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ON SOME MODAL LOGICS RELATED TO THE Ł-MODAL SYSTEM

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1 *Introduction* Five modal logics are introduced in this paper. They are denoted by $F*F$ where F = Ł, W, S, D and E. F^* denotes the semantics (see section 3) and F denotes the formal system (see section 4). Each modal logic $F*F$ is composed of four sub-logics $F^*_iF_i$ (i = 1, 2, 3, 4) corresponding to four different kinds of provability and rejection, namely F_i-provability and F_i-rejection.

Since the idea of these modal logics arose from certain semantical considerations rather than from formal ones, some questions on the semantics of the Ł-modal system and 3-valued logic are mentioned in section 2. These questions help to provide the motivation for the semantics F^* and a semantics Ł^*_3 for the Ł-modal system in particular.

The formal treatment uses an adaption of Smullyan's method of the analytic tableaux [6] and is illustrated for Ł in section 4. In section 5, the semantical consistency and completeness proofs for $\text{Ł}*\text{Ł}$ are given. The sub-systems F_1, Ł_2, W_2, S_2 violate some of the laws of Łukasiewicz's basic modal logic [1]. Halldén's incompleteness property [5] holds in the sub-systems F_3. Also, the sub-systems F_4 are formally inconsistent (see section 7). The connection between all these formal properties and the underlying semantics is discussed in section 7.

2 *Some questions and comments on the Ł-modal system and 3-valued logic*

Question 1 Considering Łukasiewicz's four truth-values underlying his semantics for the Ł-modal system, what do the four truth-values mean?

Comment It is interesting to note that when Łukasiewicz is referring to the semantics in [1], [2], he is basically talking in a 2-valued idiom, i.e., he simply uses the words 'true' and 'false' (*cf.* Łukasiewicz's truth-values '1' and '4'). Concerning the values '2' and '3', Łukasiewicz in his paper [2] refers to them as "denoting possibility, but nevertheless both values represent one and the same possibility in two different shapes."

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Question 2 What meanings should be attached to the words 'possible' and 'necessary' in [1], [2], and how do such meanings link up with the 4-valued truth-tables for '*Mp*' and '*Lp*'?

Comment Łukasiewicz's meanings of 'It is possible that *p*' (for '*Mp*') and 'It is necessary that *p*' (for '*Lp*') are unclear, since we are unable, on the basis for his intended meanings, to then go on and calculate the truth-tables for '*Mp*' and '*Lp*'. In contrast, note that the truth-tables for '*Np*', '*Kpq*', etc. in **PC** can be calculated, once the intended meaning of these functors '*N*', and '*K*' has been given.

Question 3 What is the connection, if any, between the meanings to be attached to '*Mp*' and '*Lp*' in the Ł-modal system and the '*Mp*' and '*Lp*' introduced into Łukasiewicz's 3-valued logic [3], [4]?

Comment In contrast to what has been said about the semantics of '*Mp*' and '*Lp*' in Q2, the meanings of '*Mp*' and '*Lp*' in [3] are reasonably clear.

Question 4 Why does Halldén's incompleteness property [5] arise in the Ł-modal system?

Comment The answer to Q5 lies in the semantics of the Ł-modal system, but Łukasiewicz's semantics do not seem adequate to provide an explanation.

Question 5 What semantical meaning can be attached to the tertium function '*Tp*' of [4]?

Comment Słupecki's idea of introducing the tertium function '*Tp*' into Łukasiewicz's 3-valued logic was to produce a system that is 'full' and 'complete' like classical **PC**. However, the function '*Tp*' still requires that a semantical interpretation be given to complete the semantics for the system.

In the concluding section "Philosophical implications of modal logic" at the end of [1], Łukasiewicz argues that: "There are no true apodeictic propositions, and from the standpoint of logic there is no difference between a mathematical and an empirical truth." This view (supported by Quine[1], Quine[2], White[3]) is relevant for the semantics **F***.

3 *The Semantics* F*(F = Ł, W, S, D, E)

3.1 Before proceeding to the semantics, we give the definition of well-formed formulae (wff) for the various formal systems **F**.

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[1] W. V. Quine, "Two dogmas of empiricism," in *From a Logical Point of View*, Harvard University Press, Cambridge (1953), pp. 20-46.

[2] W. V. Quine, "Necessary truth" in *The Ways of Paradox*, Random House, New York (1966).

[3] M.G.White, "The analytic and the synthetic: An untenable dualism" in L. Linsky, *Semantics and the Philosophy of Language*, University of Illinois Press, Urbana (1952), pp. 272-286.

The undefined logical connectives for **Ł**, **W** and **D** are *N*, *M*, *L* (monadic) and *C*, *A*, *K*, *E* (dyadic). Also, the monadic functors *T*, *F* are added for **S** and **E**. We are also given a denumerable number of propositional variables *p*, *q*, *r*, *s*, ... A wff in each system **F** is defined recursively in the usual way.

3.2 *The idea of a truth-value in the semantics* **F*** These semantics **F*** presuppose that every proposition needs to be considered under two logically distinct categories namely, the relative category C_1, and the absolute category C_2. Following Frege's ideas of the 'sense' and 'reference' of a proposition, we replace these by the following four notions: C_1-sense, C_1-reference, C_2-sense and C_2-reference.

The C_1-reference of a proposition is called its C_1-truth-value or relative truth-value or truth-value under the relative category.

The C_2-reference of a proposition is called its C_2-truth-value or absolute truth-value or truth-value under the absolute category.

The C_1-truth-values of propositions are considered to be truth-values based on experience in the world. Hence every proposition (including the mathematical and logical varieties), considered under C_1, is thought of in a quite general sense, i.e., C_1-sense, as an empirical proposition. On the other hand, the C_2-truth-values of propositions are not truth-values based on possible human experience in the world—they are meant as truth-values in a transcendental or absolute sense, i.e., C_2-sense. Consequently, in contrast to the C_1-truth-value of any proposition, its C_2-truth-value is by definition unknown and unknowable. The idea of a C_2-truth-value can be thought of as a regulative idea in Kant's[4] sense.

