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Three Existence Principles in a Modal Calculus Without Descriptions Contained in A. Bressan's MC^ν

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1 Introduction In this paper an axiomatization is provided for an interpreted modal calculus similar to MC^ν. Several variants of this axiomatization—which I have striven to make as concise as possible—are also proposed.*

Essentially, we extend the logical calculus without descriptions that was defined and named K^λ in [8] by adding to it the axiom of choice. Recall that K^λ is equivalent to MC^ν deprived of four axioms (namely, AS12.20, AS12.23, AS25.1, AS45.1 in [1], pp. 46, 48, 95, 184).

The existence principle for functions (AS12.17 in [1], p. 45; AS.3.16 in [8]) does not appear among the axioms of our calculus, which are no stronger than those of MC^ν. Nonetheless, our short version of the axiom of choice makes it possible to infer that principle, along with the most customary version of the axiom of choice. More interestingly, it allows us to simplify considerably the axiom that was introduced in [8] in order to eliminate descriptions from MC^ν (named, in its new formulation of this paper, the "existence principle for descripta").

The axiom about the existence of predicates (AS12.19 in [1], p. 46) has also been simplified in this paper and the new version is shown to be as strong as the original one.

A formal analogue for the concept of elementary possible case (or "Γ-case") is defined within our calculus. Although we are making no assumptions about the plurality of Γ-cases, the new definition seems to be as powerful as

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the one first stated by Bressan ([1], pp. 197-202). In fact our calculus is compatible with the nonexistence of a contingent proposition (AS12.23 in [1], p. 48).

Another axiom of MC^ν has not been taken into account, concerning the number of individuals in different Γ-cases ([1], p. 95, AS25.1). From a semantical standpoint, we are interpreting our calculus along the directions given in [1] (pp. 96-97). I would make a conjecture similar to that made in [1] (pp. 271-272) in connection with the calculus MC'^ν. I think that our calculus has a property of relative completeness with respect to the extensional semantical system hinted at above, similar to that enjoyed by MC^ν (see [1], pp. 269-270).

The modal language without descriptions considered in this paper differs from the one considered in [8] because implication is adopted as the only primitive connective. Using implication and a logically false statement, all other connectives can be defined ([5], p. 78). Our choice of implication as the only primitive connective makes a more concise axiomatization of the sentential calculus available.

Finally, the axioms about universal and modal quantification have been replaced by axioms that are independent of recursive notions such as "free occurrence of a variable within a wff", "modally closed wff", and "term free for a variable in a wff".

*2 **Semantical preliminaries*** In this section we define a modal language whose terms are partitioned into infinitely many distinct types. Some useful shorthand notations are introduced by means of metalinguistic conventions to stand for terms or well-formed formulas (wffs). The following concepts are defined: (intensional) designatum of a term or a wff, logical truth (or falsity) of a wff. We single out a logically false wff and a list of logically true wff schemas. Although we are adopting $\supset$ as a primitive connective instead of $\sim$ and $\wedge$, our language can be identified with the largest sublanguage of Bressan's ML^ν that is devoid of descriptions ([1], pp. 10-12).

As preliminaries to the notions of designatum and logical truth, we introduce the following concepts: elementary possible case (or "Γ-case"), extension, quasi-intension (*QI*), *cv*-valuation. Our definitions do not match those given in [1] (pp. 18-21) in three respects:

(a) We are *not* assuming that there are at least two Γ-cases
(b) We are *not* assuming that the set of extensions of a particular type is the same in every Γ-case (in this respect our semantical analysis is similar to that outlined in [1], pp. 96-97)
(c) We uniformly define the *QI*'s of any particular type as being functions defined on the set of Γ-cases.

Throughout this paper we use n as a metavariable running over positive integers, while ν stands for a *fixed* positive integer.

The set τ^ν of the *term types* is defined recursively as follows:

Definition 2.1 (a) $1, \ldots, \nu$ belong to τ^ν and are called *individual types*.

(b) If $t_i \in \tau^\nu$ for $i = 0, 1, \ldots, n$ then the following two $(n + 1)$-tuples both belong to τ^ν: $\langle t_1, \ldots, t_n, 0\rangle$, which is denoted by $(t_1, \ldots, t_n)$ and is called a *relator type*; and $\langle t_1, \ldots, t_n, t_0 \rangle$, which is denoted by $(t_1, \ldots, t_n : t_0)$ and is called a *functor type.*

Definition 2.2 The *primitive symbols* of our language are: the *variables* v_{tn} with $t \in \tau^\nu$; the *constants* c_{tn} with $t \in \tau^\nu$; the *symbols for universal and modal quantification* $\forall$ and N; the *connective for implication* $\supset$; the *identity symbol* $=$; the *parentheses* and the *comma.*

Notation 2.1 (a) We regard ( ), [ ], and {} as three typographical variants of the same pair of primitive symbols, called parentheses.

(b) Usually variables are represented by the metalinguistic symbols x, x_n, y, z, f, g, F, and G. Occasionally other Latin or Greek letters are used for the same purpose. The signs X and Y stand for finite nonempty strings of distinct variables separated by commas.

(c) The sign U stands for either N or an expression of the form $(\forall x)$ (which is called, as usual, a universal quantifier); Q stands for a finite, possibly empty, string of universal quantifiers and N's.

Now we want to define the set T_t^ν of the *terms having the type t,* for every $t \in \tau^\nu$. All the sets T_t^ν are simultaneously characterized by the following recursive definition:

Definition 2.3 For every $t \in \tau^\nu$:

(a) $v_{tn} \in T_t^\nu$ and $c_{tn} \in T_t^\nu$
(b) if $t_i \in \tau^\nu$ and $\Delta_i \in T_{t_i}^\nu$ for $i = 1, \ldots, n$, and furthermore $\Delta \in T_{(t_1,\ldots,t_n:t)}^\nu$, then $\Delta(\Delta_1, \ldots, \Delta_n) \in T_t^\nu$.

Definition 2.4 The set of *wffs*, or *expressions of the type* 0, is the smallest set W such that:

(a) if $\Delta_0 \in T_t^\nu$ and $\Delta_1 \in T_t^\nu$, where $t \in \tau^\nu$, then $\Delta_0 = \Delta_1 \in W$
(b) if $t_i \in \tau^\nu$ and $\Delta_i \in T_{t_i}^\nu$ for $i = 1, \ldots, n$, and $\Delta \in T_{(t_1,\ldots,t_n)}^\nu$, then $\Delta(\Delta_1, \ldots, \Delta_n) \in W$
(c) if $p \in W$ and $q \in W$, and U is a universal quantifier or N, then $(p \supset q) \in W$ and $Up \in W$.

Notation 2.2 (a) p, p_n, q, r, and s stand for wffs

(b) Unless it occurs as part of a longer wff, $(p \supset q)$ is often abbreviated $p \supset q$.

