslant} \mathsf{Cyl}_{\mathfrak{B}} \underset{\mathsf{F}^{-1}}{\leqslant} \mathsf{Cyl}_{\mathfrak{A}} \underset{\mathsf{F}}{\leqslant} \mathfrak{A}, \text{ i.e., } \mathfrak{B} \underset{\mathsf{F}}{\leqslant} \mathfrak{A}.$$

Theorem 3.6 *Every* **FAD** (*on* A) *contains a maximal* **FAOD**.

Proof: Consider $\mathfrak{A}_{\mathsf{F}}$ and $\mathsf{Cyl}_{\mathfrak{A}\mathsf{F}^{-1}}$. Obviously, $\mathsf{Cyl}_{\mathfrak{A}\mathsf{F}^{-1}}$ is contained in $\mathfrak{A}_{\mathsf{F}}$. Now, let $\mathfrak{B} \in \mathfrak{A}_{\mathsf{F}}$ be in any **FAOD**, say in $\mathfrak{B}_{\mathsf{F}^{-1}}$. Since $\mathfrak{B} \in \mathfrak{A}_{\mathsf{F}}$, we have $\mathfrak{B} \underset{\mathsf{F}}{\leqslant} \mathfrak{A}$, and by (ii) of Lemma 3.1 $\mathfrak{B} \underset{\mathsf{F}^{-1}}{\leqslant} \mathsf{Cyl}_{\mathfrak{A}}$.

Let us remark that (iii) of Lemma 3.1 establishes an order-homomorphism from the ordering $\underset{\mathsf{F}}{\leqslant}$ into the ordering $\underset{\mathsf{F}^{-1}}{\leqslant}$. All this shows that finitary reducibility of atlases is an appropriate extension of the reducibility of enumerations. Let me remark that "$\mathfrak{B} \underset{\mathsf{F}}{\leqslant} \mathfrak{A}$" is equivalent with "$\langle A, \mathfrak{B}\rangle$ is effectively a quasi-submanifold of $\langle A, \mathfrak{A}\rangle$". This suggests placing our manifolds $\langle A, \mathfrak{A}\rangle$, $\langle A, \mathfrak{B}\rangle$, $\langle A, \mathfrak{C}\rangle$, . . ., inside one fixed larger manifold.

Thus, I should now have a fixed **REM** $\langle M, \mathfrak{M}\rangle$, $\mathfrak{M} = \{\mu_t \mid t \in T\}$, M_t range of μ_t, $M = \bigcup_{t \in T} M_t$, and that $A \subset M$. I shall consider atlases on A (obviously, I suppose that A is non-empty) which are *finitely reducible* to $\mathfrak{M}$ in the obvious sense: $\mathfrak{A} \underset{\mathsf{F}}{\leqslant} \mathfrak{M}$ iff each A_p can be covered by finite many M_t's, say by $M_{t_1}, \ldots, M_{t_s}$, and there are p.r. functions $f_i^{(p)}$, $i = 1, \ldots, s$, such that

$$\alpha_p(n) = \mu_{t_i}(f_i^{(p)}(n)) \text{ for } n \in \alpha_p^{-1}(M_{t_i}),$$

and $i = 1, \ldots, s$. (Consequently, I shall suppose that **REM**'s $\langle A, \mathfrak{A}\rangle$, $\langle A, \mathfrak{B}\rangle$, $\langle A, \mathfrak{C}\rangle$, . . ., are effectively quasi-submanifolds of $\langle M, \mathfrak{M}\rangle$.)

In a similar way I can extend the notion of finitary reducibility to subsets A_0 of A, i.e., to the atlases on such subsets.

Definition 3.6 The atlas $\mathfrak{A}$ (on $A \subset M$) is *principal* iff $\mathfrak{A} \underset{\mathsf{F}}{\leqslant} \mathfrak{M}$ and, for every atlas $\mathfrak{B}$ on A, the relation $\mathfrak{B} \underset{\mathsf{F}}{\leqslant} \mathfrak{M}$ implies $\mathfrak{B} \underset{\mathsf{F}}{\leqslant} \mathfrak{A}$.

The existence of principal atlases on A depends on the recursive structure of A in $\langle M, \mathfrak{M}\rangle$, and on the structure of $\mathfrak{M}$.