Thus, by truth-value of a proposition in **F***, we mean an ordered pair of truth-values. The first component is the C_1-truth-value and the second component is the C_2-truth-value. In the most general cases **D*** and **E***, there are six basic semantical notions C_1-truth, C_1-falsity, C_1-indeterminateness, C_2-truth, C_2-falsity and C_2-indeterminateness, which are denoted by t_1, f_1, i_1, t_2, f_2 and i_2 respectively.

3.3 *The semantics for the five systems* The question of the logical structure of the world is resolved into two aspects:

(i) the logical structure of the world under C_1, and
(ii) the logical structure of the world under C_2.

The different systems arise by making different assumptions concerning (i) and (ii) here. These assumptions are not stated explicitly for the different systems, but are clear from the truth-tables given below (Tables I-VII).

For all the systems the functors '*N*', '*C*', '*K*', '*A*' and '*E*' are interpreted in the usual way as 'not', 'if-then', 'and', 'or' and 'if and only if'. The C_1- and C_2-truth-tables given below for some of these functors are based on the 2- and 3-valued propositional logics already mentioned, and only tables for *N*, *M*, *C*, *L*, *T* and *F* are presented.

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[4] I. Kant, *Critique of Pure Reason*, translated by N. K. Smith, MacMillan and Co., London (1933).

TABLE I

Truth-tables for Ł*

C	t_1t_2	f_1t_2	t_1f_2	f_1f_2	*N*	*M*	*L*
t_1t_2	t_1t_2	f_1t_2	t_1f_2	f_1f_2	f_1f_2	t_1t_2	f_1t_2
f_1t_2	t_1t_2	t_1t_2	t_1f_2	t_1f_2	t_1f_2	t_1t_2	f_1t_2
t_1f_2	t_1t_2	f_1t_2	t_1t_2	f_1t_2	f_1t_2	t_1f_2	f_1f_2
f_1f_2	t_1t_2	t_1t_2	t_1t_2	t_1t_2	t_1t_2	t_1f_2	f_1f_2

TABLE II

C_1-truth-tables for Ł*

C	t_1f_1	*N*	*M*	*L*
t_1	t_1f_1	f_1	t_1	f_1
f_1	t_1t_1	t_1	t_1	f_1

TABLE III

C_2-truth-tables for Ł*

C	t_2f_2	*N*	*M*	*L*
t_2	t_2f_2	f_2	t_2	t_2
f_2	t_2t_2	t_2	f_2	f_2

TABLE IV

C_1-truth-tables for **W***, **S***

C	$t_1f_1i_1$	*N*	*M*	*L*	*T*	*F*
t_1	$t_1f_1i_1$	f_1	t_1	f_1	i_1	i_1
f_1	$t_1t_1t_1$	t_1	t_1	f_1	i_1	i_1
i_1	$t_1i_1t_1$	i_1	t_1	f_1	i_1	i_1

TABLE V

C_2-truth-tables for **W***, **S***

C	t_2f_2	*N*	*M*	*L*	*T*	*F*
t_2	t_2f_2	f_2	t_2	t_2	t_2	f_2
f_2	t_2t_2	t_2	f_2	f_2	f_2	t_2

Note: For **W*** we omit the functors *T* and *F*.

TABLE VI

C_1-truth-tables for **D***, **E***

C	$t_1f_1i_1$	*N*	*M*	*L*	*T*	*F*
t_1	$t_1f_1i_1$	f_1	t_1	f_1	i_1	i_1
f_1	$t_1t_1t_1$	t_1	t_1	f_1	i_1	i_1
i_1	$t_1i_1t_1$	i_1	t_1	f_1	i_1	i_1

TABLE VII

C_2-truth-tables for **D***, **E***

C	$t_2f_2i_2$	*N*	*M*	*L*	*T*	*F*
t_2	$t_2f_2i_2$	f_2	t_2	t_2	t_2	f_2
f_2	$t_2t_2t_2$	t_2	f_2	f_2	f_2	t_2
i_2	$t_2i_2t_2$	i_2	t_2	f_2	f_2	f_2

Note: For **D*** we omit the functors *T* and *F*.

The composite truth-tables for the 4-valued Ł* (Table I) are given as well as the C_1- and C_2-truth-tables, but for the remaining systems, i.e., the 6-valued **W***, **S*** and the 9-valued **D***, **E***, only C_1- and C_2-truth-tables are presented (Tables II-VII).

We now give the interpretations of '*Mp*', '*Lp*', '*Tp*', and '*Fp*', for the various systems.

For **Ł***, **W*** and **S***:

‘*Mp*’ is interpreted as ‘It is possible that *p* is C_2-true’.
‘*Lp*’ is interpreted as ‘It is necessary that *p* is C_2-true’.

For **D*** and **E***:

‘*Mp*’ is interpreted as ‘It is possible that *p* is or will be C_2-true’.
‘*Lp*’ is interpreted as ‘It is necessary that *p* is or will be C_2-true’.

The 4-valued matrix for **Ł***, Table I, is isomorphic to the 4-valued truth-tables given for the Ł-modal system in [1], Table $\mathcal{M}9$, p. 168. Also, from Table VII, the C_2-tables for ‘*Mp*’ and ‘*Lp*’ correspond to those for ‘*Mp*’ and ‘*Lp*’ in Łukasiewicz’s 3-valued logic (see [7], p. 55).

For **S***, **E***:

‘*Tp*’ is interpreted as ‘*p* is C_2-true’.
‘*Fp*’ is interpreted as ‘*p* is C_2-false’.

We can refer to the modal functors ‘*T*’ and ‘*F*’ as the verum and falsum operators respectively. As Łukasiewicz notes in [3], p. 41 of [7], these two modes, verum and falsum, were cited by the logicians in the Middle Ages, “However, these modes were given no further consideration, as the modal propositions corresponding to them ‘it is true that *p*’ and ‘it is false that *p*’, were regarded as being equivalent to the propositions ‘*p*’ and ‘*Np*’.” The situation regarding ‘*Tp*’ and ‘*Fp*’ here is different, since, in general, these are not equivalent to ‘*p*’ and ‘*Np*’. These modalities ‘*Tp*’ and ‘*Fp*’ and the C_1-truth-tables IV and VI also suggest that on purely logical grounds **Ł*Ł** is not a fully adequate modal logic, because these functors ‘*T*’ and ‘*F*’ with the above interpretations cannot be admitted in **Ł*Ł**. Comparing ‘*T*’ (or ‘*F*’) in Table VI with ‘*T*’ in Słupecki [4], the above gives a possible semantical interpretation, under C_1, for the tertium function.