(c) By means of the sign $=_{df}$, we shall introduce into our language shorthand notations for wffs. For instance we set

$(\forall x_1, \ldots, x_n)p =_{df} (\forall x_1) \ldots (\forall x_n)p,$
$(p \supset^{\cap} q) =_{df} N(p \supset q)$, and
$\Delta_0 =^{\cap} \Delta_1 =_{df} N\Delta_0 = \Delta_1.$

The left-hand side of any metadefinition of this kind can replace an occurrence of the right-hand side within a wff.

Convention 2.1 (a) When two or more of the signs x, y, z, f, g, F, and G occur within the same wff schema, they are supposed to stand for pairwise distinct variables. Furthermore, f and g (respectively F and G) always stand for functor (relator) variables.

(b) Whenever p and v occur within the same wff schema, v being one of the signs z, f, and F, we assume that the variable represented by v does not occur in p.

We assume that the reader is familiar with the notions of free (bound) occurrence of a variable within a wff and knows the meaning of the statement "the term Δ is free for x in p".

Definition 2.5 A wff p is said to be $(\forall x)$-*closed* (or just x-*closed*) if x has no free occurrences in p; N-*closed,* or *modally closed,* if it has the form Nq, or either of the two forms $(s \supset r)$ and $(\forall x)r$ where both s and r are N-closed.

Notation 2.3 (a) When for no variable y occurring in $x = \Delta$ does the quantifier $(\forall y)$ occur in p, then we denote by $p!^x_\Delta$ the result of replacing x with Δ in all of its occurrences in p.

(b) When Δ is free for x in p, then we denote by p^x_Δ the result of replacing x by Δ in all of its free occurrences in p.

In order to attach meanings to terms and wffs, we first consider a nonempty domain Γ (whose elements are called Γ-*cases*) and ν functions $\hat{D}_1, \ldots, \hat{D}_\nu$ from Γ to nonempty sets.

By the following recursive definition, the set QI^ν_t of the *quasi-intensions of type* t and the set $E^\nu_t(\gamma)$ of the *extensions of type* t *relative to* γ are simultaneously defined for $t \in \tau^\nu \cup \{0\}$ and $\gamma \in \Gamma$.

Definition 2.6 (a) $E^\nu_0(\gamma)$ is the set of truth values $\{0, 1\}$ (where 0 stands for "true" and 1 for "false"), for all $\gamma \in \Gamma$

(b) $E^\nu_t(\gamma) = \hat{D}_t(\gamma)$ for all $\gamma \in \Gamma$ and $t = 1, \ldots, \nu$

(c) QI^ν_t is the Cartesian product $\prod_{\gamma \in \Gamma} E^\nu_t(\gamma)$, for all $t \in \tau^\nu \cup \{0\}$

(d) if $t_1, \ldots, t_n \in \tau^\nu$, $t \in \tau^\nu \cup \{0\}$, and $\gamma \in \Gamma$, then $E^\nu_{\langle t_1,\ldots,t_n,t\rangle}(\gamma)$ is the class of all functions from $\prod_{i=1}^{n} QI^\nu_{t_i}$ into $E^\nu_t(\gamma)$.

Definition 2.7 Let $\mathscr{J}$ be the set of all variables and constants and $I_{c_{tn}} = I_{v_{tn}} = QI^\nu_t$. A member of $\prod_{s \in \mathscr{J}} I_s$ is called a *cv-valuation.* For every variable x, we denote by $\sim_x$ the equivalence relation that holds between two *cv*-valuations if and only if they differ at most in x.

In connection with a *cv*-valuation V each term or wff Δ will now be assigned a *quasi-intensional designatum,* i.e., a *QI* $\widetilde{des}_V(\Delta)$—in short $\tilde{\Delta}$—having the same type as Δ.

Definition 2.8 For every γ, $\tilde{\Delta}(\gamma)$ is defined by the following designation rules:

if Δ *is*	*then* $\tilde{\Delta}(\gamma)$ *is*
v_{tn} or c_{tn}	$V(\Delta)(\gamma)$
$\Delta_0(\Delta_1, \ldots, \Delta_n)$	$\tilde{\Delta}_0(\gamma)(\tilde{\Delta}_1, \ldots, \tilde{\Delta}_n)$
$\Delta_0 = \Delta_1$	0 if and only if $\tilde{\Delta}_0(\gamma) = \tilde{\Delta}_1(\gamma)$
$(p \supset q)$	$(1 - \tilde{p}(\gamma))\cdot \tilde{q}(\gamma)$
Np or $(\forall x)p$	$\max\limits_{\gamma\epsilon\Gamma} (\tilde{p}(\gamma))$ or $\max\limits_{V'\sim_x V} (\widetilde{des}_{V'}(p)(\gamma))$, respectively.

We say that p is *logically true* (respectively: *logically false*) if $\widetilde{des}_V(p)(\gamma) = 0$ ($\widetilde{des}_V(p)(\gamma) = 1$) for every choice of $\hat{D}_1, \ldots, \hat{D}_\nu$, Γ, V, and all $\gamma \epsilon \Gamma$.

After observing that $(\forall v_{(1)1}, v_{11})v_{(1)1}(v_{11})$ is logically false, we give the definition:

Definition 2.9

$$\begin{aligned}
\curlywedge &=_{df} (\forall v_{(1)1}, v_{11})v_{(1)1}(v_{11})\\
\sim p &=_{df} p \supset \curlywedge\\
(p_0 \wedge p_1 \wedge \ldots \wedge p_n) &=_{df} \sim(p_0 \supset (p_1 \supset (\ldots (p_n \supset \curlywedge) \ldots)))\\
(p \equiv q) &=_{df} [(p \supset q) \wedge (q \supset p)]\\
(p \equiv^{\cap} q) &=_{df} N(p \equiv q)\\
\Diamond p &=_{df} \sim N \sim p\\
(\exists X)p &=_{df} \sim(\forall X) \sim p\\
(\exists_1 x)p &=_{df} [(\exists x)p \wedge (\exists z)(\forall x)(p \supset x = z)]\\
(\exists_1^{\cap} x)p &=_{df} [(\exists x)p \wedge (\exists z)(\forall x)(p \supset x =^{\cap} z)]
\end{aligned}$$

where Convention 2.1 applies.

The proof of the following proposition is left to the reader:

Proposition 2.1 *Every wff which is an instance of any of the following schemas is logically true* (see Notations 1-3 and Convention 1).