Theorem 3.7 *If* $\langle M, \mathfrak{M}\rangle$ *is positive and* A *is an* $\mathfrak{M}$-r.e. *set, then there exists at least one principal atlas on* A.

Proof: Let $T_0 \subset T$ be the set of all $t \in T$ such that $\mu_t^{-1}(A) \neq \emptyset$. Then, every set $\mu_t^{-1}(A)$, for $t \in T_0$, is a non-empty r.e. subset of N. Let it be the range of the recursive function m_t. Then, $A \cap M_t = \mu_t(M_t(N))$ for all $t \in T_0$, and $A = \bigcup_{t \in T_0} A_t$, where A_t = range of $\alpha_t = \mu_t \circ M_t$. At last, set $\mathfrak{A} = \{\alpha_t \mid t \in T_0\}$. I shall prove that $\mathfrak{A}$ is principal.

Suppose $\mathfrak{B} = \{\beta_q \mid q \in Q\}$ is an atlas on A, such that $\mathfrak{B} \underset{F}{\leqslant} \mathfrak{M}$. Let $q \in Q$ be fixed and let $\{t_0, t_1, \ldots, t_s\} \subset T$ be such that $\{M_{t_0}, M_{t_1}, \ldots, M_{t_s}\}$ covers B_q; let $f_0^{(q)}, f_1^{(q)}, \ldots, f_s^{(q)}$ be p.r. and such that, for $i = 0, 1, \ldots, s$,

$$\beta_q(n) = \mu_{t_i}(f_i^{(q)}(n)) \text{ for } n \in \beta_q^{-1}(M_{t_i}).$$

Since $B_q \subset A$, we have $B_q \cap M_{t_i} = B_q \cap A_{t_i}$. Thus,

$$\beta_q(n) = \mu_{t_i}(f_i^{(q)}(n)) \text{ for } n \in \beta_q^{-1}(A_{t_i}), \tag{3.8}$$

$(i = 0, 1, \ldots, s)$ and $\{A_{t_0}, A_{t_1}, \ldots, A_{t_s}\}$ covers B_q. For $i = 0, 1, \ldots, s$, define g_i by $g_i(n) \simeq$ some $y \in N$ such that $\mu_{t_i}(y) = \mu_{t_i}(m_{t_i}(n))$; since $\mathfrak{M}$ is positive, each q_i is a p.r. function, and we have

$$\mu_{t_i}(n) = \alpha_{t_i}(g_i(n)) \text{ for all } n \in D_{g_i}.$$

Then, by (3.8), we obtain

$$\beta_q(n) = \alpha_{t_i}(g_{t_i}(f_i^{(q)}(n))) \text{ for } n \in \beta_q^{-1}(A_{t_i}),$$

and $i = 0, 1, \ldots, s$, which proves that $\mathfrak{B} \underset{F}{\leqslant} \mathfrak{A}$.

Corollary 3.7.1 *If* $\langle M, \mathfrak{M}\rangle$ *is an* **IREM** (*an* **IRM**) *and* $A \subset M$ *an* $\mathfrak{M}$-r.e. *set, such that each non-empty* $\mu_t^{-1}(A)$ *is infinite, then there is a principal atlas* $\mathfrak{A}$ *on* A, *such that* $\langle A, \mathfrak{A}\rangle$ *is an* **IREM** (*an* **IRM**).

If both $\mathfrak{A}$ and $\mathfrak{B}$ are principal, they are in the same **FAD**; this **FAD** is the *maximal* element of the family of **FAD**'s of all **REM**'s $\langle A, \mathfrak{C}\rangle$, which are effectively quasi-submanifolds of $\langle M, \mathfrak{M}\rangle$.

Example 3.2 Consider $\langle N, \{\mathsf{I}\}\rangle$, where I is the identity on N, as the fixed **REM** $\langle M, \mathfrak{M}\rangle$. (To be precise, $\langle N, \{\mathsf{I}\}\rangle$ is an **IRM**.) Let A be any non-empty subset of N. We shall consider at most denumerable genuine atlases on A. Let $\mathfrak{A} = \{\alpha_i \mid i \in N\}$, A_i = range of α_i, $A = \bigcup_{i=0}^{\infty} A_i$, be such an atlas. Suppose it is principal. Consider any **REM** $\langle A, \{\alpha\}\rangle$. If $\{\alpha\} \underset{F}{\leqslant} \{\mathsf{I}\}$, α must be a recursive function with range A, i.e.,

