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Eventual Permanence

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Some statements are true now and will be true from now on. Such statements are permanently true. This kind of claim figures prominently in reasoning about such diverse topics as games, natural resources, and cosmology. The significance of the notion of presently permanent truth has long been realized. It is in fact just the Diodorean conception of necessity, a notion which has been widely and thoroughly examined (see [3] and [4]).

Some statements, while not permanently true now, will sooner or later become permanently true. Claims with this modal status also figure prominently in reasoning about many diverse topics. For example, it is not now true that our oil reserves are exhausted. However, eventually it will be true that these reserves are permanently exhausted.[1]

The usefulness of the concept mentioned in the preceding paragraph extends to circumstances where the concern is not with universal permanence, but with what is permanent in a system. Games yield many examples of this kind. At a certain move in a chess match, the statement, "White has no Queen," may be false. Yet, it may also be true, because of the arrangement of the pieces on the board, that it will eventually be permanently true that White has no Queen. Relative to that move in that particular chess game, the statement, "White has no Queen," will eventually be permanently true.

These examples suggest that the concept of eventual permanence is a significant and useful one. Unlike Diodorean necessity, however, the idea of eventual permanence has been largely neglected. The only exception to this pattern of neglect with which I am familiar is Rescher and Urquhart's treatment of "Marxist necessity" ([4], pp. 135-137). Their discussion, while correct as far as it goes, is admittedly inconclusive on an important point which will be discussed below. Moreover, they restrict their investigation to a special case, a case which is not necessarily the most interesting or significant. My aim in this paper is to begin where Rescher and Urquhart leave off and to give a more

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thorough treatment of the notion of eventual permanence and its most important applications.

The task is twofold. First, it is necessary to find a precise, but intuitively satisfactory, account of the truth conditions of statements of the form, "It will eventually be permanently true that p." This proves to be a problem that is not entirely trivial. Second, I shall seek to isolate the logic of eventual permanence by providing axiomatic systems which are sound and complete relative to this concept. As is usual in such cases, what the logic of the notion is depends on what assumptions are made concerning the structure of time. In succeeding sections, I concentrate on two key cases: (a) linear, unending time; (b) branching, unending time. Some other cases are discussed more briefly.

1 ***Unending, linear time*** For temporal structures of this kind, Rescher and Urquhart supply the following truth condition (see [4], p. 135) for sentences of the form, "It will eventually be permanently true that S":

V1 $V(LS, x) = t \text{ iff } (Ey)(y > x \text{ and } (z)(\text{if } z > y, \text{ then } V(S, z) = t))$.

For the case of linear, unending time, V1 provides a satisfactory explication of the truth-conditions for eventual permanence. The truth of LS implies that, eventually, it will always be true that S; the falsity of LS implies that it will never be the case that S is permanently true.

Rescher and Urquhart conjecture that the logic of eventual permanence is, in this case, represented by the modal system *D5*. This logic is axiomatized in the following way:

AS0 All truth-functional tautologies.
A1 $L(S \supset T) \supset .LS \supset LT$
A2 $LS \supset MS$
A3 $MS \supset LMS$
Def. $(MS = \sim L \sim S)$
Rules. Substitution, modus ponens, necessitation.

Obviously, *D5* is sound relative to V1. However, Rescher and Urquhart leave the completeness of *D5* as an open question. That *D5* is indeed complete is a consequence of Segerberg's generalization of Scroggs's Theorem ([5], pp. 126, 190). Segerberg shows that the only consistent extensions of *D5* are *D5*, *D5* Alt_n ($n \geqslant 2$), *S5*, and *S5* Alt_n ($n \geqslant 1$). Here the system *D5* Alt_n (respectively, *S5* Alt_n) is formed by adding the axiom:

$\mathbf{Alt}_n$ $LS_1 \vee L(S_1 \supset S_2) \vee L((S_1 \,\&\, S_2) \supset S_3) \vee \ldots \vee L((S_1 \,\&\, S_2 \,\&\, \ldots \,\&\, S_n) \supset S_{n+1})$

to the system *D5* (*S5*). The correct logic for eventual permanence is not an extension of *S5*, since eventual permanence does not imply truth. Because *D5* is sound for V1, it follows that the correct logic is *D5* or *D5* Alt_n, for some n. To see that no *D5* Alt_n is correct, consider the structure $\langle N, <\rangle$, where N is the natural numbers and $<$ is "less than". This structure validates *D5*, when "L" is interpreted via V1. To falsify Alt_n, consider the valuation V_n for which:

$$V_n(S_i, j) = f \text{ iff } i \equiv j \bmod (n + 1).$$

On this valuation, no disjunct of Alt_n is true at 0. Consider, for example, that k^{th} disjunct $L((S_1 \mathbin{\&} \ldots \mathbin{\&} S_{k-1}) \supset S_k)$. For every m such that $k \equiv m \bmod (n+1)$, $V_n(S_1 \mathbin{\&} \ldots \mathbin{\&} S_{k-1}, m) = t$ and $V_n(S_k, m) = f$. Since there are infinitely many numbers of this kind, the k^{th} disjunct is false at 0. So, the logic of eventual permanence is not *D5* Alt_n, for any n. The correct logic is therefore *D5*.

One further point is worthy of note. The dual of eventual permanence in unending, linear time is an interesting concept in its own right. Using V1, the operator "M" has the following truth-condition:

V1′ $V(MS, x) = t$ iff (y) (if $y > x$, then $(Ez)(z > y$ and $V(S, z) = t))$.

Consequently, the truth of MS implies that S never permanently ceases to be true. Put in positive and more familiar terms, the truth of MS implies (and is implied by) the claim that the truth of S will eternally recur. The dual of eventual permanence in infinite linear time is therefore eternal recurrence, and the logic of the notion of eternal recurrence in this sort of temporal structure is *D5*.

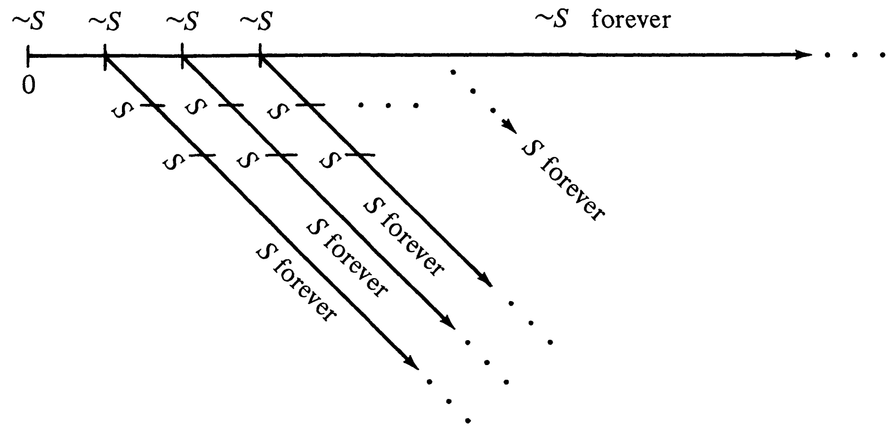
2 ***Unending, branching time*** Rescher and Urquhart consider the notion of eventual permanence only for the case of linear time. But, of course, it is not true that the concept of eventual necessity presupposes the linearity of time. It is clearly consistent to maintain that there are and will be distinct, alternative future courses of events, even though certain conditions are eventually permanent. Recall the chess example mentioned at the beginning of the paper. In that case, it can be maintained that "White has no Queen" will eventually be permanently true and yet that there are distinct, alternative ways in which the match might proceed. Consequently, an investigation of eventual permanence in unending, branching time is a natural and important extension of Rescher and Urquhart's work.