A1	$Q\{[(p \supset q) \supset r] \supset [(r \supset p) \supset (s \supset p)]\}$
A2	$Q(\curlywedge \supset p)$
A3	$Q[U(p \supset q) \supset (Up \supset Uq)]$
A4	$Q(p \supset (\forall z)p)$
A5	$Q[(\forall x)p \supset p!^x_\Delta]$
A6	$Q(Np \supset p)$
A7	$Q[(Np \supset Nq) \supset N(Np \supset Nq)]$
A8	$Q[(\forall x)Np \supset N(\forall x)p]$
A9	$N(\forall x, y)\{x =^{\cap} y \equiv (\forall F)[F(x) \supset F(y)]\}$
A10	$N(\forall x, y, z)[x = y \supset (y = z \supset z = x)]$
A11	$N(\forall F, G)\{F = G \equiv (\forall X)[F(X) \equiv G(X)]\}$
A12	$N(\forall f, g)[f = g \equiv (\forall X)f(X) = g(X)]$
A13	$Q(\exists z)N[(\exists y)p \supset (\exists y)(z = y \wedge p)]$
A14	$Q(\exists F)(\forall X)\{[F(X) \supset p] \wedge [p \supset NF(X)]\}$
A15	$N(\forall F)(\exists f)(\forall X)[(\exists y)F(X, y) \supset F(X, f(X))]$

3 Syntactical preliminaries In this section we start considering the modal calculus whose only inference rule is modus ponens (or *MP*) and whose axioms are all the instances of the schemas A1 to A15 listed in Section 2. We concentrate, for now, on the first set of Axioms A1 to A12. It turns out readily that every theorem that can be inferred by exclusive use of these axiom schemas can also be derived in Bressan's calculus MC^ν by use of AS12.1-AS12.15 (see [1], pp. 42-44). Less trivially, the axiom schemas AS12.1-AS12.15 of MC^ν are deducible in our calculus. The following, in particular, are theorems (they are proven as Proposition 3.1(i), (vi), (vii), and (viii), respectively, and in the discussion preceding Proposition 3.2):

(a) every instance of a tautology of ordinary (extensional) sentential calculus (hence AS12.1-3 of MC^ν, which are tautological schemas, follow)

(b) the universal specification schema (AS12.8 of MC^ν), which includes our A5 as a subschema

(c) every wff of the form $p \supset Up$, where p is U-closed (hence AS12.6-7 of MC^ν, which are obtained from this schema by replacing $(\forall x)$ and N for U, follow)

(d) the wff schemas expressing the usual properties of identity: reflexivity, symmetry, transitivity, and interchangeability of terms which are strictly identical (AS12.10-13 of MC^ν).

(Those among AS12.1-15 of MC^ν which are not mentioned in (a)-(d) above occur also among A1-A12.)

These results will make all of the theory developed in [1], Nn. 29-33, available. In particular, we may use the deduction, generalization, equivalence, and replacement theorems; and rules G and C can be employed to shorten deductions.

Definition 3.1 The notation $p_1, \ldots, p_m \vdash q$, where $m \geqslant 0$, indicates that q belongs to the smallest set, $\mathcal{C}$, of wffs that contains the axioms and the p_i's and satisfies the *MP*-closure property: p, $p \supset q \in \mathcal{C}$ implies $q \in \mathcal{C}$. If $\vdash q$ holds, q is said to be a *theorem.* The notation $p \vdash\dashv q$ indicates that both $\vdash p \supset q$ and $\vdash q \supset p$ hold.

Depending on the class of axioms one chooses, the definition above characterizes various calculi, i.e., various interpretations for the relation, denoted by $\vdash$, of deducibility of q from $p_1, \ldots, p_m$. We have chosen A1 to A15 to be our axioms, but we are often interested in considering different sets of axioms. In particular, when we say that a new wff schema S_2 is equivalent to one, say S_1, of the old schemas A1 to A15 we mean that $\vdash$ is invariant under the replacement of S_2 for S_1.

Proposition 3.1 *Consider a calculus (whose only inference rule is MP) where* $\vdash$Ai *holds for* $i = 1, \ldots, 8$ *and* $\vdash Qp$ *holds whenever* p *is an axiom. Then:*

(i) *every substitution instance of a tautology is a theorem*
(ii) *if* $\vdash q$, *then* $\vdash Qq$; *if* $\vdash r \supset s$, *then* $\vdash Qr \supset Qs$
(iii) $\vdash\dashv$ *is an equivalence relation on the set of wffs*
(iv) *if* s *is obtained from* r *by replacing* p *with* q *in one or more of its occurrences in* r, *then:* $p \vdash\dashv q$ *implies* $r \vdash\dashv s$, *hence* $\vdash r$ *implies* $\vdash s$
(v) *if neither of* $(\forall x)$ *and* z *occurs in* p, *then* $(\forall x)p \vdash\dashv (\forall z)p_z^x$

(vi) *if Δ is free for x in p, then* $\vdash(\forall x)p \supset p^x_\Delta$
(vii) *if p is x-closed, then* $p \vdash\dashv (\forall x)p$
(viii) *if p is N-closed, then* $p \vdash\dashv Np$.

Proof: When Q is empty, A1 and A2 constitute a well-known complete axiomatization for an ordinary extensional sentential calculus based on the connective $\supset$ and the false constant $\mathscr{f}$ (see [5], p. 140). Thus (i) holds. The clauses (ii), (iii), and (iv) are proven by ordinary methods. The main steps of a proof that $\vdash(\forall x)p \supset (\forall z)p!^x_z$ are:

(1) $(\forall z)((\forall x)p \supset p!^x_z)$ [A5]
(2) $(\forall z, x)p \supset (\forall z)p!^x_z$ [(1), A3]
(3) $(\forall x)p \supset (\forall z, x)p$ [A4]
(4) $(\forall x)p \supset (\forall z)p!^x_z$ [(2), (3), (i)]

The converse proof is symmetrical: since $p!^x_z!^z_x$ is p, thus (v) is proven. In order to prove (vi) and (vii), let ℓ be the number of occurrences of $\forall$, N, and $\supset$ in p. We define $[p{:}q]$ recursively for every q: if $\ell = 0$, then $[p{:}q] =_{df} p$; if p is Nr, then $[p{:}q] =_{df} N[r{:}q]$; if p is $r \supset s$, then $[p{:}q] =_{df} [r{:}q] \supset [s{:}q]$; if p is $(\forall v)r$, then $[p{:}q] =_{df} (\forall z)[r{:}(\forall v)q]^v_z$, where z is the first variable having the same type as v that does not occur in either $[r{:}(\forall v)q]$ or q. By easy induction on ℓ, one proves that:

(a) p and $[p{:}q]$ have the same free variables and for no variable y that occurs in q, does $(\forall y)$ occur in $[p{:}q]$
(b) if Δ is free for x in p and for no y that occurs in $x = \Delta$, does $(\forall y)$ occur in $[p{:}q]$, then $p^x_\Delta \vdash\dashv [p{:}q]^x_\Delta$.