Set against the intended meanings of C_1- and C_2-truth-values, the truth-tables for ‘*Mp*’, ‘*Lp*’, ‘*Tp*’ and ‘*Fp*’ are intuitively plausible, and can be calculated on the basis of those intended meanings.

These provisional semantics require an explicit statement on the role of the notions of time, knowledge, belief, and a fuller account of the idea of the categories. For example, the interpretation of ‘*Tp*’ given above and the third truth-value C_1-indeterminate occurring therein, clearly involve epistemic notions. In contrast, the value C_2-indeterminate occurring in **D*** and **E*** accords well with Łukasiewicz’s original idea of a third truth-value associated with propositions about future contingent events (discussed in §6,7 of [3]).

3.4 *The Four Kinds of Tautologies* The system **Ł*Ł** is used to illustrate what we mean by interpretations, truth-value of wff under an interpretation, and the different kinds of tautologies arising.

Definition 1 By an ŁC_1-interpretation, of a wff *X* of Ł, we mean a mapping which assigns to every propositional variable occurring in *X* one of the two values t_1 or f_1.

Definition 2 By truth-value of a wff X under an ŁC_1-interpretation we mean the C_1-truth-value obtained on the basis of the ŁC_1-truth-tables (Table II) and the particular values t_1 or f_1 assigned to each propositional variable under the ŁC_1-interpretation.

Definition 3 A wff X is ŁC_1-satisfiable iff X is C_1-true under at least one ŁC_1-interpretation.

Definition 4 A wff X is an ŁC_1-tautology iff X is C_1-true under every ŁC_1-interpretation.

(The idea of truth-value of a wff X under an ŁC_1-interpretation can be made explicit by using Smullyan's ideas of 'sub-formulas' of X and 'formation tree' for X given in [6], pp. 8-11.)

Definitions of ŁC_2-interpretations, etc. are analogous to Definitions 1-4, above, by putting C_2 for C_1 and t_2, f_2 for t_1, f_1 respectively. ŁC_1-tautologies and ŁC_2-tautologies are called fundamental tautologies. We now define two kinds of derived tautologies.

Definition 5 A wff X is an ŁC_1C_2-tautology iff X is an ŁC_1-tautology and X is an ŁC_2-tautology.

Definition 6 A wff X is an ŁC_1/C_2-tautology iff X is an ŁC_1-tautology or X is an ŁC_2-tautology.

We can introduce four sub-systems Ł$^*_i$Ł$_i$ ($i = 1, 2, 3, 4$) each of which is associated with an Ł$_i$-tautology ($i = 1, 2, 3, 4$) corresponding to the ŁC_1-, ŁC_2-, ŁC_1C_2- and ŁC_1/C_2-tautologies respectively.

In general, for the systems F*F (F = Ł, W, S, D, E) we have analogously FC_1-, FC_2-interpretations and tautologies and FC_1C_2-, FC_1/C_2-tautologies. Also, associated with each F*F we have four sub-systems F$^*_i$F$_i$ ($i = 1, 2, 3, 4$) corresponding to FC_1-, FC_2-, FC_1C_2-, FC_1/C_2-tautologies respectively.

4 *The formal systems* F

4.1 The formal systems F (F = Ł, W, S, D, E) are based on an adaption of Smullyan's method of the analytic tableaux given in chapters 1, 2 of [6]. Much of these two chapters is presupposed and will be referred to on several occasions in what follows. The system Ł is treated in some detail and the modifications for the other four systems are indicated. We have already introduced the basic syntax and definition of wff for the systems F in 3.1.

For Ł, we also add to the syntax the four symbols, called signs 'T_1', 'F_1', 'T_2', and 'F_2'. We define an Ł-signed wff (Ł-swff) as an expression of the form T_1X, F_1X, T_2X or F_2X where X is a wff of Ł. The first two swff are called ŁC_1-swff and the latter two ŁC_2-swff. The formal interpretation of Ł-swff is shown in Table VIII and in the following definitions. Although wff in Ł are given both ŁC_1- and ŁC_2-interpretations, ŁC_1-swff are given only ŁC_1-interpretations and ŁC_2-swff are given only ŁC_2-interpretations. We let SX denote any swff, and S_1X, S_2X any ŁC_1-swff and ŁC_2-swff respectively.

TABLE VIII

	T_1	F_1		T_2	F_2
t_1	t_1	f_1	t_2	t_2	f_2
f_1	f_1	t_1	f_2	f_2	t_2

Definition 1 An ŁC_1-swff S_1X is said to be ŁC_1-satisfiable iff S_1X is C_1-true under some ŁC_1-interpretation.

Definition 2 A set of ŁC_1-swff $\mathcal{L}_1$ is said to be ŁC_1-satisfiable iff there exists at least one ŁC_1-interpretation of all propositional variables occurring in any of the swff in $\mathcal{L}_1$ under which every swff in $\mathcal{L}_1$ is C_1-true.

Definitions of ŁC_2-satisfiability are analogous to definitions 1 and 2, (put C_2 for C_1 and $\mathcal{L}_2$ for $\mathcal{L}_1$). For the other four systems, we also add to the basic syntax, 'I_1' to **W** and **S** and 'I_1', 'I_2' to **D** and **E**, and get analogous definitions to Definitions 1, 2, using Tables VIII and IX where appropriate.

TABLE IX

	T_1	F_1	I_1
t_1	t_1	f_1	f_1
f_1	f_1	t_1	f_1
i_1	f_1	f_1	t_1

	T_2	F_2	I_2
t_2	t_2	f_2	f_2
f_2	f_2	t_2	f_2
i_2	f_2	f_2	t_2

4.2 *Definition of tableaux and provable sentences in* Ł We now state the rules for the construction of Ł-tableaux in schematic form (analogous to **PC**-tableaux in [6], p. 17). There are two sets of rules.