The first problem is to provide a formal truth condition for this concept that is appropriate in a forward branching temporal structure. It is clear that V1 of Section 1 will not work since, on that definition, it is possible that $V(LS, x) = t$ and $V(L{\sim}S, x) = t$. The trouble with V1 is that it doesn't require enough for the truth of LS. On V1, LS can be true if S becomes always true in some alternative future. But this latter fact is not sufficient to imply that the truth of S will eventually be permanent.

A natural first idea for improvement is the thought that the right-hand side of the biconditional in V1 should be made to hold for every $u > x$. Thus, one might propose:

V2 $V(LS, x) = t$ *iff* $(u)($*if* $u > x$, *then* $(Ey)(y > u$ *and* $(z)($*if* $z > y$, *then* $V(S, z) = t)))$.

This definition succeeds in ruling out the possibility that both LS and $L{\sim}S$ are true at x. However, it is still not an intuitively satisfactory definition. For if S is to be eventually permanent in a branching structure, then, no matter what course the future takes, S must sooner or later turn out to be always true. But the right-hand side of V2 can be true even if there is an alternative course of events in which the falsity of S eternally recurs. This possibility is realized in a branching structure of the following sort:



In this model, LS is true by V2 at 0, although, intuitively speaking, it is not true at 0 to say that it will eventually be permanently true at S. So, V2 isn't a satisfactory definition, either.

To obtain an adequate truth condition, it is necessary to introduce the concept of a temporal branch. Given an irreflexive, transitive ordering of moments, a temporal branch is a possible course that events might actually take. Such a course is represented by a set whose elements are linearly ordered with respect to one another and which is maximal among such sets. That is, a set B is a temporal branch in a branching structure iff:

1. For distinct x and y in B, $x < y$ or $y < x$
2. If $B \subset B^*$, then there are distinct x and y in B^* for which neither $x < y$ nor $y < x$ holds.

Using the concept of a temporal branch, a correct characterization of eventual permanency can be obtained. For it to be true at time x that S will eventually be permanently true it must be the case that no matter which branch through x is considered, the truth of S sooner or later becomes permanent on that branch. This is expressed in V3:

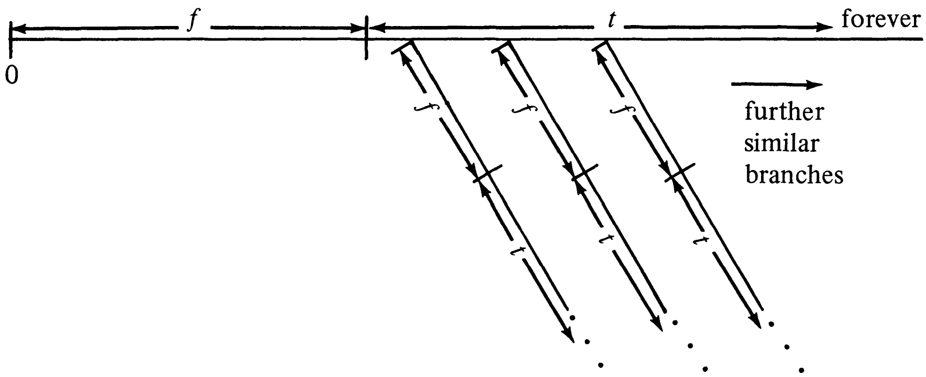
V3 *$V(LS, x) = t$ iff for every branch B, if $x \in B$, then $(Ey)(y \in B$ and $y > x$ and (z) (if $z \in B$ and $z > y$, then $V(S, z) = t))$.*

Before proceeding to determine the logic of eventual permanency so defined, it will be worthwhile to distinguish eventual permanence from a closely related concept. This concept is given by the truth condition:

V3* *$V(NS, x) = t$ iff for every branch B, if $x \in B$, then $(Ey)(y \in B$ and $y > x$ and (z) (if $z > y$, then $V(S, z) = t))$.*

V3* defines a concept appropriately described as eventual Diodorean necessity. For, if NS is true at x, then no matter which branch through x is considered, we will sooner or later arrive at a point at which S is necessary, in the Diodorean sense.