The key steps in a proof of (vi) are:

(1) $p \vdash\dashv [p{:}x = \Delta]$ [(b), with x substituted for Δ and $x = \Delta$ for q, and (a)]
(2) $(\forall x)p \vdash\dashv (\forall x)\ [p{:}x = \Delta]$ [(1), (iv)]
(3) $\vdash(\forall x)[p{:}x = \Delta] \supset [p{:}x = \Delta]!^x_\Delta$ [(a), A5]
(4) $[p{:}x = \Delta]!^x_\Delta \vdash\dashv p^x_\Delta$ [(b), (iii), and (a)]
(5) $(\forall x)p \vdash\dashv p^x_\Delta$ [(2), (3), (4), (i)]

The key steps in a proof of (vii) are:

(1) $p \vdash\dashv [p{:}(\forall x)p]$ [(b), where Δ is x]
(2) $(\forall x)p \vdash\dashv (\forall x)[p{:}(\forall x)p]$ [(1), (iv)]
(3) $(\forall x)[p{:}(\forall x)p] \vdash\dashv [p{:}(\forall x)p]$ [(vi), where Δ is x, and A4]
(4) $p \vdash\dashv (\forall x)p$ [(iii), (1), (3), (2)]

(in (1) and (3) we are using the fact that x has no occurrences in $[p{:}(\forall x)p]$, by (a)).

As a preliminary toward the proof of (viii) observe that if $p_1 =_{df} Nq$ and $p_2 =_{df} (p_1 \supset r)$, then the following are tautologies: $p_1 \supset (p_2 \supset r)$, $(Np_2 \supset p_2) \supset ((p_1 \supset (p_2 \supset r)) \supset (p_1 \supset (Np_2 \supset r)))$. Using these and A6, we derive the first of the following:

(1) $\vdash Nq \supset [N(Nq \supset r) \supset r]$
(2) $\vdash Nq \supset \Diamond Nq$ [special case of (1), with $\mathscr{f}$ replacing r]

(3) $\not{\ell} \vdash\dashv N\not{\ell}$ [A2, A6]
(4) $\vdash \sim Nq \supset N \sim Nq$ [A7 with $\not{\ell}$ replacing r, (3), (iv)]
(5) $\vdash \Diamond Nq \supset N \Diamond Nq$ [(4) with $\sim Nq$ replacing q]
(6) $\vdash \Diamond Nq \supset Nq$ [by transposition of (4), using (i)]
(7) $\vdash N \Diamond Nq \supset NNq$ [(2), (6), (iv)]
(8) $\vdash Nq \supset NNq$ [(2), (5), (7), (i)]

Now (viii) follows easily, by induction on the number of occurrences of $\supset$ and $\forall$ in p, using (8), A6, (iv), A7, and A8.

To conclude this section we make a few remarks about Axioms A9 to A12. It turns out easily that A9 could be replaced in our calculus by the following pair of schemas:

A9′ $(\forall x)x =^{\cap} x$, *where x has an individual type*
A9″ $(\forall x, y, F)\{x =^{\cap} y \supset [F(x) \supset^{\cap} F(y)]\}$.

The deduction of A9 from A9′ and A9″ involves the use of A11, A12, and A14. In fact from A9′, A11, and A12 we derive that $\vdash x =^{\cap} x$, where x can have any type. Hence it follows that

$$\vdash(\forall y)[F(y) \equiv x =^{\cap} y] \supset \{[F(x) \supset F(y)] \supset x =^{\cap} y\}.$$

On the other side, by use of A14, we find that

$$\vdash(\exists F)(\forall y)[F(y) \equiv x =^{\cap} y].$$

Thus

$$\vdash(\exists F)\{[F(x) \supset F(y)] \supset x =^{\cap} y\}$$

whence we infer that

$$\vdash(\forall F)[F(x) \supset F(y)] \supset x =^{\cap} y$$

because F does not occur in $x =^{\cap} y$. The converse implication easily follows from A9″. We note also that, thanks to A11 and A12, we might set the restriction on A10 that x, y, and z have an individual type. It is well-known that A10 is equivalent to:

A10′ $N(\forall x, y)(x = y \supset y = x)$;
A10″ $N(\forall x, y, z)[x = y \supset (y = z \supset x = z)]$.

Using A9″ and A10′ one derives $\vdash x =^{\cap} y \supset (\forall F)[F(x) \equiv F(y)]$. Hence, $\vdash x =^{\cap} y \supset (p \equiv p^{x}_{y})$ (i.e., AS12.13 of MC^{ν} is a theorem of our calculus), because A14 yields that $\vdash(\exists F)(\forall x)[F(x) \equiv p]$.

A final result about identity is the following, whose proof we omit:

Proposition 3.2 *If y does not occur in the term Δ, which in turn is free for y in p, then*

(i) $(\exists y)(\Delta =^{\cap} y \wedge p) \vdash\dashv p^{y}_{\Delta}$;
(ii) $(\exists^{\cap}_{1} x)p \vdash [(\forall x)(p \supset q) \equiv (\exists x)(p \wedge q)]$.

4 The existence principle for attributes—Formal characterizations of the actual elementary possible case and of extensions In this section we consider a few equivalent versions of the axiom schema A14 on the existence of attributes. A14 enables us to characterize, within our calculus, both the actual elementary possible case and the extensions of any particular type. These characterizations play an important role in subsequent sections (see proofs of Propositions 5.3, 6.1, and 6.2). One of the variants of A14 that we are going to consider, namely A14″, is identical with the axiom schema AS12.19 of MC^ν (cf. [1], p. 46). Thus the present section incidentally shows that any of our versions of A14 could replace AS12.19 in MC^ν.

Definition 4.1 Let F, G, c, x, and e be variables of the respective types t, t, (1), θ, and (θ), where $t = (t_1, \ldots, t_n)$. Furthermore let H and y be the first two variables of the respective types t and θ which are distinct from F and x, respectively. Finally, let X stand for the expression $x_1, \ldots, x_n$, where x_j is the first variable of the type t_j that does not occur in $x_1, \ldots, x_{j-1}$ $(j = 1, \ldots, n)$. We define:

$$\begin{aligned}
F \subseteq G &=_{df} (\forall X)[F(X) \supset G(X)]\\
F \subseteq^\cap G &=_{df} NF \subseteq G\\
MConst_t(F) &=_{df} (\forall X)[\Diamond F(X) \supset NF(X)]\\
MMin_t(F) &=_{df} (\forall H)(F \subseteq H \supset F \subseteq^\cap H)\\
AEC(c) &=_{df} [MMin_{(1)}(c) \wedge (\forall v_{11})c(v_{11})]\\
Ex_\theta(e, x) &=_{df} \{MMin_{(\theta)}(e) \wedge (\forall y)[e(y) \equiv x = y]\}.
\end{aligned}$$

Remarks 4.1 (a) Usually we omit the subscripts and write just *MConst*, *MMin*, and *Ex* instead of $MConst_t$, $MMin_t$, and Ex_θ.