(i) *If a singleton atlas* $\{\alpha\}$ *on* A *is finitely reducible to* $\{\mathsf{I}\}$, *then* A *must be a* r.e. *set and* α *a recursive function.*

Therefore, let us start with the case in which A is a r.e. set. Then any recursive function α: $N \to A$, with A as range, defines an atlas $\{\alpha\}$ on A such that $\{\alpha\} \underset{F}{\leqslant} \{\mathsf{I}\}$. Since $\mathfrak{A}$ is principal, we obtain at once: A must be covered by at most finite many A_i's. However, this contradicts the supposition that $\mathfrak{A}$ is genuine. So, we have:

(ii) *If* A *is* r.e. *then no genuine infinite atlas* $\mathfrak{A} = \{\alpha_i \mid i \in N\}$ *on* A *can be principal.*

Thus, if principal, $\mathfrak{A}$ must be finite. But then, by Theorem 3.3, $\mathfrak{A}$ can be replaced by a singleton atlas $\{\alpha\}$, where α is recursive and has A as range. Now, let $\mathfrak{A} = \{\alpha\}$, $\alpha: N \to A$, α recursive. Let $\mathfrak{B} = \{\beta_i \mid i \in N\}$ be any atlas on A, such that $\mathfrak{B} \leq_F \{\mathrm{I}\}$. This implies: each β_i is recursive. Define then

$$f_i(n) = \mu_y(\alpha(y) = \beta_i(n)).$$

Then $\beta_i(n) = \alpha(f_i(n))$, i.e., $\mathfrak{B} \leq_F \{\alpha\}$, and we obtain

(iii) *Every principal atlas on A, in case A is* r.e., *can be reduced to a singleton atlas* $\{\alpha\}$, *with recursive* α. *Every such atlas is then principal.*

(The last statement in (iii) should not be astonishing, in view of Theorem 3.6.) Now suppose that A is not r.e. Remark that it cannot be immune if it admits any atlas $\mathfrak{A} \leq_F \{\mathrm{I}\}$ which contains at least one infinite local neighborhood A_p (since, then α_p must be recursive). Thus, we have to consider two cases: A immune, and A non-immune.

Let first A be immune. Then, every atlas $\mathfrak{A} \leq_F \{\mathrm{I}\}$ on A, must contain only finite local neighborhoods A_p, and so must be infinite. Let $\mathfrak{A} = \{\alpha_i \mid i \in N\}$, where each α_i is recursive, with finite range, and suppose that $\mathfrak{A}$ is genuine. Thus, if $\mathfrak{B} = \{\beta_i \mid i \in N\}$ is any atlas on A which is finitely reducible to $\{\mathrm{I}\}$, we will have $\overline{\overline{B}}_i < \infty$, for all $i \in N$. Also the relation $\mathfrak{B} \leq_F \{\mathrm{I}\}$ implies that each β_i is recursive (with finite range). Therefore, to each $i \in N$ there corresponds finite many numbers $i_0, i_1, \ldots, i_s$, such that A_{i_0}, $A_{i_1}, \ldots, A_{i_s}$ cover B_i. Define f_{i_μ} by

$$f_{i_\mu}(n) \simeq \text{any } y \in N \text{ such that } \alpha_{i_\mu}(y) = \beta_i(n).$$

Then each f_{i_μ} is partial recursive, $\mu = 0, \ldots, s$, and

$$\beta_i(n) = \alpha_{i_\mu}(f_{i_\mu}(n)) \text{ for } n \in \beta_i^{-1}(A_{i_\mu}),$$

$\mu = 0, \ldots, s$, i.e., $\mathfrak{B} \leq_F \mathfrak{A}$. Thus:

(iv) *If A is immune, then every atlas* $\mathfrak{A} = \{\alpha_i \mid i \in N\}$ *on A, where each* α_i *is recursive, with finite range, is principal, and every principal atlas on A is of this type.*