Despite the similarity between V3 and V3* they do not coincide. To see this, consider the model on which 'S' is assigned these values:



In this model, LS is true at 0 according to V3. No matter which branch through x is chosen, it ends in an unbroken string of t's. However, V3* assigns NS false at 0; for no matter what point on the top branch is chosen, there is always a later point (not on the top branch) at which S is false.

The preceding model suggests an example which brings out the distinction between what will eventually be permanently true and what will eventually be necessary in the Diordorean sense of necessity. Suppose that there are only ten years of oil reserves left and that the reserves cannot be replenished. Assume also that we may postpone indefinitely (because of alternative energy sources) our use of this reserve. In this circumstance, it is correct to claim that the statement, "Oil is not being used as an energy source," will eventually be permanently true. If in the future we never choose to use oil again, then in that case the claim is clearly a permanent truth. On the other hand, should we decide at some point to use the reserve, then after ten years it will be exhausted. Thereafter, the claim, "Oil is not being used as an energy source," will be permanently true.

Note, however, that it is not correct to claim that "Oil is not being used as an energy source" will eventually be necessary, in the Diodorean sense of necessity. For suppose we postpone using the reserve forever. In that case, the claim that oil is not being used is permanently true. However, it is not necessary since at any future point there is an alternative future course in which oil is used for a while as an energy source.[2]

Let me return now to the task of isolating the logic of eventual permanence, as defined in V3, for a branching, unending temporal structure. The correct logic is a system which I shall call *D4C*. This system is axiomatized, as follows:

AS0 All truth-functional tautologies.
A1 $L(S \supset T) \supset LS \supset LT$
A2 $LS \supset MS$
A3 $LS \supset LLS$

A4 $L(LS \supset S)$

Rules. substitution, modus ponens, necessitation.

It seems worthwhile to comment briefly on some of these axioms. A2 reflects the fact each branch is unending. A3 says that if S is eventually necessary, then S's eventual permanence will eventually be permanent (in fact, it is presently permanent). A4 is perhaps the most interesting. It is clear, of course, that $LS \supset S$ is not valid for eventual permanence. It need not be *true* that something which will eventually be permanent need be true now. However, as A4 says, it will *eventually be permanently true* that whatever is eventually permanent is true. For if this were not so then something might be going to be permanently true forever without ever actually becoming permanently true. And this last is impossible.

3 ***Completeness*** The remarks of the preceding paragraph form the core of a proof that every theorem of *D4C* is valid for eventual permanence in unending, branching time. It remains to show that *D4C* is complete.

Consider the Scott-Lemmon canonical frame $\langle W_{D4C}, R_{D4C}\rangle$ for *D4C*, where W_{D4C} is the class of all *D4C* maximal consistent sets and for all X, Y in *D4C*, $XR_{D4C}Y$ iff $\{A: LA \in X\} \subseteq Y$. An irreflexive, transitive ordering $<$ is imposed on this frame using the following definitions:

D1 For $X, Y \in W_{D4C}$, $X \sim Y$ iff $X = Y$ or both $XR_{D4C}Y$ and $YR_{D4C}X$.

D2 The cluster $[X] = \{Y: X \sim Y\}$.

D3 $[X]$ precedes $[Y]$ iff $XR_{D4C}Y$ and not-$YR_{D4C}X$.

To define the ordering relation $<$ on W_{D4C}, first select for each cluster $[X]$ an arbitrary linear order $<_{[X]}$ on $[X]$. The relation $<$ is defined by cases:

1. If $[X] = [Y]$, $X < Y$ iff $X <_{[X]} Y$
2. If $[X] \neq [Y]$, $X < Y$ iff $[X]$ precedes $[Y]$.

It is clear that $<$ is irreflexive and asymmetric; $<$ is transitive since R_{D4C} is. (The ideas used here derive from Segerberg [5], p. 75.)