(b) Intuitively *MConst*(F) (to be read as "F is modally constant") says that the extension of $\tilde{F}$ is independent of the Γ-case; *MMin*(F) (to be read as "F is modally minimal") says that $\tilde{F}$ is empty in all Γ-cases, except at most the actual Γ-case; *AEC*(c) says that $\tilde{c}$ may represent the actual elementary case or Γ-case (this formula roughly corresponds to Bressan's formula $x \in ElR \wedge x = 1$; see [1], pp. 197-202): an equivalent definition for *AEC* would be

$$AEC(c) =_{df} (\forall k)[(\forall v_{11})k(v_{11}) \equiv c \subseteq^\cap k].$$

Ex(e, x) says that $\tilde{e}$ may represent the extension of $\tilde{x}$ in the actual Γ-case, since e denotes, in the actual Γ-case, the set of all *QI*'s that share the same extension with $\tilde{x}$; in every other Γ-case, the empty set.

(c) Note that the identity symbol does not occur in the definiens of either *MMin*(F) or *AEC*(c). We will see by the end of this section that if we set

$$u =^\wedge v =_{df} (\forall H)[H(u) \supset H(v)],$$

then

$$\begin{aligned}
&\vdash x = y \equiv (\exists F, G)\{MMin(F) \wedge MMin(G) \wedge\\
&(\forall z)[(F(z) \equiv z =^\wedge x) \wedge (G(z) \equiv z =^\wedge y)] \wedge\\
&(\iota z)F(z) =^\wedge (\iota z)G(z)\}
\end{aligned}$$

holds in MC^ν.

This incidentally shows that identity is eliminable from MC^ν. The eliminability of identity from MC^ν by use of the description operator was discovered by Schorch (for an alternative proof see [2], N16).

Below we state three alternative versions of A14 (here Convention 2.1 applies):

A14′ $Q(\exists F)\{MMin(F) \wedge (\forall X)[F(X) \equiv p]\}$
A14″ $Q(\exists F)\{MConst(F) \wedge (\forall X)[F(X) \equiv p]\}$
A14‴ (I) $Q(\exists F)(\forall X)[F(X) \equiv^{\cap} p]$
(II) $N(\exists v_{(1)1})AEC(v_{(1)1})$.

It is trivial (by the way *MConst* is defined) that A14″ yields A14. It is also easy to derive A14‴(I) from either A14 or A14′. In fact both A14 and A14′ (along with A11) yield readily that

$$\vdash N(\exists_1 F)(\forall X)[F(X) \equiv p].$$

Now we can apply A13 to get A14‴(I).

We are going to show that A14 yields A14‴(II) (see Proposition 4.1); A14′ yields A14″ (see Proposition 4.2); A14‴ yields A14′ (see Proposition 4.4). Hence the equivalence of the four schemas above will follow. In our proofs we may make free use of A14‴(I).

The adoption of A14‴ as an axiom schema seems especially attractive when assumptions are made about the plurality of Γ-cases. In fact A14‴(II) can be easily strengthened so it will express such assumptions. For instance, the existence of a contingent proposition can be stated by

$$(\exists v_{(1)1})[AEC(v_{(1)1}) \wedge \Diamond \sim AEC(v_{(1)1})].$$

Notice also that A14‴ (unlike A14, A14′, or A14″) allows us to replace A13 by its subschema

$$(\forall F)(\exists z)N\{(\exists y)F(y) \supset (\exists y)[z = y \wedge F(y)]\}.$$

Proposition 4.1 *A14 yields A14‴(II).*

Proof: Set:

$$p =_{df} \{(\forall x)k(x) \wedge N[(\exists x)k(x) \supset (\forall x)k(x)]\}$$
$$p_1 =_{df} (\forall k)\{[F(k) \supset p] \wedge [p \supset NF(k)]\}$$
$$p_2 =_{df} (\forall x)\{c(x) \equiv^{\cap} (\forall k)[F(k) \supset (\exists x)k(x)]\}$$
$$p_3 =_{df} c \subseteq k$$
$$p_4 =_{df} \Diamond \sim c \subseteq k$$
$$p_5 =_{df} (\forall x)[k'(x) \equiv^{\cap} c \subseteq k]$$

where x, c, k, k', and F stand for v_{11}, $v_{(1)1}$, $v_{(1)2}$, $v_{(1)3}$, and $v_{((1))1}$, respectively. The main steps in a proof are:

(1) $p_1, p_2 \vdash (\forall x)c(x)$	
(2) $p_1, \ldots, p_5 \vdash \not{f}$	
(3) $p_1, \ldots, p_4 \vdash \not{f}$	[(2), A14‴(I)]
(4) $p_1, p_2 \vdash \sim(p_3 \wedge p_4)$	[(3), deduction thm.]
(5) $p_1, p_2 \vdash [(\forall x)c(x) \wedge (\forall k)(c \subseteq k \supset c \subseteq^{\cap} k)]$	[(1), (4)]
(6) $\vdash (\exists c)AEC(c)$	[(5), A14‴(I), A14]

Proposition 4.2 *A14′ yields A14″.*

Proof: The main steps in a proof are:

(1) $MMin(F), \Diamond F(X), \sim F(X), (\forall Y)\{G(Y) \equiv^{\cap} [\sim F(X) \wedge F(Y)]\} \vdash \nleftarrow$

(2) $MMin(F) \vdash (\forall X)[\Diamond F(X) \supset F(X)]$ [(1), A14‴(I)]

(3) $MMin(F), (\forall X)[F(X) \equiv p], (\forall X)[G(x) \equiv^{\cap} \Diamond F(X)] \vdash$
$\{MConst(G) \wedge (\forall X)[G(X) \equiv \Diamond F(X)] \wedge (\forall X)[\Diamond F(X) \equiv F(X)]\}$

(4) $MMin(F), (\forall X)[F(X) \equiv p] \vdash (\exists G)\{MConst(G) \wedge (\forall X)[G(X) \equiv p]\}$
[(3), A14‴(I), deduction thm.]

(5) $(\exists G)\{MConst(G) \wedge (\forall X)[G(X) \equiv p]\}$ [(4), A14′]

Proposition 4.3 *A14‴(I) yields*

$$\vdash \Diamond[AEC(c) \wedge p] \supset N[AEC(c) \supset p].$$

Proof: The main steps in a proof are:

(1) $\Diamond[AEC(c) \wedge p], \Diamond[AEC(c) \wedge \sim p], (\forall x)\{k(x) \equiv^{\cap} [AEC(c) \wedge p]\} \vdash \nleftarrow$,
where neither k nor x occurs in p

(2) $\Diamond[AEC(c) \wedge p], \Diamond[AEC(c) \wedge \sim p] \vdash \nleftarrow$ [(1), A14‴(I)]

(3) $\vdash \sim\{\Diamond[AEC(c) \wedge p] \wedge \Diamond[AEC(c) \wedge \sim p]\}$ [(2), deduction thm.]

which is equivalent to our thesis.