At last, suppose A infinite, non-recursively enumerable and non-immune. Then, no singleton atlas $\{\alpha\}$ on A can satisfy $\{\alpha\} \leq \{\mathrm{I}\}$, and no finite atlas can do it either. Thus, exactly all genuine infinite atlases $\mathfrak{A} = \{\alpha_i \mid i \in N\}$, with all α_i recursive and such that $\langle A_i \rangle_{i \in N}$ is not a recursively enumerable sequence of r.e. sets, satisfy $\mathfrak{A} \leq_F \{\mathrm{I}\}$. Here, some A_i may be infinite; in fact,

if $\mathfrak{A}$ *is to be principal, at least one* A_i *must be infinite.*

To see this, remark that A contains an infinite r.e. set, say B, which is the range of the injective, recursive function β_0. Now, construct the atlas $\mathfrak{B} = \{\beta_i \mid i \in N\}$ by taking every β_i for $i \geq 1$ to be identically b_i, where $b_1, b_2, b_3, \ldots$, is an enumeration of $A - B_0$. Since $\mathfrak{B} \leq_F \{\mathrm{I}\}$ we must have

$\mathfrak{B} \leq_F \mathfrak{A}$; this implies that B_0 can be covered by finite many A_i's; thus, at least one of those has to be infinite.

It should be obvious that definite characterization of principal atlases in this last case depends very much on the nature of A. Thus, I will leave this characterization for a special study.

Let us say that an atlas $\mathfrak{A}$ on $A \subset M$ is *finitary with respect to* $\mathfrak{M}$ (the atlas on M) iff each M_t meets at most finite many A_p's.

Theorem 3.8 *Let* $\mathfrak{M}$ *be positive and let* $\mathfrak{A}$ *be a principal atlas on* $A \subset M$, *which is finitary with respect to* $\mathfrak{M}$. *Let* $A_0 \subset A$ *and let* $\langle A_0, \mathfrak{B}\rangle$ *be any* **REM** *which is effectively a quasi-manifold of* $\langle M, \mathfrak{M}\rangle$. *Then* $\mathfrak{B} \leq_F \mathfrak{A}$.

Proof: We suppose $\mathfrak{B} = \{\beta_q | q \in Q\}$. If $M_{t_0}, \ldots, M_{t_s}$ cover B_q, let $f_i^{(q)}$, $i = 0, \ldots, s$, be partial recursive and such that

$$\beta_q(n) = \mu_{t_i}(f_i^{(q)}(n)) \text{ for } n \in \beta_{q_i}^{-1}(M_{t_i}).$$

Then $\{A \cap M_{t_0}, \ldots, A \cap M_{t_s}\}$ covers B_q. Now, by the condition on $\mathfrak{A}$,

$$A \cap M_{t_i} = \{A_{p_{i,1}} \cup, \ldots, \cup A_{p_{i,s_i}}\} \cap M_{t_i}$$

for $i = 0, \ldots, s$; thus, $\bigcup_{i=0}^{s} \bigcup_{j=0}^{s_i} A_{p_{i,j}}$ covers B_q. Moreover, there are partial recursive functions $f_{p_{i,j}}$, $i = 0, \ldots, s$, $j = 1, \ldots, s_i$ such that

$$\alpha_{p_{i,j}}(n) = \mu_{t_i}(f_{p_{i,j}}(n)) \text{ for } n \in \alpha_{p_{i,j}}^{-1}(M_{t_i}).$$

Applying the same method as in the proof of the second part of Theorem 3.6 we obtain the proof of this theorem.

A slight variant of Theorem 3.7 is

Theorem 3.9 *Let* $\mathfrak{A} = \{\alpha_t | t \in T\}$ *be such that* $A_t \subset M_t$ *and that there is a family* $\{f_t \mid t \in T\}$ *of* p.r. *functions, satisfying for all* $t \in N$

$$\mu_t^{-1}(A) \subset D_{f_t}$$

and

$$\mu_t(n) = \alpha_t(f_t(n)) \text{ for all } n \in \mu_t^{-1}(A).$$

Then $\mathfrak{A}$ *is principal* $\left(\text{on } A = \bigcup_{t \in T} A_t\right)$.

Proof: We have to prove only that $\mathfrak{A} \leq_F \mathfrak{M}$. Define g_t by

$$g_t(n) \simeq \text{any } y \in N \text{ such that } f_t(y) = n.$$

Since $R_{f_t} = N$, g_t is recursive and

$$\alpha_t(n) = \mu_t(g_t(n)) \text{ for all } n \in N,$$

i.e., $\mathfrak{A} \leq_F \mathfrak{M}$.

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To be continued

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