Given the structure $\langle W_{D4C}, <\rangle$, we define a sequence of such structures whose union is the frame for the model required to show completeness. First, we extend the definitions of $\sim$ and [ ] to pairs of the form $\langle U, X\rangle$ where U is a set and X is in W_{D4C}.

D4 $\langle U, X\rangle \sim \langle U', Y\rangle$ iff $U = U'$ and $X \sim Y$.

D5 The cluster $[\langle U, X\rangle] = \{\langle U, Y\rangle: \langle U, X\rangle \sim \langle U, Y\rangle\}$.

Next, we define by recursion a sequence of structures $\langle W_n, <_n\rangle$.

A1. $W_0 = \{\langle 0, X\rangle: X \in W_{D4C}\}$.

2. if $\langle 0, X\rangle, \langle 0, Y\rangle \in W_{D4C}$, then $\langle 0, X\rangle <_0 \langle 0, Y\rangle$ iff $X < Y$.

B1. $W_{n+1} = W_n \cup \{\langle U, X\rangle$: U is a cluster in W_n of the form $[\langle V, Y\rangle]$, Y is a reflexive member of W_{D4C}, and either $[\langle V, Y\rangle] = [\langle V, X\rangle]$ or $\langle V, Y\rangle <_n \langle V, X\rangle\}$.

2a. if $U, U' \in W_n$, $U <_{n+1} U'$ iff $U <_n U'$.

2b. if $\langle U, X\rangle, \langle U', Y\rangle \in W_{n+1} - W_n$, then $\langle U, X\rangle <_{n+1} \langle U', Y\rangle$ iff $U = U'$ and $X < Y$.

2c. if $\langle U, X\rangle \in W_{n+1} - W_n$ and $\langle U', Y\rangle \in W_n$, then $\langle U', Y\rangle <_{n+1} \langle U, X\rangle$ iff there is a cluster $[\langle U'', Z\rangle]$ such that $U = [\langle U'', Z\rangle]$ and either $\langle U', Y\rangle \sim \langle U'', Z\rangle$ or $\langle U', Y\rangle <_n \langle U'', Z\rangle$.

W_{n+1} extends W_n and $<_{n+1}$ extends $<_n$. We define the frame $\langle W^*, <^*\rangle$ by setting $W^* = \cup W_n$ and defining $U <^* U'$ iff for some n, $U <_n U'$.

The relation $<^*$ is clearly irreflexive and asymmetric. Transitivity of $<^*$ requires a simple induction which we omit. It also needs to be established that each branch in W^* is unending. To do this, we prove first that if X is an irreflexive element of W_{D4C}, then X occurs only in pairs of the form $\langle 0, X\rangle$. Suppose X does not occur as right-hand member of any element in $W_{k+1} - W_k$, for any $k < m$. If X *does* occur as a right-hand member of an element of $W_{m+1} - W_m$, then X occurs in an element of the form $\langle[\langle V, Y\rangle], X\rangle$. Here $[\langle V, Y\rangle]$ is a cluster in W_m, either $[\langle V, Y\rangle] = [\langle V, X\rangle]$ or $\langle V, Y\rangle <_m \langle V, X\rangle$, and Y is a reflexive element in W_{D4C}. Since X is irreflexive, $[\langle V, Y\rangle] \neq [\langle V, X\rangle]$; so, $\langle V, Y\rangle <_m \langle V, X\rangle$. Since $<_m$ is defined on W_m, $\langle V, X\rangle$ is in W_m. By the induction hypothesis, $\langle V, X\rangle$ is not in $W_{k+1} - W_k$ for any $k < m$. Thus $\langle V, X\rangle$ is in W_0 and so has the form $\langle 0, X\rangle$, and $\langle V, Y\rangle$ has the form $\langle 0, Y\rangle$. Since $\langle 0, Y\rangle <_m \langle 0, X\rangle$, we have $\langle 0, Y\rangle <_0 \langle 0, X\rangle$, and so $Y < X$. Now $Y < X$ implies that $YR_{D4C}X$. Since X is irreflexive, there is a wff B such that $LB \in X$ and $\sim B \in X$. But $L(LB \supset B)$ is a theorem of $D4C$ and so $L(LB \supset B)$ is in Y. Thus, $LB \supset B$ is in X, a contradiction, thereby establishing that X occurs only as right-hand members of elements of W_0.