ŁC_1-Rules

1. $\dfrac{T_1NX}{F_1X}$ 2. $\dfrac{F_1NX}{T_1X}$

3. $\dfrac{T_1MX}{T_1X \mid F_1X}$

4. $\dfrac{F_1LX}{T_1X \mid F_1X}$

5. $\dfrac{T_1CXY}{F_1X \mid T_1Y}$ 6. $\dfrac{F_1CXY}{\begin{matrix} T_1X \\ F_1Y \end{matrix}}$

7. $\dfrac{T_1KXY}{\begin{matrix} T_1X \\ T_1Y \end{matrix}}$ 8. $\dfrac{F_1KXY}{F_1X \mid F_1Y}$

ŁC_2-Rules

13. $\dfrac{T_2NX}{F_2X}$ 14. $\dfrac{F_2NX}{T_2X}$

15. $\dfrac{T_2MX}{T_2X}$ 16. $\dfrac{F_2MX}{F_2X}$

17. $\dfrac{T_2LX}{T_2X}$ 18. $\dfrac{F_2LX}{F_2X}$

19. $\dfrac{T_2CXY}{F_2X \mid T_2Y}$ 20. $\dfrac{F_2CXY}{\begin{matrix} T_2X \\ F_2Y \end{matrix}}$

21. $\dfrac{T_2KXY}{\begin{matrix} T_2X \\ T_2Y \end{matrix}}$ 22. $\dfrac{F_2KXY}{F_2X \mid F_2Y}$

ŁC$_1$-Rules (cont'd.)

9. $$\frac{T_1AXY}{T_1X \mid T_1Y}$$

10. $$\frac{F_1AXY}{\begin{array}{c} F_1X \\ F_1Y \end{array}}$$

11. $$\frac{T_1EXY}{\begin{array}{c|c} T_1X & F_1X \\ T_1Y & F_1Y \end{array}}$$

12. $$\frac{F_1EXY}{\begin{array}{c|c} T_1X & F_1X \\ F_1Y & T_1Y \end{array}}$$

ŁC$_2$-Rules (cont'd.)

23. $$\frac{T_2AXY}{T_2X \mid T_2Y}$$

24. $$\frac{F_2AXY}{\begin{array}{c} F_2X \\ F_2Y \end{array}}$$

25. $$\frac{T_2EXY}{\begin{array}{c|c} T_2X & F_2X \\ T_2Y & F_2Y \end{array}}$$

26. $$\frac{F_2EXY}{\begin{array}{c|c} T_2X & F_2X \\ F_2Y & T_2Y \end{array}}$$

We can express these rules succinctly by introducing a unifying notation (analogous to the α, β introduced in Smullyan [6], p. 20) to get the condensed rules:

Rule A: $$\frac{\alpha}{\begin{array}{c} \alpha_1 \\ \alpha_2 \end{array}}$$

Rule B: $$\frac{\beta}{\beta_1 \mid \beta_2}$$

Rule C: $$\frac{\gamma}{\begin{array}{c|c} \gamma_1 & \gamma_2 \\ \gamma_3 & \gamma_4 \end{array}}$$

An Ł-tableau for a swff SX is an ordered dyadic tree $\mathcal{T}(SX)$ (see [6], pp. 3-4) whose points are occurrences of swff and which is constructed as follows:

We take SX as the origin. Suppose now $\mathcal{T}(SX)$ is an Ł-tableau for SX which has already been constructed. Let ρ, a swff be an end-point on some branch of $\mathcal{T}(SX)$, then we may extend $\mathcal{T}(SX)$ by either one of the three kinds of operations which correspond to Rules A, B and C stated above:
(A) If some α occurs on the path P_ρ, then we adjoin either α_1 or α_2 as the sole successor of ρ.
(B) If some β occurs on the path P_ρ, then we may adjoin simultaneously β_1 as the first successor of ρ and β_2 as the second successor of ρ. (ρ is then a branch point.)
(C) If some γ occurs on the path P_ρ, then we may adjoin γ_1 or γ_3 as the first successor of ρ and γ_2 or γ_4 as the second successor of ρ.

This definition can be made explicit as in Smullyan [6], p. 24, Ł-tableaux with T_1X or F_1X as origin, i.e., $\mathcal{T}(T_1X)$ or $\mathcal{T}(F_1X)$ are called ŁC$_1$-tableaux and those with T_2X or F_2X as origin are called ŁC$_2$-tableaux. As can be seen from the rules, ŁC$_1$-tableaux involve only ŁC$_1$-swff and ŁC$_2$-tableaux involve only ŁC$_2$-swff.

We now give some definitions.

Definition 1 A branch of a tableau is said to be incompatible if it contains at least two swff as points which differ only in the sign appearing in them.

Definition 2 A branch of an Ł-tableau is said to be broken if the swff of the form F_1MX or T_1LX occurs as a point in it.

Definition 3 A branch is said to be closed if it is either incompatible or broken.

Definition 4 A branch is said to be open if it is not closed.

Definition 5 A branch $\tau(SX)$ of an Ł-tableau $\mathcal{T}(SX)$ is said to be complete if, for every α which occurs in τ, α_1 and α_2 both occur in τ, for every β

which occurs in τ, either β_1 or β_2 occurs in τ, and for every γ which occurs in τ, γ_1 and γ_3 both occurs in τ or γ_2 and γ_4 both occur in τ.

Definition 6 A tableau is said to be closed if every branch is closed.

Definition 7 A tableau is said to be open if at least one branch is open.

Definition 8 A tableau is said to be completed if every branch is either closed or complete.

4.2.1 *Provability and Rejection in* **Ƚ** We define four kinds of provable wff in **Ƚ**, in terms of syntactical closure properties of certain **Ƚ**-tableaux, as follows:

A wff X is **Ƚ**C_1-provable iff there exists a closed tableau $\mathcal{T}(F_1X)$, and we write **Ƚ**$C_1 \vdash X$ if such a tableau exists otherwise we write **Ƚ**$C_1 \dashv X$ and say X is **Ƚ**C_1-rejectable. Analogously we define X is **Ƚ**C_2-provable (and X is **Ƚ**C_2-rejectable) by putting C_2 for C_1 and F_2 for F_1 in the above.