Proposition 4.4 *A14‴ yields A14′.*

Proof: The main steps in a proof are:

(1) $AEC(c) \vdash (\forall X)\{[AEC(c) \wedge p] \supset G(X)\} \supset (\forall X)\{[AEC(c) \wedge p] \supset^{\cap} G(X)\}$
[Prop. 4.3]

(2) $AEC(c), (\forall X)\{F(X) \equiv^{\cap} [AEC(c) \wedge p]\} \vdash (\forall X)[F(X) \equiv p]$

(3) $AEC(c), (\forall X)\{F(X) \equiv^{\cap} [AEC(c) \wedge p]\} \vdash \{MMin(F) \wedge (\forall X)[F(X) \equiv p]\}$
[(1), (2)]

(4) $\vdash$A14′ [(3), A14‴(I), A14‴(II)]

Now we state various results to be used in the subsequent sections (we omit the proof, which is rather mechanical).

Proposition 4.5

(i) $MMin(F), MMin(G), F = G \vdash F =^{\cap} G$

(ii) $\vdash (\exists_1^{\cap} F)\{MMin(F) \wedge (\forall X)[F(X) \equiv p]\}$

(iii) $\vdash (\exists_1^{\cap} e)Ex(e, x)$

(iv) $\vdash (\exists_1^{\cap} c)AEC(c)$

(v) $(\exists c)[AEC(c) \wedge p] \vdash\dashv (\forall c)[AEC(c) \supset p]$

(vi) $Ex(e, x_1), Ex(e, x_2) \vdash x_1 = x_2$

(vii) $Ex(e_1, x_1), Ex(e_2, x_2), x_1 = x_2 \vdash e_1 =^{\cap} e_2$.

Observe in particular that Proposition 4.5(i) can be obtained from the following proposition, by putting F and G for Φ and Ψ, respectively.

Proposition 4.6 *If Φ and Ψ are terms of the same type, then we have* $MMin(F), MMin(G), \Phi = \Psi, [F = G \supset^{\cap} \Phi = \Psi] \vdash \Phi =^{\cap} \Psi$.

Proof: Let H, X be a string of distinct variables separated by commas, none of which occurs in $\Phi = \Psi$, such that $(\exists H)(\forall X)[H(X) \equiv^{\cap} \Phi = \Psi]$ is well-formed. Since this formula is a theorem (cf. A14‴(I)), it is sufficient for us to derive $\Phi =^{\cap} \Psi$ from the hypotheses:

(1) $(\forall X)[H(X) \equiv^{\cap} \Phi = \Psi]$
(2) $MMin(F)$
(3) $MMin(G)$
(4) $\Phi = \Psi$
(5) $F = G \supset^{\cap} \Phi = \Psi$

The main steps in a proof are:

(6) $(\forall X)H(X)$ [(1), (4)]
(7) $F \subseteq H,\ G \subseteq H$
(8) $F \subseteq^{\cap} H,\ G \subseteq^{\cap} H$ [(7), (2), (3), def. of *MMin*]
(9) $(\exists X)H(X) \supset^{\cap} \Phi = \Psi$ [(1)]
(10) $\sim\Phi = \Psi \supset^{\cap} (\forall X) \sim H(X)$
(11) $(\forall X) \sim H(X) \supset^{\cap} (\forall X)[\sim F(X) \wedge \sim G(X)]$ [(8)]
(12) $(\forall X)H(X) \supset^{\cap} F = G$ [(11), A11]
(13) $\sim\Phi = \Psi \supset^{\cap} \Phi = \Psi$ [(10), (12), (5)]
(14) $\Phi =^{\cap} \Psi$

To obtain (6) and (9) we have used the hypothesis that none of the variables in X occurs in $\Phi = \Psi$.

We are ready, now, to prove the fact, hinted at in Remark 4.1(c), that identity is eliminable from MC^{ν} by use of the descriptor ι. Since A9 is a theorem schema in MC^{ν}, it is sufficient for us to show that

$$\vdash x = y \equiv (\exists F, G)\{MMin(F) \wedge MMin(G) \wedge (\forall z)[(F(z) \equiv z =^{\cap} x) \wedge (G(z) \equiv z =^{\cap} y) \wedge (\iota z)F(z) =^{\cap} (\iota z)G(z)\}.$$

I recall that in MC^{ν} the following are theorems:

(1) $(\forall z)[F(z) \equiv z =^{\cap} x] \supset x = (\iota z)F(z)$
(2) $(\forall z)[G(z) \equiv z =^{\cap} y] \supset y = (\iota z)G(z)$

so that

$$(\forall z)\{[F(z) \equiv z =^{\cap} x] \wedge [G(z) \equiv z =^{\cap} y]\},\ (\iota z)F(z) = (\iota z)G(z) \vdash x = y$$

whence it readily follows that the second member of the equivalence above yields the first member. From (1) and (2) we derive also

$$(\forall z)\{[F(z) \equiv z =^{\cap} x] \wedge [G(z) \equiv z =^{\cap} y]\},\ x = y \vdash (\iota z)F(z) = (\iota z)G(z).$$

From this and the analogue of Proposition 4.6, with $(\iota z)F(z)$ and $(\iota z)G(z)$ replacing Φ and Ψ, respectively, we obtain

$$x = y,\ MMin(F),\ MMin(G),\ (\forall z)\{[F(z) \equiv z =^{\cap} x] \wedge [G(z) \equiv z =^{\cap} y]\} \vdash (\iota z)F(z) =^{\cap} (\iota z)G(z)$$

because $F = G \supset^\cap (\imath z)F(z) = (\imath z)G(z)$ is a theorem of MC^ν. Hence we see that the first member of the equivalence under inspection yields the second member; in fact, A14′ is a theorem schema of MC^ν (because, as we have seen in this section, the axiom schema A14″ of MC^ν yields A14′) and therefore

$$\vdash(\exists F, G)(MMin(F) \wedge MMin(G) \wedge (\forall z)\{[F(z) \equiv z =^\cap x] \wedge [G(z) \equiv z =^\cap y]\}).$$

5 ***The axiom of choice and the existence principle for functions*** Our version of the axiom of choice is A15. In this section, five schemas that slightly differ from A15 are shown to be equivalent to it. The version of the axiom of choice that was adopted in MC^ν is derived as a theorem in our calculus (see Proposition 5.3).

MC^ν includes an axiom schema concerning the existence of functions (cf. [1], AS12.17, p. 45); this had to be restated in [8], when descriptions were eliminated from MC^ν. By use of A15, we derive in our calculus both those schemas on the existence of functions (see Proposition 5.2).