To show that every branch in W^* is unending, let $\langle V, Y\rangle$ be a member of a branch T. Either Y is irreflexive or reflexive. If it is irreflexive, $\langle V, Y\rangle = \langle 0, Y\rangle$. Since $LA \supset MA$ is in $D4C$, there is an X such that $YR_{D4C}X$. By the transitivity of X, not-$XR_{D4C}Y$. Thus $Y < X$ and $\langle 0, Y\rangle <^* \langle 0, X\rangle$. If Y is reflexive, $\langle V, Y\rangle <^* \langle[\langle V, Y\rangle], Y\rangle$.

To define the canonical countermodel v^* on $\langle W^*, <^*\rangle$, we set:

For each sentence letter p, $v^*(p, \langle V, X\rangle) = t$ iff $p \in X$.

It must be shown that:

For each sentence A, $v^*(A, \langle V, X\rangle) = t$ iff $A \in X$.

The only problem concerns the case where $A = LB$. So, suppose $LB \in X$ and let T be a branch in W^* containing $\langle V, X\rangle$. Assume $\langle U, Y\rangle$ is on T and $\langle V, X\rangle <^* \langle U, Y\rangle$. We wish to show that $B \in Y$. This will follow from the fact that for any n, if $\langle V, X\rangle <_n \langle U, Y\rangle$, then $XR_{D4C}Y$. If $n = 0$, the result follows because $\langle 0, X\rangle <_0 \langle 0, Y\rangle$ iff $X < Y$. Next, assume the result holds if $k \leqslant m$. If both $\langle V, X\rangle$ and $\langle U, Y\rangle$ are in $W_{m+1} - W_m$, the result holds because $\langle V, X\rangle <_{m+1} \langle U, Y\rangle$ only if $X < Y$. Suppose then that $\langle V, X\rangle \in W_m$ and $\langle U, Y\rangle \in W_{m+1} - W_m$. There must be a cluster $[\langle U'', Z\rangle]$ such that $U = [\langle U'', Z\rangle]$ and either $\langle V, X\rangle \sim \langle U'', Z\rangle$ or $\langle V, X\rangle <_m \langle U'', Z\rangle$. Suppose the former: then $XR_{D4C}Z$. Now, $\langle[\langle U'', Z\rangle], Y\rangle$ is in $W_{m+1} - W_m$ only if either $\langle U'', Z\rangle = \langle U'', Y\rangle$ or $\langle U'', Z\rangle <_m \langle U'', Y\rangle$. In the first case, $ZR_{D4C}Y$; so $XR_{D4C}Y$, by transitivity. If $\langle U'', Z\rangle <_m \langle U'', Y\rangle$, then $ZR_{D4C}Y$ by induction hypothesis, and so $XR_{D4C}Y$ as before. If instead we assume that $\langle V, X\rangle <_m \langle U'', Z\rangle$, we obtain $XR_{D4C}Z$ by induction hypothesis and proceed as in the previous case. Thus, $\langle V, X\rangle <^* \langle U, Y\rangle$ implies that $XR_{D4C}Y$. It follows that $B \in Y$; so, by the induction on rank, $v^*(B, \langle U, Y\rangle) = t$ for each $\langle U, Y\rangle$ succeeding $\langle V, X\rangle$ on T. Thus, $v^*(LB, \langle V, X\rangle) = t$.