A wff X is **Ƚ**C_1C_2-provable iff there exist closed tableaux $\mathcal{T}(F_1X)$ and $\mathcal{T}(F_2X)$, and we write **Ƚ**$C_1C_2 \vdash X$ if such tableaux exist, otherwise we write **Ƚ**$C_1C_2 \dashv X$ and say X is **Ƚ**C_1C_2-rejectable. A wff X is **Ƚ**C_1/C_2-provable iff there exists a closed tableau $\mathcal{T}(F_1X)$ or $\mathcal{T}(F_2X)$ and we write **Ƚ**$C_1/C_2 \vdash X$ if such a tableau exists, otherwise we write **Ƚ**$C_1/C_2 \dashv X$ and say X is **Ƚ**C_1/C_2-rejectable. Associated with these four kinds of provable (and rejectable) wff we can introduce four sub-systems **Ƚ**$_i$ ($i = 1, 2, 3, 4$) where we define $\vdash_{\text{Ƚ}_i} X$ ($i = 1, 2, 3, 4$) by **Ƚ**$C_1 \vdash X$, **Ƚ**$C_2 \vdash X$, **Ƚ**$C_1C_2 \vdash X$ and **Ƚ**$C_1/C_2 \vdash X$ respectively. We can also introduce $\dashv_{\text{Ƚ}_i}$ analogously.

4.3 *Definitions of tableaux and provable sentences in* **F** For **W**, **D**, **S** and **E** we can introduce rules analogous to Rules 1 - 26 for **Ƚ**. One example, for **E**-tableaux, considering the functors 'M' and 'L', instead of 3 and 4 for **Ƚ**, we would have

$$\frac{T_1MX}{T_1X \mid F_1X \mid I_1X} \qquad \text{and} \qquad \frac{F_1LX}{T_1X \mid F_1X \mid I_1X}$$

and instead of 15 and 18 for **Ƚ**, we would have

$$\frac{T_2MX}{T_2X \mid I_2X} \qquad \text{and} \qquad \frac{F_2LX}{F_2X \mid I_2X}$$

Also for the functor 'T' we would have the rules:

$$\frac{I_1TX}{T_1X \mid F_1X \mid I_1X}, \qquad \frac{T_2TX}{T_2X} \qquad \text{and} \qquad \frac{F_2TX}{F_2X \mid I_2X}$$

A wff X in **E** is **E**C_1-provable iff there exist closed tableaux $\mathcal{T}(F_1X)$ and $\mathcal{T}(I_1X)$. If such tableaux exist we write **E**$C_1 \vdash X$, otherwise we say X is **E**C_1-rejectable and we write **E**$C_1 \dashv X$. **E**$C_2 \vdash$, **E**$C_1C_2 \vdash$ and **E**$C_1/C_2 \vdash$ are defined analogously. In general for **F** we can give rules and definitions for **F**-tableaux with modifications to Definitions 2 and 5 in **4.2**.

For **W**, **S**, **D**, **E** we require in addition to the Rules A, B, C of **4.2** for **Ƚ**, the following two rules:

$$\text{Rule D:}\quad \begin{array}{c|c|c} \multicolumn{3}{c}{\delta} \\ \hline \delta_1 & \delta_2 & \delta_3 \\ \delta_4 & \delta_5 & \delta_6 \end{array} \qquad \text{Rule E:}\quad \begin{array}{c|c|c|c} \multicolumn{4}{c}{\epsilon} \\ \hline \epsilon_1 & \epsilon_2 & \epsilon_3 & \epsilon_4 \\ \epsilon_5 & \epsilon_6 & \epsilon_7 & \epsilon_8 \end{array}$$

where δ, ϵ, δ_i $(i = 1, 2, \ldots, 6)$, ϵ_i $(i = 1, 2, \ldots, 8)$ are swff of certain kinds. We also have properties analogous to P1-P6 of **5.1** holding for δ and ϵ. We can introduce four sub-systems $\mathbf{F}_i$, associated with $\mathbf{F}_i$-provability and $\mathbf{F}_i$-rejection ($i = 1, 2, 3, 4$).

5 *Semantical investigations*

5.1 *Semantical consistency and completeness proofs for* Ł*Ł We note the following properties hold for the unifying notation α, β, γ (*cf.* properties for α, β in [6], pp. 20-22). Under any ŁC_1-interpretation:

P1 For any ŁC_1-swff of type α, α is C_1-true iff α_1 and α_2 are both C_1-true.
P2 For any ŁC_1-swff of type β, β is C_1-true iff β_1 or β_2 is C_1-true.
P3 For any ŁC_1-swff of type γ, γ is C_1-true iff γ_1 and γ_3 are both C_1-true or γ_2 and γ_4 are both C_1-true.

Analogous properties labelled P4, P5, P6 respectively hold for the different types of ŁC_2-swff, under any ŁC_2-interpretation, by putting C_2 for C_1 in the above.

Consider an ŁC_1-tableau $\mathcal{T}(S_1X)$ and an ŁC_1-interpretation $\mathcal{I}_1$ whose domain includes at least all the propositional variables which occur in any point of $\mathcal{T}(S_1X)$. If $\tau(S_1X)$ is a branch of $\mathcal{T}(S_1X)$ then we say $\tau(S_1X)$ is ŁC_1-satisfiable under $\mathcal{I}_1$ if every ŁC_1-swff which occurs as a point in $\tau(S_1X)$ is C_1-true under $\mathcal{I}_1$. Also, an ŁC_1-tableau $\mathcal{T}(S_1X)$ is ŁC_1-satisfiable under $\mathcal{I}_1$, iff at least one branch of $\mathcal{T}(S_1X)$ is ŁC_1-satisfiable. Also ŁC_2-satisfiability of ŁC_2-tableaux and branches are defined as above with C_2 for C_1, S_2 for S_1 and $\mathcal{I}_2$ for $\mathcal{I}_1$.

To facilitate proofs and definitions by induction, we define the degree of a wff of Ł as the number of occurrences of the logical connectives, thus:

1. A propositional variable is of degree 0.
2. If X is of degree n, then NX, MX and LX are of degree $n + 1$.
3. If X, Y are of degree n_1, n_2 respectively then CXY, KXY, AXY and EXY are of degree $n_1 + n_2 + 1$.
4. The degree of a swff SX is the same as the degree of X.