Consider the following six schemas (where Convention 2.1 applies):

A15	$N(\forall F)(\exists f)(\forall X)[(\exists y)F(X, y) \supset F(X, f(X))]$
A15$^{\text{i}}$	$N(\forall F)(\exists f)(\forall X)\{(\exists y)F(X, y) \supset (\exists y)[f(X) =^\cap y \wedge F(X, y)]\}$
A15$^{\text{ii}}$	$Q(\exists f)(\forall X)\{(\exists y)p \supset (\exists y)[f(X) =^\cap y \wedge p]\}$
A15$^{\text{iii}}$	$Q\{(\forall X)(\exists y)p \supset (\exists f)(\forall X)(\exists y)[f(X) =^\cap y \wedge p]\}$
A15$^{\text{iv}}$	$N(\forall F)\{(\forall X)(\exists y)F(X, y) \supset (\exists f)(\forall X)(\exists y)[f(X) =^\cap y \wedge F(X, y)]\}$
A15$^{\text{v}}$	$N(\forall F)[(\forall X)(\exists y)F(X, y) \supset (\exists f)(\forall X)F(X, f(X))]$.

We claim that the above schemas are pairwise equivalent. The equivalence of A15 and A15$^{\text{i}}$ (and similarly of A15$^{\text{v}}$ and A15$^{\text{iv}}$) is immediate from Proposition 3.2(i). The equivalence of A15$^{\text{i}}$ and A15$^{\text{ii}}$ (and similarly of A15$^{\text{iv}}$ and A15$^{\text{iii}}$) is proven using A14‴(I).

Finally, to show the equivalence of A15$^{\text{ii}}$ and A15$^{\text{iii}}$, observe that A15$^{\text{ii}}$ yields A15$^{\text{iii}}$ because in general (using A1 to A8) if x does not occur free in Qp, then

$$\vdash(\exists x)Q(p \supset q) \supset [Qp \supset (\exists x)Qq].$$

Conversely,

Proposition 5.1 *A15$^{\text{iii}}$ yields A15$^{\text{ii}}$.*

Proof: Let us show, using A15$^{\text{iii}}$, that

$$(\forall X)(\forall y)\{G(X, y) \equiv^\cap [(\exists y)p \supset p]\} \vdash \text{A15}^{\text{ii}}$$

where G and f do not occur in p. Thus, by A14‴(I), it follows that $\vdash$A15$^{\text{ii}}$. The main steps in a deduction are:

(1) $(\forall X)(\exists y)G(X, y)$ [from the hypothesis]
(2) $(\exists f)(\forall X)(\exists y)[f(X) =^\cap y \wedge G(X, y)]$ [(1), A15$^{\text{iii}}$]
(3) $(\exists f)(\forall X)(\exists y)\{f(X) =^\cap y \wedge [(\exists y)p \supset p]\}$
[(2), hypothesis, equivalence thm.]
(4) $(\exists f)(\forall X)\{(\exists y)p \supset (\exists y)[f(X) =^\cap y \wedge p]\}$

where (4) follows from (3) because $[q \wedge (r \supset p)] \vdash r \supset (q \wedge p)$.

Proposition 5.2 *If f does not occur in p or Δ, then*

(i) $\vdash (\forall X)(\exists_1^{\cap} y)Np \supset (\exists f)(\forall X)(\exists y)N[f(X) = y \wedge p]$
(ii) $\vdash (\exists f)(\forall X)f(X) =^{\cap} \Delta$
(iii) $\vdash (\exists f)(\forall X)f(X) = \Delta$

Proof: A15$^{\text{iii}}$ readily yields (i). From (i) and the theorem $\vdash (\exists_1^{\cap} y)y =^{\cap} \Delta$ one derives that

$$\vdash (\exists f)(\forall X)(\exists y)N[f(X) = y \wedge y = \Delta]$$

whence (ii) and (iii) follow.

We note that (ii) and (iii) are Thm. 40.1.(46)$_2$ and AS12.17 in [1] (pp. 166 and 45): although the existence axiom for functions (AS12.17 of [1]) has no close resemblance with any of our axioms A1 to A15, nevertheless it can be deduced as a theorem of our calculus thanks to the version A15 of the axiom of choice.

In the proposition below and within its proof, the following abbreviating definitions apply, where x, F, G, Φ, Ψ, and f are distinct variables having the respective types t, (t), (t), $((t))$, $((t))$, and $(((t)):t)$:

$p_1 =_{df} (\forall F)[\Psi(F) \supset (\exists x)F(x)]$
$p_2 =_{df} (\forall \Phi)(\forall x)(\forall F)\{[f(\Phi) =^{\cap} x \wedge \Psi(F) \wedge Ex_{(t)}(\Phi, F)] \supset F(x)\}$
$p_3 =_{df} (\forall x)\{G(x) \equiv (\exists \Phi)(\exists F)[f(\Phi) =^{\cap} x \wedge \Psi(F) \wedge Ex_{(t)}(\Phi, F)]\}$
$p_4 =_{df} (\forall F)(\forall G)\{[\Psi(F) \wedge \Psi(G) \wedge (\exists x)(F(x) \wedge G(x))] \supset F = G\}$
$p_5 =_{df} (\forall F)\{\Psi(F) \supset (\exists_1^{\cap} x)[F(x) \wedge G(x)]\}$

Proposition 5.3 $\vdash (p_1 \wedge p_4) \supset (\exists G)p_5$.

Proof: The main steps in a proof are:

(1) $p_1 \vdash (\exists f)p_2$	[Prop. 4.5(vi), A15$^{\text{iii}}$, Prop. 3.2(ii)]
(2) $p_1 \vdash (\exists f)(\exists G)(p_2 \wedge p_3)$	[(1), A14$'''$(I)]
(3) $p_2, p_3, \Psi(F) \vdash (\exists x)[F(x) \wedge G(x)]$	[Prop. 4.5(iii)]
(4) $p_2, p_3, p_4, \Psi(F) \vdash (\exists_1^{\cap} x)[F(x) \wedge G(x)]$	[(3), Prop. 4.5(vii)]
(5) $p_4 \vdash (\exists f)(\exists G)(p_2 \wedge p_3) \supset (\exists G)p_5$	[(4)]
(6) $p_1, p_4 \vdash (\exists G)p_5$	[(2), (5)]

The above proposition readily yields

$$\vdash (p_1 \wedge p_4) \supset (\exists G)(\forall F)\{\Psi(F) \supset (\exists_1 x)[F(x) \wedge G(x)]\}$$

which is the version of the axiom of choice that was adopted in MC^{ν}.

*6 **The existence principle for descripta*** In this section we show that A13 is equivalent to each of the following two schemas:

A13$'$ $Q[N(\exists y)p \supset (\exists z)N(\exists y)(z = y \wedge p)]$
A13$''$ $Q[N(\exists_1 y)p \supset (\exists z)N(\exists y)(z = y \wedge p)]$.

In A13, A13$'$, and A13$''$ we follow Convention 2.1 and may assume, with no loss in generality, that y has an individual or relator type (see Proposition 6.2).

There is a striking analogy between A13$'$ and the version A15$^{\text{iii}}$ of the axiom of choice. Note also that A13 turns out to be a theorem in MC^{ν}, since

A13$''$ is readily derived in MC^ν using the main axiom about descriptions (AS38.1(I) in [1], p. 154).