Next, assume that $LB \notin X$. We must show that there is a branch T through $\langle V, X\rangle$ such that $\sim B$ eternally recurs on T after $\langle V, X\rangle$. If $LB \notin X$, then $M \sim B \in X$. Hence, there is a Y in W_{D4C} such that $XR_{D4C}Y$ and $\sim B \in Y$. Since $XR_{D4C}Y$ and $L(LA \supset A)$ is a theorem of $D4C$, we have $YR_{D4C}Y$. Let $Q_1, Q_2, \ldots$ be the sequence of elements of W^* defined by:

$$Q_1 = \langle [\langle V, Y\rangle], Y\rangle$$
$$Q_{n+1} = \langle [Q_n], Y\rangle.$$

Now, $\langle V, X\rangle <^* Q_1$. For let n be the least number such that $\langle V, X\rangle \in W_n$. Since $XR_{D4C}Y$, either $X \sim Y$ or $X < Y$. In either case, the definition of $<_{n+1}$ guarantees that $\langle V, X\rangle <_{n+1} Q_1$. By a similar argument, we have $Q_k <^* Q_{k+1}$ for each k. Let T be a branch through $\langle V, X\rangle$ which terminates with the members of $[Q_1]$, followed by the members of $[Q_2]$, and so on. By the induction on rank, $v^*(\sim B, Q_i) = t$. Thus, for each U on T, there is a Q_i after U which falsifies B. So, $v^*(LB, \langle V, X\rangle) = f$.

Since every nontheorem of $D4C$ is falsified somewhere in W^* on v^*, it follows that every nontheorem of $D4C$ can be falsified in an unending, branching structure, where LS means eventual permanence. This establishes the completeness of $D4C$.

Several facts about $D4C$ are worth noting. Although $D4C$ has not been discussed previously in connection with modal or tense logic, it has been suggested by Hintikka as a candidate for a satisfactory deontic logic ([1], p. 185). The system is sound and complete for the class of frames $\langle W, R\rangle$ where R meets the conditions:

1. $(x)(Ey)(xRy)$
2. $(x)(y)$ (if xRy, then yRy)
3. $(x)(y)(z)$ (if xRy and yRz, then xRz).

Using the method of filtrations, it is apparent that $D4C$ has the finite model property relative to the class of frames just described. So, $D4C$ is decidable. Furthermore, $D4C$ has the same structure of modalities as $S4$, excepting the null modality. Finally, a natural deduction formulation of $D4C$ is obtained by making two changes in the usual rules for $S4$ (see [2], pp. 331-334): (a) add the rule, from LA and $L \sim A$, infer B; (b) restrict applications of the necessity elimination rule to strict subderivations.

I should add that I believe the methods employed in this section have a significance beyond the particular problem considered here. These methods have enabled us to gain insight into a modal operator whose structure is of greater complexity than traditionally considered modal operators. To see this, reflect on the quantificational structure of the truth conditions of the usual operators and of eventual permanence. Necessity, in its various guises, has a truth condition of the form $(x)(\ldots x \ldots)$. On the other hand, the truth conditions V3 for eventual permanence have roughly the form $(x)(\exists y)(z)(\ldots x \ldots y \ldots z \ldots)$.

It seems to me not unlikely that other interesting concepts have truth-conditions with this or a similar structure. So, methods like the one used above may well be significant analytical tools.

4 Concluding remarks Throughout the discussion, I have assumed that the temporal structures are unending. If this assumption is dropped, then the truth-conditions V1 and V3 should be revised. For, as these conditions stand, both *LS* and $L \sim S$ would be true at any final moment. The natural intuition here however, I think, is that truth, permanence, and eventual permanence collapse at a final moment. This difficulty can be surmounted by replacing strict inequalities in V1 and V3 by $\leqslant$.