Theorem 1 (Semantical Consistency for Ł*Ł) *If* $\vdash_{\text{Ł}_i} X$ *then* X *is an* Ł$_i$-*tautology* ($i = 1, 2, 3, 4$).

Proof: Case $i = 1$. Let $\mathcal{T}_1(S_1X)$ be an ŁC_1-tableau, and consider $\mathcal{T}_2(S_1X)$ the immediate extension of $\mathcal{T}_1(S_1X)$. We show that $\mathcal{T}_2(S_1X)$ is ŁC_1-satisfiable under every ŁC_1-interpretation $\mathcal{I}_1$ for which $\mathcal{T}_1(S_1X)$ is ŁC_1-satisfiable, as follows:

If $\mathcal{T}_1(S_1X)$ is ŁC_1-satisfiable under an ŁC_1-interpretation $\mathcal{I}_1$, then it contains a branch $\tau(S_1X)$ say, which is ŁC_1-satisfiable under $\mathcal{I}_1$. Now $\mathcal{T}_2(S_1X)$ was obtained from $\mathcal{T}_1(S_1X)$ by one application of one of the operations (A), (B),or (C), applied to some branch $\tau_1(\mathrm{S}_1X)$ of $\mathcal{T}_1(\mathrm{S}_1X)$. If now τ_1 is distinct from τ, then τ is still a branch of $\mathcal{T}_2(\mathrm{S}_1X)$ and hence $\mathcal{T}_2(\mathrm{S}_1X)$ contains a branch τ which is ŁC_1-satisfiable under $\mathcal{I}_1$.

Alternatively, suppose τ_1 is identical with τ, then if τ was extended by (A), then some α appears in τ and τ has been extended to $\tau - \alpha_1$ or $\tau - \alpha_2$. Hence either $\tau - \alpha_1$ or $\tau - \alpha_2$ is the extended branch τ_1 of $\mathcal{T}_2(\mathrm{S}_1X)$. Also since τ is ŁC_1-satisfiable under $\mathcal{I}_1$, α is C_1-true under $\mathcal{I}_1$ and hence by P1 (*cf.* **5.1**), α_1 and α_2 are both C_1-true under $\mathcal{I}_1$. Thus $\mathcal{T}_2(\mathrm{S}_1X)$ contains an ŁC_1-satisfiable branch $\tau - \alpha_1$ or $\tau - \alpha_2$ under $\mathcal{I}_1$. Similarly, if τ was extended by (B) or (C) we can show using properties P2 or P3 above that $\mathcal{T}_2(\mathrm{S}_1X)$ contains a branch ŁC_1-satisfiable under $\mathcal{I}_1$. Thus, the immediate extension $\mathcal{T}_2(\mathrm{S}_1X)$ of $\mathcal{T}_1(\mathrm{S}_1X)$ contains a branch which is ŁC_1-satisfiable.

It follows by mathematical induction that for any ŁC_1-tableau $\mathcal{T}(\mathrm{S}_1X)$, if the origin S_1X is C_1-true under a given ŁC_1-interpretation $\mathcal{I}_1$, of the tableau $\mathcal{T}_1(\mathrm{S}_1X)$, then $\mathcal{T}(\mathrm{S}_1X)$ must be ŁC_1-satisfiable under $\mathcal{I}_1$.

Suppose now $\vdash_{Ł_i} X$, i.e., Ł$\mathrm{C}_1 \vdash X$. Then by definition there exists a closed tableau $\mathcal{T}(F_1X)$. But $\mathcal{T}(F_1X)$ cannot be ŁC_1-satisfiable under $\mathcal{I}_1$, since each branch is either incompatible or broken (see Tables VIII and IX). Hence the origin F_1X must be C_1-false under $\mathcal{I}_1$. Hence X must be C_1-true under every $\mathcal{I}_1$, i.e., X is an ŁC_1-tautology.

Case $\mathbf{i} = \mathbf{2}$ is similar. Cases $\mathbf{i} = \mathbf{3}, \mathbf{4}$, follow directly.

Lemma *Every complete open branch of any* ŁC_j*-tableau is* ŁC_j*-satisfiable* ($j = 1, 2$).

Proof: Case $j = 1$. Let $\tau(\mathrm{S}_1X)$ be a complete open branch of a tableau $\mathcal{T}(\mathrm{S}_1X)$. Let S_τ be the set of all ŁC_1-swff which occur as points in the branch τ. We wish to find an ŁC_1-interpretation in which every swff $\epsilon\ S_\tau$ is C_1-true. Assign to each variable p which occurs in at least one element of S_τ a C_1-truth-value as follows:

(i) If $T_1p \in S_\tau$, assign p the value C_1-true.
(ii) If $F_1p \in S_\tau$, assign p the value C_1-false.
(iii) If neither T_1p nor F_1p is an element of S_τ, then assign p the value C_1-true.

Now since τ is open, then for no variable p do T_1p and F_1p both occur in τ. Thus every element of S_τ of degree 0 is C_1-true under this ŁC_1-interpretation. Now consider an element S_1X of S_τ of degree greater than 0, and suppose all elements of S_τ of lower degree than S_1X are C_1-true. We wish to show that S_1X is then C_1-true.

Since S_1X is of degree greater than 0, it must be either some α or β or γ, for, since τ is open, then no swff of the forms F_1MX or T_1LX can occur in τ. We consider these cases in turn. Suppose S_1X is an ŁC_1-swff of type α. Then α_1 and α_2 must both be in S_τ, since τ is complete. But α_1 and α_2 are of lower degree than α. Hence by inductive hypothesis α_1 and α_2 are both C_1-true. This implies by P1 that α is C_1-true, i.e., S_1X is C_1-true. Similarly, if S_1X is of type β or γ, using P2, P3 we get S_1X is C_1-true. Therefore by mathematical induction, the branch $\tau(\mathrm{S}_1X)$ is ŁC_1-satisfiable, and hence case $j = 1$ is proved. Case $j = 2$ is similar to case $j = 1$.