We would name the version A13$''$ of A13 the "axiom on the existence of descripta". In fact, it is possible to define a natural translation of ML^ν into our language (see [8], N.4) and then regard A13$''$ as an axiom which asserts the existence of those objects designated by the descriptions of ML^ν. The original version of this axiom—somewhat different from those being considered here—was proposed in [8] when descriptions were eliminated from MC^ν.

The proof of the equivalence of A13 and A13$'$ is quite analogous to the proof, given in Section 5, that A15$^{\text{ii}}$ and A15$^{\text{iii}}$ are equivalent. In fact, the same argument we used there to show that A15$^{\text{ii}}$ yields A15$^{\text{iii}}$ shows also that A13 yields A13$'$. Conversely, to prove that A13$'$ yields A13 let us demonstrate (using A13$'$ and mimicking the proof of Proposition 5.1) that

$$(\forall y)\{G(y) \equiv^{\cap} [(\exists y)p \supset p]\} \vdash \text{A13},$$

where G and z do not occur in p. The main steps in a deduction are:

(1) $N(\exists y)G(y)$ [from the hypothesis]
(2) $(\exists z)N(\exists y)[z = y \wedge G(y)]$ [(1), A13$'$]
(3) $(\exists z)N(\exists y)\{z = y \wedge [(\exists y)p \supset p]\}$ [(2), hypothesis, equivalence thm.]
(4) $(\exists z)N[(\exists y)p \supset (\exists y)(z = y \wedge p)]$.

Trivially, A13$'$ yields A13$''$. The following lemma is a preliminary toward the proof that A13$''$ yields A13$'$.

Lemma 6.1 *Assume that x, c, f, and i are distinct variables having the types t,(1),((1):t), and ((1):(t)), respectively. Let f, c, and i have no occurrences in p. Then:*

(i) $N(\exists x)p \vdash (\forall c)(\exists x)N[AEC(c) \supset p]$
(ii) $N(\exists x)p \vdash (\exists f)(\forall c)N[AEC(c) \supset p^x_{f(c)}]$
(iii) $N(\exists x)p \vdash (\exists i)(\forall c)N\{AEC(c) \supset (\exists x)[p \wedge Ex(i(c), x)]\}$.

[Intuitively, (iii) asserts—under the hypothesis—the existence of an intension that associates with every Γ-case γ the extension of a ξ satisfying p in γ.]

Proof: The main steps in a proof of (i) are:

(1) $\Diamond AEC(c) \supset (\exists x)\Diamond[AEC(c) \wedge p]$ [from the hypothesis]
(2) $\Diamond AEC(c) \supset (\exists x)N[AEC(c) \supset p]$ [(1), Prop. 4.3]
(3) $(\exists x)N[AEC(c) \supset p]$ [(2)]

From (i) we easily obtain (ii) using A15$^{\text{iii}}$ and Proposition 3.2(i). Now observe that $N(\exists x)p \vdash N(\exists e)(\exists x)[p \wedge Ex(e, x)]$ by Proposition 4.5(iii), if we choose an e having no occurrences in p. Hence we get readily (iii), by substituting in (ii): e for x, $(\exists x)[p \wedge Ex(e, x)]$ for p.

Proposition 6.1 *A13$''$ yields A13$'$.*

Proof: By Lemma 1(iii), it suffices to prove that A13$''$ yields

$$(\forall c)N\{AEC(c) \supset (\exists y)[p \wedge Ex(i(c), y)]\} \vdash (\exists z)N(\exists y)(z = y \wedge p).$$

From the hypothesis, by Proposition 4.5(v) we derive

$$N(\exists y)(\exists c)[AEC(c) \wedge p \wedge Ex(i(c), y)]$$

hence

$$N(\exists_1 y)(\exists c)[AEC(c) \wedge p \wedge Ex(i(c), y)] \qquad \text{[Prop. 4.5(iv), (vi)]}$$

Now we apply A13″ to get

$$(\exists z)N(\exists y)\{z = y \wedge (\exists c)[AEC(c) \wedge p \wedge Ex(i(c), y)]\}$$

and finally $(\exists z)N(\exists y)(z = y \wedge p)$.

Proposition 6.2 $N(\exists g)p \vdash (\exists f)N(\exists g)(f = g \wedge p)$ *holds even if we assume an instance of A13 to be an axiom only for y and z variables of an individual or relator type.*

Proof: By Lemma 6.1(ii), it suffices to prove that

$$(\forall c)N[AEC(c) \supset p^g_{\phi(c)}] \vdash (\exists f)N(\exists g)(f = g \wedge p)$$

From the hypothesis, by Proposition 4.5(v) we derive

$$(\forall X)N(\exists c)[\phi(c)(X) =^{\cap} \phi(c)(X) \wedge AEC(c) \wedge p^g_{\phi(c)}].$$

The subsequent main steps in a deduction are:

(1) $(\forall X)N(\exists z)(\exists c)[z =^{\cap} \phi(c)(X) \wedge AEC(c) \wedge p^g_{\phi(c)}]$
(2) $(\forall X)(\exists z)N(\exists c)[z = \phi(c)(X) \wedge AEC(c) \wedge p^g_{\phi(c)}]$
[(1) and either A13′ or inductive hypothesis]
(3) $(\exists f)(\forall X)N(\exists c)[f(X) = \phi(c)(X) \wedge AEC(c) \wedge p^g_{\phi(c)}]$
[(2), A15$^{\text{iii}}$, Prop. 3.2(i)]
(4) $(\exists f)N(\forall c)(\forall X)\{AEC(c) \supset [f(X) = \phi(c)(X) \wedge p^g_{\phi(c)}]\}$ [(3), Prop. 4.5(v)]
(5) $(\exists f)N(\forall c)\{AEC(c) \supset [f = \phi(c) \wedge p^g_{\phi(c)}]\}$ [(4), A12]
(6) $(\exists f)N(\exists c)(\exists g)[AEC(c) \wedge f = g \wedge p]$ [(5), Prop. 4.5(v)]
(7) $(\exists f)N(\exists g)(f = g \wedge p)$.

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Added in proof:

Proposition 2.2 *Let p be the following wff:*

A16 $(\exists F)(\forall x)N(\exists y)\{x = y \wedge F(y) \wedge (\forall x)[(\Diamond y = x \wedge \Diamond F(x)) \supset x =^{\cap} y]\}$

where x, y, *and* F *are* v_{t1}, v_{t2}, *and* $v_{(t)1}$, *respectively. Then* $\tilde{p}(\gamma) = 0$ *holds for all* $\gamma \in \Gamma$ *if and only if the cardinality of* $D_t(\gamma)$ *is independent of* γ $(t = 1, \ldots, \nu)$.

The wff A16 roughly corresponds to the axiom schema AS25.1 in [1], p. 95, and is only mentioned here for the sake of completeness. Proofs are left to the reader.