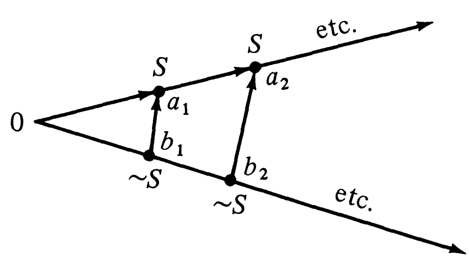
What logics are obtained by dropping the "no last moment" assumption and revising V1 and V3 as indicated? The answer is the same as before; linear time with modified V1 yields *D5*; branching time with modified V3 yields *D4C*.

Another interesting question concerns assumptions intermediate between linearity and branching. The most familiar assumption of this kind is future convergence:

FC $(x)(y)(Ez)(x \leqslant z \text{ and } y \leqslant z)$.

A first question to face is: what truth-condition is appropriate, V1 or V3? The answer to this question isn't obvious. My criticism of V1 in branching time is not correct if applied to future convergent time. So I shall leave this question undecided.

It is important to note, however, that the choice of truth-condition affects the logic of eventual permanence. If V1 is used, then the logic of eventual permanence in unending, FC time is *D5*. On the other hand, if V3 is used, the logic is weaker. To see this note that $MLS \supset LS$ is false at 0 in the following structure:

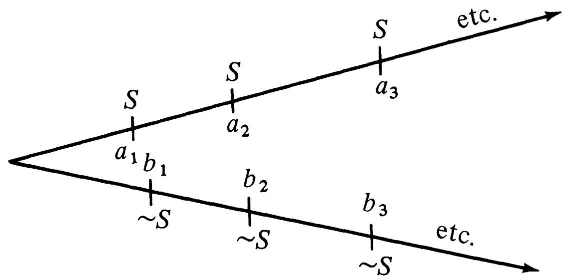


Moreover, the correct logic is stronger than *D4C*, since the familiar postulate $MLS \supset LMS$ is valid on V3 in unending, convergent time. My conjecture is that the correct logic is the logic *D4.2C* obtained by adding the usual convergence postulate to *D4C*.

Finally, I want to return to the notion of eternal recurrence. I suggested earlier that eternal recurrence was the dual of eventual permanence in linear time. This duality doesn't hold in branching time. The dual of '*L*' on V3 is given by the condition:

V3′ *$V(MS, x) = t$ iff there is a branch B such that x is in B and for every $y > x$, if $y \in B$, then there is a $z > y$ and $z \in B$ for which $V(S, z) = t$.*

In my opinion, this condition is too weak to adequately represent eternal recurrence. Condition V3′ makes MS true at 0 in the model:



However, my intuition is that it is false to claim at 0 that S will eternally recur. To correctly claim at 0 that S will eternally recur, it is necessary that S eternally recur, no matter what course the future may take. I thus suggest the following as a satisfactory truth condition for eternal recurrence ('R'):

V4 *$V(RS, x) = t$ iff for every branch B, if $x \in B$, then for every $y > x$, if $y \in B$, there is a z, $z > y$, $z \in B$ and $V(S, z) = t$.*

The reader should note that 'R' is not the dual of 'L' in branching time, since $\sim R \sim S \supset LS$ is not valid.

What logic correctly describes the logic of eternal recurrence in branching time? I don't know, but it is easy to see that 'R' is not a familiar modal operator. To see this, note that 'R' differs from necessity-like operators in that neither "$RS \supset \sim R \sim S$" nor "$(RS \;\&\; RT) \supset R(S \;\&\; T)$" are valid. On the other hand, 'R' differs from possibility-like operators in that "$R(RA \supset A)$" is valid and "$R(A \vee B) \supset (RA \vee RB)$" is not valid. I therefore leave the question of axiomatizing 'R' open.

NOTES

1. It is natural at first sight to think that the concept of eventual permanence should be identified with the concept of eventual necessity, in the Diodorean sense of necessity. Reasons challenging this identification emerge in Section 2.

2. My remarks do not mean that the 'internal logic' of 'N' differs from 'L'. It does not. The difference would emerge in a combined system where '$NS \supset LS$' would be valid, but not '$LS \supset NS$'.

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