We use this lemma to prove:

Theorem 2 (Semantical Completeness Theorem for Ł*Ł) *If X is an* Ł_i-*tautology then* $\vdash_{\text{Ł}_i} X$ ($i = 1, 2, 3, 4$).

Proof: Case $i = 1$. Suppose $\mathcal{T}(F_1X)$ is a completed tableau. If $\mathcal{T}(F_1X)$ is open then there exists a complete open branch $\tau(F_1X)$ say, which is $\text{Ł}C_1$-satisfiable under some $\text{Ł}C_1$-interpretation $\mathcal{I}_1$. In particular, F_1X is C_1-true under $\mathcal{I}_1$, hence X is C_1-false under $\mathcal{I}_1$. Therefore, if X is an $\text{Ł}C_1$-tautology then $\mathcal{T}(F_1X)$ must be closed. If now X is an $\text{Ł}C_1$-tautology then every completed tableau $\mathcal{T}(F_1X)$ is closed. In particular, there exists one closed tableau $\mathcal{T}(F_1X)$. Hence $\vdash_{\text{Ł}_1} X$. Case $i = 2$ is analogous. Cases $i = 3, 4$ then follow as simple corollaries.

We can use Theorems 1 and 2 to get:

Theorem 3 ${}_{\text{Ł}_i}\dashv X$ *iff X is not an* Ł_i-*tautology* ($i = 1, 2, 3, 4$).

This follows since for any wff X, and any fixed i, $\vdash_{\text{Ł}_i} X$ or ${}_{\text{Ł}_i}\dashv X$ and X is an Ł_i-tautology or X is not an Ł_i-tautology.

5.2 We state without proofs the theorems for F*F.

Theorem 1 (Semantical Consistency Theorem for F*F)

(i) *If* $\vdash_{F_i} X$ *then X is an* F_i-*tautology.*
(ii) *If* ${}_{F_i}\dashv X$ *then X is not an* F_i-*tautology.*

Theorem 2 (Semantical Completeness Theorem for F*F)

(i) *If X is an* F_i-*tautology, then* $\vdash_{F_i} X$.
(ii) *If X is not an* F_i-*tautology, then* ${}_{F_i}\dashv X$.

where, for both Theorems 1 and 2, $F = \text{Ł}, W, S, D, E$, $i = 1, 2, 3, 4$, *X is a wff of* F *and* F_i-($i = 1, 2, 3, 4$) *denotes* FC_1-, FC_2-, FC_1C_2- *and* FC_1/C_2 *respectively.*

The proofs of Theorems 1 and 2 follow along the same lines as Theorems 1, 2, 3 of **5.1** for Ł*Ł.

6 *Some sample proofs in* Ł In general to establish $\vdash_{\text{Ł}_1} X$, we are required to construct one closed $\text{Ł}C_1$-tableau $\mathcal{T}(F_1X)$, and to show ${}_{\text{Ł}_1}\dashv X$ we need only construct a complete open branch $\tau(F_1X)$. Similar remarks apply for showing $\vdash_{\text{Ł}_2} X$ (or ${}_{\text{Ł}_2}\dashv X$). Two examples are shown in Figures 1 and 2 (see p. 204).

In Figure 1, $\mathcal{T}(F_11)$ indicates that an $\text{Ł}C_1$-tableau is being constructed with the wff F_11 as origin, where '1' is used to number the wff on RHS of $\vdash_{\text{Ł}}$. Also, in Figure 2, $\tau(F_12)$ indicates that a branch construction follows with F_12 as origin. It is convenient to present the tableaux and branches as linear sequences of swff, with the rules (see **4.2**) used in the construction indicated. To do this we associate with each swff in the tableaux (or branches) a number of the form $n_1n_2n_3 \ldots n_s$ which defines its location in the tableau. Here, n_1 denotes the level of the point in the tableau. A sole successor of a point numbered $n_1n_2n_3 \ldots n_s$ is denoted by $(n_1 + 1)\, n_3 \ldots n_s$. If the point $n_1n_2n_3 \ldots n_s$ is a branch point, its first successor is denoted by

$(n_1 + 1)n_3 \ldots n_s1$, and second successor by $(n_1 + 1)n_3 \ldots n_s2$. Also, for a point $n_1n_2n_3 \ldots n_s$ where $n_1 \geqslant 10$, we write each $n_i (i = 1, 2, \ldots, s)$ for which $n_i \geqslant 10$ as (n_i). We illustrate the use of this notation in Figure 3 for $\mathcal{T}(F_21)$ from Figure 1 (see section **8**).

7 *Some properties of the sub-systems* $\mathsf{F}_i(\mathsf{F} = \text{Ł}, \mathsf{W}, \mathsf{S}, \mathsf{D}, \mathsf{E}, i = 1, 2, 3, 4)$.

(i) In the sub-systems F_1, we have $\vdash_{\mathsf{F}_1} Mp$ and $\vdash_{\mathsf{F}_1} NLp$, which violate two laws of Łukasiewicz's basic modal logic, *cf.* [1] and [2]. Also in F_1, no theorem of the form LX occurs. In fact $\dashv_{\mathsf{F}_1} LX$.

(ii) In the sub-systems F_2, there are theorems of the form LX and the rule of necessitation holds, i.e., *if* $\vdash_{\mathsf{F}_2} X$ *then* $\vdash_{\mathsf{F}_2} LX$. Also, in Ł_2, W_2, S_2, we have $\vdash CMpp$ and $\vdash CpLp$ which again violate some of the laws of basic modal logic, *cf.* [1], [2].

(iii) For the sub-systems F_3, Halldén's incompleteness property holds, e.g., take $X = MKpNp$, $Y = LCqq$, then we have $\vdash_{\mathsf{F}_3} AXY$ but $\dashv_{\mathsf{F}_3} X$ and $\dashv_{\mathsf{F}_3} Y$. Hughes and Cresswell[5] suggest that for the Lewis Systems S1, S2, S3, this property is paradoxical. The semantics F^* in section **3** above, indicate why this property arises in F_3. It arises because F_3-tautologies are derived tautologies and not fundamental ones. I will refer to this formal incompleteness in F_3 as *A*-incompleteness.

The sub-system Ł_3 corresponds to the Ł-modal system in the sense that X is a theorem of the Ł-modal system iff $\vdash_{\text{Ł}_3} X$. This follows from the isomorphism between the truth-tables Table I in **3.3**, Table $\mathcal{M}\mathit{9}$ of [1] already referred to, and the consistency and completeness theorems for $\text{Ł}^*\text{Ł}$. Hence the semantics Ł_3^* provides semantics for the Ł-modal system, but with this consequence that the Ł-modal system should be viewed, not as a full system on its own right, but rather as a sub-system.

(iv) The sub-systems F_4 are formally inconsistent, in this sense:

A system Σ is said to be formally consistent iff there are no wff X s.t. $\vdash_{\Sigma} X$ and $\vdash_{\Sigma} NX$ (*cf.* Rosser and Turquette's[6] definition of *N*-consistency). If such a wff X exists then we say Σ is formally inconsistent.

Take $X = LCpp$, then $\vdash_{\mathsf{F}_4} LCpp$ and $\vdash_{\mathsf{F}_4} NLCpp$. Hence F_4 are formally inconsistent. This property can be thought of as an incompleteness property related to the Halldén property, but with respect to the functor '*K*' rather than '*A*'. That is, in F_4 there are wff of the form KXY such that $\vdash_{\mathsf{F}_4} X$ and $\vdash_{\mathsf{F}_4} Y$, but $\dashv_{\mathsf{F}_4} KXY$. We will call this property *K*-incompleteness. As in (iii) above, considering the semantics F_4^*, the properties of *K*-incompleteness and formal inconsistency arise because an F_4-tautology is a derived tautology, and not a fundamental one.

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[5] G. E. Hughes and M. J. Cresswell, *An Introduction to Modal Logic,* Methuen, London (1968), p. 268.

[6] J. B. Rosser and A. R. Turquette, *Many-Valued Logics,* North-Holland, Amsterdam (1952).

(v) It seems reasonable to conjecture that each of the sub-systems $\mathbf{F}_i$ (i = 1, 2, 3, 4) is axiomatisable. However, for the full systems **F**, the meaning of axiomatisation needs to be reviewed. Considering the axiomatisation of $\mathbf{F}_4$, one feature of interest is that modus ponens would not hold. E.g.: Take $X = MNCpp$, $Y = NCpp$. Then we have $\vdash_{\mathbf{F_4}} CMNCppNCpp$ and $\vdash_{\mathbf{F_4}} MCNpp$ but $\vdash_{\mathbf{F_4}} NCpp$. Consequently, these logics **F*F** indicate a possible limitation in the purely axiomatic approach to formal logic.

8 *Philosophical applications* Łukasiewicz's Ł-modal system [1], [2], grew out of a close reading and study of Aristotle's[7] work on modalities. These formal logics **F*F** may be of help in the difficult problem of providing a coherent interpretation and account of Aristotle's treatment of modalities. Also, some of these formal logics may be suitable as logical groundings for a certain class of metaphysics.

I would like to thank my supervisor Dr. R. M. Dicker for his help in the writing of this paper.

Figure 1

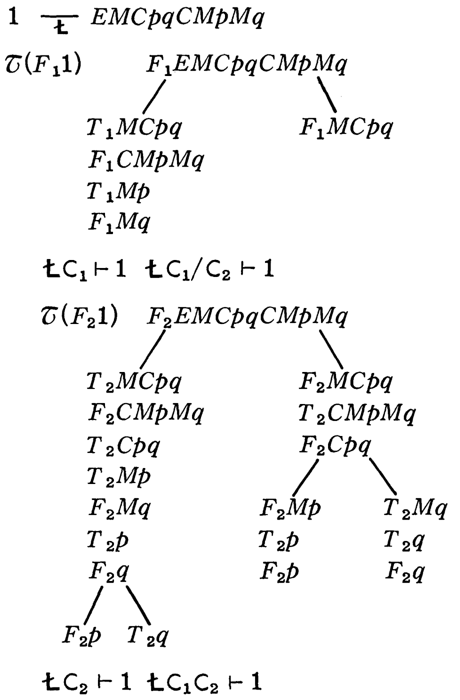
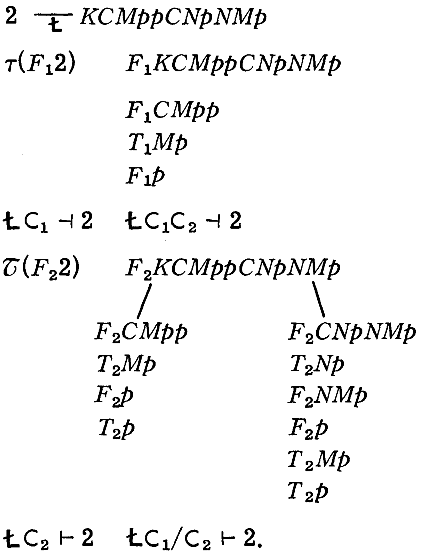


Figure 2



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[7] Aristotle, *Prior Analytics*, Book I, Chapters 3 and 8-22, and *De Interpretatione*, Chapter 9.

Figure 3

$\mathcal{T}(F_2 1)$

$F_2 EMCpqCMpMq$ 1

$T_2 MCpq$	21	26	1
$F_2 MCpq$	22	26	1
$F_2 CMpMq$	31	26	1
$T_2 CMpMq$	32	26	1
$T_2 Cpq$	41	26	21
$F_2 Cpq$	42	16	22
$T_2 Mp$	51	20	31
$F_2 Mp$	521	19	32
$T_2 Mq$	522	19	32
$F_2 Mq$	61	20	31
$T_2 p$	621	20	42
$T_2 q$	622	15	522
$T_2 p$	71	15	51
$F_2 p$	721	16	521
$F_2 q$	722	20	42
$F_2 q$	81	16	61
$F_2 p$	911	19	41
$T_2 q$	912	19	41

Ł$C_2 \vdash 2$

Note: In the above proof 521 is the number of the swff $F_2 Mp$. It is the first successor of $F_2 Cpq$, i.e., 42, and arises from $T_2 CMpMq$, i.e., 32, by applying Rule 19 of **4.2**.

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